Response of a Grounded Dielectric Slab to an Impulse Line Source Using Leaky Modes

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Abstract—This paper describes how expansions in leaky (or improper) modes may be used to represent the continuous spectrum in an open radiating waveguide. The technique requires a thorough knowledge of the life history of the improper modes as they migrate from improper to proper Riemann surfaces. The method is illustrated by finding the electric field resulting from an impulsively forced current located in the free space above a grounded dielectric slab.

I. INTRODUCTION

ONE of the classic problems in antenna theory seeks the radiation field resulting from a point source above a grounded dielectric slab. It is now known that the formal solution consists of two parts: a discrete spectrum arising from the proper modes or normal modes and a continuous spectrum expressed as an integral along the branch cut associated with the proper Riemann surface. Because this integral is too complicated to be evaluated analytically, approximate methods have been used. During the deformation of the original contour to the steepest descent, the new path leads onto the improper Riemann surface and poles from this Riemann surface are sometimes crossed. When this occurs, contributions from these singularities must be added to the contributions from the steepest descent. These improper modes (if they exist) are generally called "leaky waves" because they literally leak energy away from the open waveguide. Further details are provided in [2], Section 11.8.

Several years ago, Veselov *et al.* [14] found the Green's function for TE-waves or TM-waves for a one-dimensional dielectric slab within a semi-infinite half-space with electric or magnetic "wall" boundary conditions. Their analysis was unique because it involved the solution of an initial-value problem using Laplace transforms and a Dirac delta function for the temporal forcing. This analysis yielded two interesting results. First, the entire transient solution consisted of discrete spectral functions that increased without bound in space when moving away from the slab's surface; that is, there was no branch cut integration. They noted the similarities between their solutions and the improper or nonmodal complex oscillations associated with leaky waves. Second, the unbounded spatial growth could never be realized because of the concurrent exponential decay of the solution with time. Their work suggests that if we solve the two-dimensional dielectric waveguide problem using Laplace transforms, then we might be able to construct the

Green's function in terms of improper as well as proper modes and eliminate the need for the branch cut integral.

Additional support for this idea was given in a study by Tsuk [13] who found the transient response resulting from a line-current being placed on the surface of a dielectric layer which lay on a grounded plane. Using the joint transform method, he inverted the transform by deforming the original contours to their steepest descent paths. This deformation of contours resulted in the major portion of the response being expressed as a sum of modes. Under certain circumstances this sum of modes gave the entire solution.

A similar situation occurs in geophysical waveguides, such as an oceanic layer overlying an infinitely deep half-space, where joint Laplace-Hankel transforms are used in finding the Green's function for the open waveguide. Over the last decade, Haddon [5]–[9], in a series of papers, has shown how (by deforming both the frequency and wavenumber contours of the inversion integrals) the continuous spectrum may be completely represented in terms of leaky waves without the need for a branch cut integral. That such a technique might occur in an open, electromagnetically radiating waveguide has been suggested in the work of Carpentier and dos Santos [1], Shatrov [10]–[11] and Tesler and Eichmann [12]. However, none of these authors developed a practical method to implement their ideas. The purpose of this paper is to show how Haddon's technique is used to construct leaky wave expansions. Here we choose the problem of a line source in the presence of a grounded slab.

II. FORMULATION

Consider a dielectric slab of depth a and index of refraction n which is placed on a perfectly conducting plane as shown in Fig. 1. A filament of current is located at $x = 0$ and $z = b$. Assuming that the magnitude of the current element is such that it is a unit source function, the electric fields in regions 1 and 2 satisfy the wave equations

$$\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_1}{\partial t^2} = -\delta(z-b)\delta(x)\delta(t), \quad z > a, \quad (1)$$

and

$$\frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial z^2} - \frac{n^2}{c^2} \frac{\partial^2 E_2}{\partial t^2} = 0, \quad 0 < z < a, \quad (2)$$

where c is the speed of light in the free space, x is the horizontal distance, z is the vertical distance and t is time. Because the dielectric is located on a conducting half-plane, $E_2(x, 0, t) = 0$ at $z = 0$. At infinity, the radiation condition

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Fig. 1. A dielectric slab on a perfectly conducting half-space with a line source of current above it.

