Exact slow-fast decomposition of the Hamilton-Jacobi equation of singularly perturbed systems

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Abstract

We study a Hamilton-Jacobi partial differential equation, arising in optimal control problem for an affine nonlinear singularly perturbed system. This equation is solvable iff there exists a special invariant manifold of the corresponding Hamiltonian system. We obtain exact slow-fast decomposition of the Hamiltonian system and of the special invariant manifold into the slow and the fast ones. We get sufficient conditions for the solvability of the Hamilton-Jacobi equation in terms of the reduced-order slow submanifold, or in the hyperbolic case, in terms of a reduced-order slow Riccati equation. On the basis of this decomposition we construct asymptotic expansions of the optimal state-feedback, optimal trajectory and optimal open-loop control in the powers of a small parameter.

1 Introduction

Nonlinear optimal control problem for singularly perturbed systems leads to a high dimensional Hamilton-Jacobi partial differential equation of two-time-scales for an optimal controller evaluation [3]. To alleviate the difficulties caused by the high dimensionality and the stiffness that result from the interaction of slow and fast dynamical modes, a composite controller was designed [3], [6]. This ϵ -

independent controller was based on the reduced-order slow and fast subproblems. Also series expansion method has been developed for approximate solution of Hamilton-Jacobi equations. It was shown that truncated expansion satisfies equations to $O(\epsilon^k)$ -accuracy. However near-optimality of the approximate solution (in the sense of its closeness to the exact one) has not been studied yet. It was proved in [1] that the leading term in the approximation converges to the exact solution as $\epsilon \rightarrow 0$.

In the linear case high-order numerical approximations were constructed in [10] on the basis of the exact decomposition of the full-order Riccati equation into the reduced-order Riccati and linear algebraic equations. In [4] exact decomposition of the singularly perturbed H_∞ Riccati equation was obtained and asymptotic approximation to its solution was constructed.

In the present paper we get the nonlinear counterpart of [10] and [4]. We apply the geometric approach of [5], [7] which relates Hamilton-Jacobi equations with special invariant manifolds of Hamiltonian systems. We obtain the exact decomposition of the special slow-fast manifold into the reduced-order slow submanifold of the Hamiltonian system and the fast manifold of an auxiliary system. We construct a higher-order approximation to the optimal closed-loop and open-loop controls and optimal trajectory in the form of expansion in the powers of ϵ .

2 Problem Formulation

Consider the optimal control problem for the system

$$\dot{x}_1 = a_1(x_1) + A_1(x_1)x_2 + B_1(x_1)u, \quad (2.1a)$$

$$\epsilon \dot{x}_2 = a_2(x_1) + A_2(x_1)x_2 + B_2(x_1)u, \quad (2.1b)$$

with respect to the functional

$$J = \int_0^\infty [r(x_1) + s'(x_1)x_2 + x_2'Q(x_1)x_2 + u'R(x_1)u]dt, \quad (2.1c)$$

where $x_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t) \in \mathbf{R}^{n_2}$ are the state vectors, $x = \text{col}\{x_1, x_2\}$, $u(t) \in \mathbf{R}^m$ is the control input. Prime denotes the transposition of a matrix. The functions a_i, A_i, B_i, p, s and Q are smooth. We assume also that $a_i(0) = 0, r(0) = 0$ and $s(0) = 0$. Assume that $Q \geq 0, R > 0$ and $r + s'x_2 + x_2'Qx_2 \geq 0$.

The system (2.1) has a standard singularly perturbed form in the sense that it is nonlinear only on the slow variable x_1 (see e.g. [3, 6]). However in the contrast to the standard case, A_{22} is allowed to be singular.

Denote by $|\cdot|$ the Euclidian norm of a vector. Consider the Hamiltonian function

$$\begin{aligned} \mathcal{H}(x_1, x_2, p_1, p_2) = & r + s'x_2 + x_2'Qx_2 + p_1'(a_1 + A_1x_2) \\ & + p_2'(a_2 + A_2x_2) - \frac{1}{4}(p_1'B_1 + p_2'B_2)R^{-1}(B_1'p_1 + B_2'p_2), \end{aligned} \quad (2.2)$$

where p_1 and ϵp_2 play the role of the costate variables. Denote $S_{ij} = \frac{1}{2}B_iR^{-1}B_j'$. The corresponding Hamiltonian system has the form:

$$\dot{x}_1 = f_1(x_1, p_1, x_2, p_2), \quad \dot{p}_1 = f_2(x_1, p_1, x_2, p_2), \quad (2.3a, b)$$

$$\epsilon \dot{x}_2 = A_2x_2 - S_{22}p_2 + f_3(x_1, p_1), \quad (2.3c)$$

$$\epsilon \dot{p}_2 = -Qx_2 - A_2'p_2 + f_4(x_1, p_1), \quad (2.3d)$$

where

$$f_1 = a_1 + A_1x_2 - S_{11}p_1 - S_{12}p_2, \quad f_2 = -\nabla_{x_1}\mathcal{H},$$

$$f_3 = a_2 - S_{21}p_1, \quad f_4 = -A_1'p_1 - s. \quad (2.4)$$

For each $\epsilon > 0$, if $V(x) \geq 0$ is a solution of the Hamilton-Jacobi equation

$$\begin{aligned} [r + s'x_1 + x_2'Qx_2] + V_{x_1}(a_1 + A_1x_2) + \frac{1}{\epsilon}V_{x_2}(a_2 + A_2x_2) \\ - \frac{1}{4}(V_{x_1}B_1 + \frac{1}{\epsilon}V_{x_2}B_2)R^{-1}(B_1'V_{x_1} + \frac{1}{\epsilon}B_2'V_{x_2}) = 0, \end{aligned} \quad (2.5)$$

where (V_{x_1}, V_{x_2}) denotes the Jacobian matrix of V , such that $V(0) = 0$ and the controller given by

$$u = -\frac{1}{2}R^{-1}[B_1', \epsilon^{-1}B_2']V_x' \quad (2.6)$$

is stabilizing, then this controller is the minimizing one. The latter is equivalent to the existence of the invariant manifold of (2.3)

$$p_1 = Z_1(x_1, x_2), \quad p_2 = Z_2(x_1, x_2), \quad (2.7)$$

where

$$V_{x_1} = Z_1', \quad V_{x_2} = \epsilon Z_2', \quad (2.8)$$

with asymptotically stable flow

$$\dot{x}_1 = a_1 + A_1x_2 - S_{11}Z_1 - S_{12}Z_2, \quad (2.9a)$$

$$\epsilon \dot{x}_2 = a_2 + A_2x_2 - S_{21}Z_1 - S_{22}Z_2 \quad (2.9b)$$

and such that $V \geq 0$ and $V(0) = 0$ (that implies $V_x(0) = 0$). Note that the manifold of the Hamiltonian system (2.3) because (2.9) needs not to be exponentially stable [5]. Also, asymptotic stability of (2.9) implies $V \geq 0$.

3 Main results

For each $x_1 \in \mathbf{R}^{n_1}$ consider the fast linear subproblem

$$\dot{x}_2 = A_2(x_1)x_2 + B_2(x_1)u, \quad (3.1)$$

$$J = \int_0^\infty [x_2'Q(x_1)x_2 + u'R(x_1)u]dt,$$

and the corresponding algebraic Riccati equation

$$A_2'M + MA_2 + Q - MS_{22}M = 0. \quad (3.2)$$

We assume further

A1. (3.2) has a positive definite symmetric solution $M(x_1)$, continuous on $x_1 \in \mathbf{R}^{n_1}$, such that for each $x_1 \in \mathbf{R}^{n_1}$ the matrix $A_2 - B_2 R^{-1} B_2' M$ is Hurwitz.

Note that if the pair $\{A_2(x_1), B_2(x_1)\}$ is controllable uniformly in x_1 , then A1 holds [6]. Consider the matrix

$$R_{22}(x_1) = \begin{pmatrix} A_2 & -S_{22} \\ -Q & -A_2' \end{pmatrix}. \quad (3.3)$$

Under A1 R_{22} possesses the following property: it has n_2 stable eigenvalues λ , $\text{Re} \lambda < -\alpha < 0$, and n_2 unstable ones λ , $\text{Re} \lambda > \alpha$ for all $|x_1| \leq m$. Then for any $m > 0$ there exists $\epsilon_m > 0$, such that for all $\epsilon \in (0, \epsilon_m]$ and $|x_1| + |p_1| < m$ the system (2.3) has the slow manifold [6], [9]

$$\begin{pmatrix} x_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} L_3^*(x_1, p_1, \epsilon) \\ L_4^*(x_1, p_1, \epsilon) \end{pmatrix} = L^*(x_1, p_1, \epsilon). \quad (3.4)$$

The subscripts of L^* correspond to the third and the fourth variables in the system of (2.3). To avoid cumbersome notation we shall omit ϵ argument in the functions below.