requires that $\lim_{z \rightarrow \infty} |E_1(x, z, t)| \rightarrow 0$. Continuity in the electric and magnetic fields at the interface $z = a$ implies

$$E_1(x, a, t) = E_2(x, a, t) \text{ and } \frac{\partial E_1(x, a, t)}{\partial z} = \frac{\partial E_2(x, a, t)}{\partial z}. \quad (3)$$

We solve (1)–(3) by the joint application of Laplace and Fourier transforms. This joint transform of $E_j(x, z, t)$ is denoted by $\bar{E}_j(k, z, s)$ with $j = 1, 2$ where s and k are the transform variables of the Laplace and Fourier transforms, respectively. Assuming that the system is initially at rest, we have

$$\frac{d^2 \bar{E}_1}{dz^2} - \nu_1^2 \bar{E}_1 = -\delta(z - b), \quad z > a, \quad (4)$$

and

$$\frac{d^2 \bar{E}_2}{dz^2} - \nu_2^2 \bar{E}_2 = 0, \quad 0 < z < a, \quad (5)$$

along with boundary conditions $\bar{E}_2(k, 0, s) = 0$, $\lim_{z \rightarrow \infty} |\bar{E}_1(k, z, s)| \rightarrow 0$,

$$\bar{E}_1(k, a, s) = \bar{E}_2(k, a, s) \text{ and } \frac{d\bar{E}_1(k, a, s)}{dz} = \frac{d\bar{E}_2(k, a, s)}{dz} \quad (6)$$

where $\nu_1 = \sqrt{k^2 + s^2/c^2}$ and $\nu_2 = \sqrt{k^2 + n^2 s^2/c^2}$. Solutions which satisfy (4)–(6) are

$$\bar{E}_1 = \frac{e^{-\nu_1|z-b|}}{2\nu_1} - \frac{e^{-\nu_1(z+b-2a)}}{2\nu_1} \left[\frac{\nu_2 \cosh(\nu_2 a) - \nu_1 \sinh(\nu_2 a)}{\nu_2 \cosh(\nu_2 a) + \nu_1 \sinh(\nu_2 a)} \right] \quad (7)$$

where $z > a$, and

$$\bar{E}_2 = \frac{\sinh(\nu_2 z) e^{-\nu_1(b-a)}}{\nu_2 \cosh(\nu_2 a) + \nu_1 \sinh(\nu_2 a)}, \quad 0 < z < a. \quad (8)$$

Because (7)–(8) are even functions of ν_2 , no branch points are associated with ν_2 . However, there are branch points and cuts associated with ν_1 . In order to satisfy the radiation condition as $z \rightarrow \infty$, $\text{Re}(\nu_1) \geq 0$.

With (7)–(8), we may find $E_j(x, z, t)$ via the inversion integrals

$$E_j(x, z, t) = \frac{1}{4\pi^2 i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \left\{ \int_{-\infty}^{\infty} \bar{E}_j(k, z, s) e^{ikx} dk \right\} ds \quad (9)$$

where $\gamma > 0$ guarantees that the Laplace transform will converge. If we introduce $s = \omega i$, (9) may be rewritten as

$$E_j(x, z, t) = \frac{1}{2\pi^2} \text{Re} \left\{ \int_{0-\gamma i}^{\infty-\gamma i} \left[\int_{-\infty}^{\infty} \bar{E}_j(k, z, \omega) e^{i(kx+\omega t)} dk \right] d\omega \right\}. \quad (10)$$

The fields $E_j(x, z, t)$ vanish identically for $t < \tau(x, z)$ where $\tau(x, z)$ denotes the minimum travel-time for any disturbance originating at the source to reach the point (x, z) . We shall now find the solution for $t \geq \tau(x, z)$.

The solution in region 1 consists of two parts: the first term is the direct wave from the source while the second is the reflection from the dielectric slab which appears to emanate from a source located at $z = 2a - b$. The first term of (7) may be inverted exactly using the Cagniard-de Hoop method [3]; this technique yields

$$E_1^d(x, z, t) = \frac{1}{2\pi c} \frac{H(t - r/c)}{\sqrt{c^2 t^2 - r^2}} \quad (11)$$

where $r = \sqrt{x^2 + (z - b)^2}$ and $H()$ denotes the Heaviside step function. The second term in (7) and $\bar{E}_2(k, z, s)$ are conventionally evaluated by closing the line integral along the real k -axis as shown in Fig. 2. Because of symmetry about the $x = 0$ plane, we need only consider the case $x > 0$. The arcs at infinity vanish by Jordan's lemma so that the inverse Fourier transform consists of two parts: poles arising from the zeros of

$$\Delta(k, \omega) = \nu_2 \cosh(\nu_2 a) + \nu_1 \sinh(\nu_2 a) = 0 \quad (12)$$

where $\nu_1 = \sqrt{k^2 - \omega^2/c^2}$ and $\nu_2 = \sqrt{k^2 - n^2 \omega^2/c^2}$ and a branch cut integral. Therefore, in principle we can compute the electric field in two steps. First we compute the inverse Fourier transform in terms of the residues from the poles in the upper half of the k -plane, which lie between $k = -\omega/c$ and $k = -n\omega/c$ in this particular problem, and a branch cut integral for a specific (complex) value of ω along the contour shown in Fig. 2. Second, we sum up the contributions to invert the Laplace transform.

Neglecting for the moment any questions concerning how we would actually compute numerically the branch cut integral, another, more pressing problem appears as ω varies from $-\gamma i$ to $\infty - \gamma i$. Some of the poles on the improper (non-spectral) Riemann surface migrate until they cross the branch cut at the cutoff frequency and “pop” onto the proper Riemann surface. A consequence of this migration of poles is a discontinuous change in the branch cut integral at the cutoff frequency. To avoid this computational difficulty, we might deform the contour associated with the branch cut integral onto the improper Riemann surface so that we may pick up the various improper modes that eventually become normal modes (see Fig. 3). This is permissible only if the poles on

Fig. 2. Paths of integration for conventional inversion of the joint transform. The poles and branch cut occurring in the fourth quadrant have been suppressed.

Fig. 3. Deformation of the original branch cut integral so that it captures those improper (non-spectral) modes which will eventually become normal modes.

the improper Riemann surface are analytically continuous to its counterpart on the proper Riemann surface. If we remain with our original contour, then it is readily shown that this is not the case. Consequently, we must find a new ω contour which does produce an orderly, continuous evolution of poles from one Riemann surface to another. This is the essence of Haddon's technique: a deformation of *both* the k and ω contours. To sculpt such a contour, we must find those ω 's which result in various singularities on the k -plane.

First, to avoid any mathematical difficulties associated with a pass through the branch point $k = \omega/c$, our new ω contour must produce a $k_m(\omega)$ which avoids this point. This ω is found numerically by solving the equation $\Delta(\omega_m^{(C)}/c, \omega_m^{(C)}) = 0$ where we denote this ω by $\omega_m^{(C)}$ and in which m is a counter. For our problem, we find

$$k_m^{(C)} = \frac{\omega_m^{(C)}}{c} = \frac{(2m-1)\pi}{2a\sqrt{n^2-1}}, \quad m = 1, 2, 3, \dots \quad (13)$$

because $\nu_1 = 0$. Next, we choose whether the contour passes above or below the singularity. Because $\Delta(k, \omega)$ is regular at

Fig. 4. The values of $k_m(\omega)$ for $m = 3, 6, 9, \dots, 30$ along the new ω contour. As ω increases a pole $k_m(\omega)$ migrates from the improper Riemann surface (dotted line) to the proper Riemann surface (solid line).

this point, it makes no difference which choice we make.

Next, we avoid points where two poles, say $k_m(\omega)$ and $k_{m+1}(\omega)$, coalesce to form a second-order pole. This is purely for computational convenience so that we are always dealing with a simple pole. It may be shown algebraically that no such points exist in the present problem.

Another set of special ω 's are branch points where any contour around it results in an interchange of $k_m(\omega)$ for $k_m^*(\omega)$, the complex conjugate of $k_m(\omega)$. In this case, it is important to choose correctly whether to go over or under such branch points. These branch points will allow us to choose a $k_m(\omega)$ which has its origins in the upper half of the k -plane. Numerical experimentation reveals that the new ω contour must go above the singularity for this to occur. If our improper modes did not have their origin in the upper half plane, then $\text{Im}[k_m(\omega)] \rightarrow -\infty$ as $m \rightarrow \infty$ and the e^{ikx} factor would grow exponentially.