Setting (3.4) into (2.3a,b) and substituting v_1 and w_1 for x_1 and p_1 respectively, we get the $2n_1$ -dimensional system for the flow on the slow manifold:

$$\dot{v}_1 = f_1[v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)], \quad (3.5a)$$

$$\dot{w}_1 = f_2[v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)]. \quad (3.5b)$$

The function L^* can be found in the form of expansion

$$L^*(x_1, p_1, \epsilon) = \sum_{j=0}^q \epsilon^j l_j^*(x_1, p_1) + O(\epsilon^{q+1}). \quad (3.6)$$

The terms of (3.6) can be determined from the equation

$$\epsilon \frac{\partial L^*}{\partial x_1} f_1 + \epsilon \frac{\partial L^*}{\partial p_1} f_2 = \begin{pmatrix} A_2 L_3^* - S_{22} L_4^* + f_3(x_1, p_1) \\ -Q L_3^* - A_2' L_4^* + f_4(x_1, p_1) \end{pmatrix}, \quad (3.7)$$

where $f_i = f_i(x_1, p_1, L_3^*, L_4^*)$, $i = 1, 2$, by algebraic operations. Thus, $l_0^* = -R_{22}^{-1} f_0$, where $f_0 =$

$\text{col}\{f_3, f_4\}$. Note that (3.7) can be derived by differentiating on t of (3.4), where $x_1 = v_1(t)$, $p_1 = w_1(t)$, $x_2 = x_2(t)$, $p_2 = p_2(t)$, and by substituting for $\dot{v}_1$ and $\dot{w}_1$ the right sides of (3.5).

Consider the slow system (3.5). Denote by $\Omega_{m_i} = \{x_i \in \mathbf{R}^{n_i} : |x_i| < m_i\}$, $i = 1, 2$. Our next assumption is

A2. There exist $m_1 > 0$ and $\epsilon_1 > 0$ such that for all $\epsilon \in (0, \epsilon_1]$ and $v_1 \in \Omega_{2m_1}$ the system (3.5) possesses the invariant manifold

$$w_1 = N(v_1), \quad (3.8)$$

where the function $N = N(v_1, \epsilon)$ is continuous on both arguments and uniformly bounded together with its first derivative on v_1 , and $N(0) = 0$.

The restriction of (3.5) to (3.8) is governed by the n_1 -dimensional system

$$\dot{v}_1 = F_1(v_1), \quad (3.9)$$

where

$$F_i(v_1) = f_i[v_1, N(v_1), L_3^*(v_1, N(v_1)), L_4^*(v_1, N(v_1))] \quad (3.10)$$

and $i = 1$. Additionally we assume

A3. For all $\epsilon \in (0, \epsilon_1]$ equation (3.9) is asymptotically stable.

It can be shown that A2 and A3 are necessary conditions for the existence of the invariant manifold (2.7) with asymptotically stable flow (2.9). Note that the stable solutions of (2.3) are exponentially approaching to the solutions of the slow manifold (3.4) [9]. Under A1-A3 we shall construct the invariant manifold (2.7) with the stable flow by means of the slow submanifold (3.8) and a fast manifold of an auxiliary system. To get the latter system let us introduce the following change of variables:

$$\begin{pmatrix} v_2 \\ \bar{p}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} - L^*(x_1, p_1), \quad (3.11a)$$

$$\begin{pmatrix} \bar{x}_1 \\ \bar{p}_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} - \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \quad (3.11b)$$

where v_1 and w_1 satisfy (3.5). For the new variables we get the system that possesses for all small ϵ the fast (stable) manifold for $|v_2| < m'$ [9]

$$\begin{pmatrix} \bar{x}_1 \\ \bar{p}_1 \end{pmatrix} = \begin{pmatrix} \epsilon L_1^+(v_1, w_1, v_2) \\ \epsilon L_2^+(v_1, w_1, v_2) \end{pmatrix}, \quad \bar{p}_2 = L_4^+(v_1, w_1, v_2), \quad (3.12)$$

where $L_4^+ = M(v_1)v_2 + O(\epsilon)$. The solutions lying on this manifold are rapidly exponentially decaying as $t \rightarrow \infty$. Substituting (3.8) and (3.12) into (3.11) we get the algebraic equations for Z_1 and Z_2 determination:

$$x_1 = v_1 + \epsilon L_1^+[v_1, N(v_1), v_2], \quad (3.13a)$$

$$x_2 = v_2 + L_3^*[x_1, N(v_1) + \epsilon L_2^+(v_1, N(v_1), v_2)], \quad (3.13b)$$

and

$$p_1 = N(v_1) + \epsilon L_2^+[v_1, N(v_1), v_2], \quad (3.14a)$$

$$p_2 = L_4^*[x_1, N(v_1) + \epsilon L_2^+(v_1, N(v_1), v_2)] + L_4^+(v_1, N(v_1), v_2). \quad (3.14b)$$