TABLE I
LOCATIONS OF VARIOUS BRANCH POINTS STATED IN TEXT
FOR $n = 1.6$ AND $a = 3.33$ cm. UNITS ARE IN cm^{-1} .

m	$\omega_m^{(B)}/c$	$\omega_m^{(C)}/c$	$\omega_m^{(D)}/c$
1	0.00000, 0.	0.37767, 0.	0.29482, 0.13761
2	1.10679, 0.	1.13301, 0.	0.88446, 0.13761
3	1.87290, 0.	1.88835, 0.	1.47410, 0.13761
4	2.63271, 0.	2.64370, 0.	2.06374, 0.13761
5	3.39051, 0.	3.39904, 0.	2.65337, 0.13761
6	4.14741, 0.	4.15438, 0.	3.24301, 0.13761
7	4.90382, 0.	4.90972, 0.	3.83265, 0.13761
8	5.65995, 0.	5.66506, 0.	4.42229, 0.13761
9	6.41590, 0.	6.42040, 0.	5.01193, 0.13761
10	7.17171, 0.	7.17574, 0.	5.60156, 0.13761
11	7.92744, 0.	7.93109, 0.	6.19120, 0.13761
12	8.68310, 0.	8.68643, 0.	6.78084, 0.13761
13	9.43871, 0.	9.44177, 0.	7.37048, 0.13761
14	10.19427, 0.	10.19711, 0.	7.96012, 0.13761
15	10.94981, 0.	10.95245, 0.	8.54975, 0.13761
16	11.70532, 0.	11.70779, 0.	9.13939, 0.13761
17	12.46081, 0.	12.46313, 0.	9.72903, 0.13761
18	13.21629, 0.	13.21848, 0.	10.31867, 0.13761
19	13.97175, 0.	13.97382, 0.	10.90831, 0.13761
20	14.72720, 0.	14.72916, 0.	11.49795, 0.13761
21	15.48263, 0.	15.48450, 0.	12.08758, 0.13761
22	16.23806, 0.	16.23984, 0.	12.67722, 0.13761
23	16.99348, 0.	16.99518, 0.	13.26686, 0.13761
24	17.74890, 0.	17.75052, 0.	13.85650, 0.13761
25	18.50430, 0.	18.50587, 0.	14.44614, 0.13761
26	19.25971, 0.	19.26121, 0.	15.03578, 0.13761
27	20.01511, 0.	20.01655, 0.	15.62541, 0.13761
28	20.77050, 0.	20.77189, 0.	16.21505, 0.13761
29	21.52589, 0.	21.52723, 0.	16.80469, 0.13761
30	22.28128, 0.	22.28257, 0.	17.39433, 0.13761

TABLE II
CORRESPONDING WAVENUMBERS k OF THE SINGULARITIES
LISTED IN TABLE I. UNITS ARE IN cm^{-1} .

m	$k_m^{(B)}$	$k_m^{(C)}$	$k_m^{(D)}$
1	- 5.20000, 0.	- 0.37767, 0.	0., 0.
2	- 1.14681, 0.	- 1.33301, 0.	0., 0.
3	- 1.89682, 0.	- 1.88835, 0.	0., 0.
4	- 2.64978, 0.	- 2.64370, 0.	0., 0.
5	- 3.40378, 0.	- 3.39904, 0.	0., 0.
6	- 4.15826, 0.	- 4.15438, 0.	0., 0.
7	- 4.91301, 0.	- 4.90972, 0.	0., 0.
8	- 5.66791, 0.	- 5.66506, 0.	0., 0.
9	- 6.42292, 0.	- 6.42040, 0.	0., 0.
10	- 7.17800, 0.	- 7.17574, 0.	0., 0.
11	- 7.93313, 0.	- 7.93109, 0.	0., 0.
12	- 8.68829, 0.	- 8.68643, 0.	0., 0.
13	- 9.44348, 0.	- 9.44177, 0.	0., 0.
14	- 10.19870, 0.	- 10.19711, 0.	0., 0.
15	- 10.95393, 0.	- 10.95245, 0.	0., 0.
16	- 11.70918, 0.	- 11.70779, 0.	0., 0.
17	- 12.46443, 0.	- 12.46313, 0.	0., 0.
18	- 13.21970, 0.	- 13.21848, 0.	0., 0.
19	- 13.97498, 0.	- 13.97382, 0.	0., 0.
20	- 14.73026, 0.	- 14.72916, 0.	0., 0.
21	- 15.48555, 0.	- 15.48450, 0.	0., 0.
22	- 16.24084, 0.	- 16.23984, 0.	0., 0.
23	- 16.99614, 0.	- 16.99518, 0.	0., 0.
24	- 17.75144, 0.	- 17.75052, 0.	0., 0.
25	- 18.50674, 0.	- 18.50587, 0.	0., 0.
26	- 19.26205, 0.	- 19.26121, 0.	0., 0.
27	- 20.01736, 0.	- 20.01655, 0.	0., 0.
28	- 20.77277, 0.	- 20.77189, 0.	0., 0.
29	- 21.52798, 0.	- 21.52723, 0.	0., 0.
30	- 22.28330, 0.	- 22.28257, 0.	0., 0.

Locating these branch points is easy because they represent points where two solutions k_m and k_m^* merge. Consequently, the ω 's are also given by the solution of the system of equations $\Delta(k, \omega) = \Delta_k(k, \omega) = 0$. We denote these branch points as $\omega_m^{(B)}$. The corresponding $k_m^{(B)}$ lies on the improper Riemann surface.

Finally, another branch point occurs for $k = 0$. When a path is taken around $k = 0$, the root $k_m(\omega)$ is replaced by $-k_m(\omega)$. In this case, the corresponding $\omega_m^{(D)}$ is given by

$$\omega_m^{(D)} = \frac{c}{2na} \left[(2m-1)\pi + i \ln \left(\frac{n+1}{n-1} \right) \right], m = 1, 2, 3, \dots \quad (14)$$

Because our sum of residues must be a regular analytical function of ω (for reasons explained below), any closed path in the ω -plane must result in a root in the second quadrant of the k -plane which returns to its initial value at the completion of the closed contour. Constructing several simple closed paths showed that this occurs if we pass over $\omega_m^{(B)}$ and under $\omega_m^{(D)}$.

When additional multivalued functions are present, more special values of ω must be found. The interested reader is referred to several of Haddon's papers [6]–[9] in which there were three Riemann surfaces.

Finally we consider the new branch cut integral. Our deformation of the contour onto the improper Riemann surface guarantees that the branch cut integral varies continuously with ω . We will now show that our new ω contour results in the

branch cut integral vanishing. We do this by first showing that the branch cut integral is a regular analytic function along any contour that passes above $\omega_m^{(B)}$ and remains in the first quadrant of the ω -plane. The proof begins by noting that the inverse Fourier transform, with its integration along the real k axis, is a regular analytic function. This is true even after we deform the contour onto the improper Riemann surface because analytical continuity was maintained. Finally, the sum of the residues is a regular analytical function of ω in the cut right-half of the ω -plane; the cuts are associated with $\omega_m^{(B)}$ and $\omega_m^{(D)}$. Thus, the branch cut integration must be a regular analytical function of ω . The point $\omega_m^{(D)}$ is of no concern here because as long as we pass above $\omega_m^{(B)}$ no root can cross the path of the branch cut integration and $\omega_m^{(D)}$ cannot be a singularity of the branch cut integration.

Let us now return to our deformed contour which passes above $\omega_m^{(B)}$ and below $\omega_m^{(D)}$ and close it with vertical and horizontal segments along the lines $\text{Re}(\omega) = 0$, $\text{Re}(\omega) = W_1$ and $\text{Im}(\omega) = W_2$. By Cauchy's theorem,

$$\text{Re} \left[\oint_{\Omega} L_j(\omega) e^{i\omega t} d\omega \right] = 0 \quad (15)$$

where Ω is the closed contour described above and $L_j(\omega)$ denotes our branch cut integral. The contribution to the closed integral along the imaginary axis is zero because the integral is purely imaginary. The contribution from the portion of Ω along