Consider (3.14) as the system with respect to v_1 and v_2 . Using the contraction principle argument, one can prove that there exists ϵ_2 such that for $\epsilon \in (0, \epsilon_2]$, the system (3.14) has a unique solution on $\Omega_{m_1} \times \Omega_{m_2}$

$$v_1 = V_1(x_1, x_2) = x_1 + \epsilon \bar{V}_1(x_1, x_2), \quad (3.15a)$$

$$v_2 = V_2(x_1, x_2) = x_2 - L_3^*(x_1, N(x_1)) + \epsilon \bar{V}_2(x_1, x_2), \quad (3.15b)$$

where the functions $\bar{V}_1$ and $\bar{V}_2$ are Lipschitzian on x_1 and x_2 , they vanish at $(x_1, x_2) = 0$, and satisfy the inequalities $\epsilon_2 |\bar{V}_1| \leq m_1$, $\epsilon_2 |\bar{V}_2| \leq m'/2$. Further, applying the implicit function theorem one can show that V_1 and V_2 are continuously differentiable on x_1 and x_2 . Substituting (3.15) into (3.14a) and (3.14b) we get (2.7), where

$$Z_1 = N(V_1) + \epsilon L_2^+[V_1, N(V_1), V_2], \quad (3.16a)$$

$$Z_2 = L_4^*[x_1, N(V_1) + \epsilon L_2^+(V_1, N(V_1), V_2)]$$

$$+ L_4^+(V_1, N(V_1), V_2). \quad (3.16b)$$

Theorem 2. Under A1-A3 for any $m_2 > 0$ there exists $\epsilon_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2]$

(i) the $(2n_1 + 2n_2)$ -dimensional Hamiltonian system (2.3) has the invariant on $\Omega_{m_1} \times \Omega_{m_2}$ manifold (2.7) with (2.9) asymptotically stable, where continuously differentiable on x_1 and x_2 functions Z_1 and Z_2 are defined by formula (3.16) from the algebraic systems of (3.13) and (3.14);

(ii) there exists a C^2 function $V : \Omega_{m_1} \times \Omega_{m_2} \rightarrow \mathbf{R}$, satisfying the Hamilton-Jacobi equation (2.5).

To get asymptotic approximation of optimal feedback, optimal trajectory we assume

A4. For small enough ϵ and $|v_1| \leq 2m_1$ the function N can be represented in the form:

$$N(v_1, \epsilon) = \sum_{j=0}^q \epsilon^j N_j(v_1) + O(\epsilon^{q+1}).$$

Under A4 L^+ can be found in the form of asymptotic expansion. Then from (3.20) we obtain

$$Z_i = \sum_{j=0}^q \epsilon^j Z_{ij}(x_1, x_2, \epsilon) + O(\epsilon^{q+1}), \quad i = 1, 2, \quad (3.17)$$

Substituting (5.8) into (2.6) we get the following $O(\epsilon^{q+1})$ -approximation to the optimal controller :

$$u = u_q + O(\epsilon^{q+1}), \quad u_q = - \sum_{k=1}^2 \sum_{j=0}^q \epsilon^j B_k' Z_{kj}(x_1, x_2, \epsilon).$$

Finally substituting expansions of v_1 , v_2 , L^+ and L^* into (3.13) and (3.14) and expanding righthand sides of the resulting equations in the powers of ϵ we find the following approximations:

$$x(t) = \sum_{i=0}^q \epsilon^i x^{(i)}(t) + \sum_{i=0}^q \epsilon^i \Pi_1^{(i)}(\tau) + \epsilon^{q+1} R_{1q}(t, \epsilon),$$

$$u(t) = \sum_{i=0}^q \epsilon^i u^{(i)}(t) + \sum_{i=0}^q \epsilon^i \Pi_2^{(i)}(\tau) + \epsilon^{q+1} R_{2q}(t, \epsilon),$$

where Π_1 and Π_2 are boundary layer terms exponentially decaying when $\tau \rightarrow \infty$, remainders R_{1q} and R_{2q} are uniformly bounded for $t \in [0, T]$.

4 Conclusions

We have developed a geometric approach to singularly perturbed optimal control problem, nonlinear on the slow state variables. We have got the exact decomposition of the slow-fast invariant manifold of the Hamiltonian system into the reduced-order slow manifold and a fast manifold. As a result, sufficient conditions for the solvability of the optimal control problem in terms of the slow manifold have been obtained. Also, an asymptotic expansion of the optimal controller have been constructed by solving partial differential equations, depending only on the slow variables. In the same time we have obtained decomposition of the Hamiltonian system to the slow and fast subsystems. This leads to asymptotic approximation to optimal trajectory and open-loop control. The results are valid on the domains containing large values of the fast variables.

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