Fig. 5. The real and imaginary part of the residues given by (20) for $m = 1, 3, 5, \dots, 13$ as a function of the real part of ω/c and the imaginary part of ω/c equaling 0.05 cm^{-1} . The $m = 1$ mode was scaled by dividing its original value by 5.

the lines $\text{Re}(\omega) = W_1$ also vanishes as we take $W_1 \rightarrow \infty$. Thus, we must verify that

$$\lim_{W_2 \rightarrow \infty} \int_{W_2 i}^{\infty + W_2 i} L_j(\omega) e^{i\omega t} d\omega = 0. \quad (16)$$

Following Haddon [6], we expand the denominator of the integrand of $L_j(\omega)$ as a geometric series of exponentials. Because each term can be integrated separately (and none of them contain poles) the integrals can be evaluated by the method of steepest descent. Finally, each asymptotic result vanishes in the limit $W_2 \rightarrow \infty$. The reason behind this remarkable result lies with the convergence factor $e^{i\omega t}$ that was introduced with the modified contour because $\text{Im}(\omega) > 0$. It is this convergence factor that allows the transients to be completely expressed in terms of nonmodal functions.

In summary, by a suitable modification of the original ω contour (in the present case, over $\omega_m^{(B)}$ and $\omega_m^{(C)}$ and under $\omega_m^{(D)}$), each pole initially in the second quadrant of the improper (non-spectral) Riemann surface follows a trajectory which leads around $k_m^{(B)}$ and $k_m^{(C)}$ onto the proper Riemann surface. This new contour has the tremendous advantage that the entire solution is composed both of normal and of non-spectral modes.

III. NUMERICAL COMPUTATIONS

In this section we detail how to apply the theoretical results of the previous section to numerically evaluate the Green's function for a grounded dielectric slab subjected to a impulse line source. Before we deform the ω contour from the fourth quadrant of the ω -plane to the first quadrant, we must calculate $\omega_m^{(B)}$, $\omega_m^{(C)}$ and $\omega_m^{(D)}$. For a specified a , c and n we obtain these points either analytically from (13) and (14) or numerically

Fig. 6. Along the top, the temporal evolution of that portion of E_1 corresponding to the reflected wave field (consisting of all 30 modes) in region 1 for the sample point $x = 20 \text{ cm}$ and $z = 5 \text{ cm}$. Below this trace, the temporal evolution of the first ten individual modes at the same point.

using the IMSL routine NEQNF which solves nonlinear, simultaneous equations. Table I shows the various ω 's for the first thirty poles in the ω -plane; the corresponding k 's are given in Table II. In these and all other calculations, $a = 3.33 \text{ cm}$, $b = 5 \text{ cm}$ and $n = 1.6$.

Fig. 7. The temporal evolution of the reflected wave fields in region 1 for various values of x for $z = 5$ cm.

Once these ω_m 's are found, our new ω contour must pass over $\omega_m^{(B)}$ and under $\omega_m^{(D)}$ and avoid passing through $\omega_m^{(C)}$. Because $\omega_m^{(B)}$ and $\omega_m^{(C)}$ lie on the real k axis while $\text{Im}(\omega_m^{(D)}/c) = 0.13761 \text{ cm}^{-1}$, our ω contour will pass over $\omega_m^{(B)}$ and under $\omega_m^{(D)}$ if we choose a line parallel to the real ω axis with $0 < \text{Im}(\omega/c) < 0.13761 \text{ cm}^{-1}$. For convenience we choose the same line for all of the $k_m(\omega)$ poles, namely $W_s/c = \text{Im}(\omega/c) = 0.05 \text{ cm}^{-1}$. A practical consideration is to pass sufficiently far from the singularities on the real ω axis. We note that different contours could have been chosen for each m and the contour need not be a straight line.

For each ω contour, we compute $k_m(\omega)$ by Newton's method. The definition of changing from one Riemann surface to another is $|\text{Re}(\nu_1)| < 5 \times 10^{-3} \text{ cm}^{-1}$. In Fig. 4 the $k_m(\omega)$ contours are displayed for the $m = 3, 6, 9, \dots, 30$ modes for $0 < \text{Re}(\omega/c) < 25 \text{ cm}^{-1}$. A dotted line is used when the mode lies on the improper Riemann surface. Note that most of the trajectory on the improper mode is not shown in Fig. 4 because it has an imaginary part greater than 0.13 cm^{-1} .

For reasons that will become clear, we reversed the order of summation and integration so that, for example,

$$E_1(x, z, t) = \sum_m R_m \quad (17)$$

where

$$R_m = \text{Re} \left\{ \frac{i}{\pi} e^{i\omega_m^{(C)} t} \int_{W_s i}^{\infty + W_s i} [\text{Residue at } k = k_m \text{ of } \bar{\mathcal{E}}_1(k, z, \omega)] e^{i[k_m x + (\omega - \omega_m^{(C)})t]} d\omega \right\}. \quad (18)$$

We integrated (18) using a Cote's sixth-order formula with $\text{Re}(\Delta\omega/c) = 0.005 \text{ cm}^{-1}$ and $0 < \text{Re}(\omega/c) < 50 \text{ cm}^{-1}$. The result of the integration was multiplied by $ie^{i\omega_m^{(C)} t}/\pi$ before taking the real part. Similar steps were taken with $\bar{\mathcal{E}}_2(k, z, \omega)$. In order to obtain finite results in the following calculations, it is necessary to choose a well-behaved forcing function. To model the impulse forcing, we adopted the following:

$$f(t) = \frac{1}{\sqrt{\pi\epsilon}} e^{-t^2/\epsilon}, \quad F(\omega) = \frac{1}{2\pi} e^{-\epsilon\omega^2/4} \quad (19)$$

where $\epsilon c^2 = 0.01 \text{ cm}^{-2}$. This choice of forcing is convenient because it does not introduce further singularities.

In Fig. 5 we show the real and imaginary part of the residues:

$$\text{Residue}(k_m, z, \omega) = -\frac{e^{-\nu_1(z+b-2a)}}{2\nu_1} \times \left[\frac{\nu_2 \cosh(\nu_2 a) - \nu_1 \sinh(\nu_2 a)}{\Delta_k} \right] e^{ik_m x} \quad (20)$$

for the first modes $m = 1, 3, \dots, 13$ when $x = 20$ cm and $z = 5$ cm. As the figure shows, the residues are small except in the vicinity of $\omega_m^{(B)}$ and $\omega_m^{(C)}$. Consequently, by performing the algebraic trick given in (18), the integrand can be made a slowly varying function of ω where the integrand is most significant and considerable accuracy can be achieved.

Having obtained some knowledge of the behavior of the residues, we now illustrate in Fig. 6 the temporal evolution of that portion of $E_1(x, z, t)$ that corresponds to the reflected wave in region 1 (the inversion of the second term in (7)) at the sample point $x = 20$ cm and $z = 5$ cm. Strictly speaking, the

results are valid only for $ct > \sqrt{x^2 + (z + b - 2a)^2}$ because that is the least amount of time before the first reflected wave would reach the observer. After the first pulse we see smaller pulses as the multiple reflections that occur within the dielectric reach the observer as predicted by asymptotic analysis (see Section II in [4]).

Finally, in Fig. 7 we illustrate the temporal evolution of the reflected wave field in region 1 for various values of x when $z = 5$ cm. The first pulse gives the reflection of the direct wave by the interface at $z = a$. The additional pulses result from multiple reflections within the dielectric. Although the early pulses decrease in amplitude as x increases, note how the fifth pulse actually increases in amplitude.

IV. CONCLUSION

In this paper we showed that leaky modes may be used to construct the continuous spectrum of an open radiating structure. This was accomplished by deforming the traditional Bromwich contour so that a continuous trajectory of modes from the improper to proper Riemann surfaces occurs. Computationally, this replaces the cumbersome numerical evaluation of a branch cut integral by a summation of a few residue terms.

We illustrated this technique by finding the electric field inside and outside of a grounded dielectric slab which was impulsively forced in the free space. The most difficult aspect of the analysis involves following the life history of each mode. One must be sure that its evolution (with frequency) flows smoothly, avoiding branch points or coalescence with other modes. Given the proper choice of whether you pass above or below certain branch points, the contour may be shaped so that it originates in the proper half of the k -plane. Although our technique does require considerable exploratory analysis, much of this analysis would be required in a straightforward integration of the branch cut integral.

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