Theory of the Optimally Coupled Q-Switched Laser

JOHN J. DEGNAN

Abstract—**The general equations describing Q-switched laser operation are transcendental in nature and require numerical solutions. This greatly complicates the optimization of real devices. In this paper, we demonstrate that, using the mathematical technique of Lagrange multipliers, one can derive simple analytic expressions for all of the key parameters of the optimally coupled laser, i.e., one which uses an optimum reflector to obtain maximum laser efficiency for a given pump level. These parameters (which include the optimum reflectivity, output energy, extraction efficiency, pulsewidth, peak power, etc.) can all be expressed as functions of a single dimensionless variable z, defined as the ratio of the unsaturated small-signal gain to the dissipative (nonuseful) optical loss, multiplied by a few simple constants. Laser design tradeoff studies and performance projections can be accomplished quickly with the help of several graphs and a simple hand calculator. Sample calculations for a high-gain Nd:YAG and a low-gain alexandrite laser are presented as illustrations of the technique.**

I. Introduction

The equations describing the operation of rapidly Q-switched lasers were first derived over two decades ago [1], [2]. The Q-switched laser problem involves the simultaneous solution of two coupled differential equations for the time rates of change of the population inversion density and the internal photon density in the resonator. Subsequent textbook descriptions of the theory [3] provided approximate expressions for parameters which are more directly useful to the laser engineer—such as laser energy, peak power, pulse duration, etc. Unfortunately, these equations are expressed in terms of the initial and final population inversion densities which depend not only on the particular choice of output coupler, but which are also related via a cumbersome transcendental equation. Thus, in order to optimize a given laser for maximum efficiency, it is generally necessary to obtain numerical solutions on a computer.

In Appendix A, we provide a complete and consistent derivation of the fundamental Q-switched equations in order to clarify some of the assumptions and notation in earlier treatments and to derive some expressions for key device parameters which are useful to the laser engineer. In Section II of the present paper, we consider the theoretical problem of choosing the optimum output coupler reflectivity which maximizes the output energy and hence the laser efficiency at a particular pump level. We demonstrate that, using the mathematical technique of Lagrange multipliers, one can derive simple analytic expressions for all of the key parameters (optimum reflector, output energy, extraction efficiency, pulsewidth, peak external and internal powers, etc.) which describe the optimally coupled laser. The aforementioned optimized laser parameters can be expressed as functions of a single dimensionless variable z, defined as the ratio of the unsaturated logarithmic small-signal gain to the dissipative (nonuseful) optical loss, multiplied by a few simple and easily accessible constants. In Section III, we present several design curves which are useful to laser design and/or the projection of potential laser performance. The specific examples of Q-switched Nd:YAG and alexandrite lasers are considered to illustrate the use of the design curves.

Manuscript received February 16, 1988; revised October 3, 1988.
The author is with the Instrument Electro-Optics Branch, Code 723, NASA Goddard Space Flight Center, Greenbelt, MD 20771.
IEEE Log Number 8825439.

II. Theory of the Optimally Coupled Q-Switched Laser

As illustrated in Appendix A, the rapidly Q-switched laser problem reduces to the simultaneous solution of two coupled differential equations for the photon density ϕ and the population inversion density n given by

$$\frac{d\phi}{dt} = \frac{2\sigma n l \phi}{t_r} - \frac{\phi}{t_c} \tag{1}$$

and

$$\frac{dn}{dt} = -\gamma\sigma c \phi n. \tag{2}$$

In the latter equations, σ is the stimulated emission cross section, l is the length of the laser medium, c is the speed of light, $t_r = 2l'/c$ is the roundtrip transit time in the laser resonator of length l', and γ is an "inversion reduction factor" discussed at length in Appendix B. The quantity t_c in (1) is the photon decay time defined by

$$t_c = \frac{t_r}{\left[\ln\left(\frac{1}{R}\right) + L\right]} \tag{3}$$

where R is the output mirror reflectivity and L is the roundtrip dissipative optical loss defined by (A.2) in Appendix A. An expression for the laser output pulse energy is derived in Appendix A, i.e.,

$$E = \frac{h\nu A}{2\sigma\gamma} \ln\left(\frac{1}{R}\right) \ln\left(\frac{n_i}{n_f}\right) \tag{4}$$

where $h\nu$ is the laser photon energy and A is the effective beam cross-sectional area. As can be seen in the derivation presented in Appendix A, (4) is an exact result to the extent that the time dependence of the spatially averaged photon density in the resonator is truly described by the fundamental Q-switch equations (1) and (2). In situations where there is a strong longitudinal variation in photon density brought about by high gain or high transmission losses, deviations from this result might be expected. The initial and final population inversion densities n_i and n_f are related by the usual transcendental equation

$$n_i - n_f = n_t \ln\left(\frac{n_i}{n_f}\right) \quad (5)$$

where n_t is the population inversion density at threshold. It is of great practical interest to determine the value of the output mirror reflectivity which maximizes the output energy of the Q-switched laser, and hence the efficiency, for a given initial inversion density n_i. The inversion density is usually linearly proportional to the pump excitation rate—at least prior to significant pump saturation or the onset of parasitic amplified stimulated emission (ASE). It will now be demonstrated that, in spite of the transcendental nature of (5), simple analytic expressions can be obtained for all of the aforementioned parameters of the optimally coupled laser using the Lagrange multiplier technique.

We begin by defining the new variables

$$x = \ln\left(\frac{1}{R}\right) \quad (6a)$$

$$y = \frac{n_f}{n_i} \quad (6b)$$

which, when substituted in (4), yield

$$E(x, y) = -\frac{Vh\nu}{2\sigma\gamma l} x \ln y \quad (7)$$

where $v = Al$ is the active gain volume. With the help of (A.8), (5) then becomes the constraint equation

$$\varphi(x, y) = 1 - y + \frac{(x + L)\ln y}{2\sigma n_i l} = 0. \quad (8)$$

Using the Lagrange multiplier method [4], we now wish to solve the simultaneous equations

$$\frac{\partial E}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \quad (9a)$$

and

$$\frac{\partial E}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \quad (9b)$$

where λ is the Lagrange multiplier. These equations yield the solutions

$$\lambda = \frac{Vh\nu n_i}{\gamma} \quad (10a)$$

and

$$y = \frac{L}{2\sigma n_i l} \equiv \frac{L}{2g_0 l} \equiv \frac{1}{z} \quad (10b)$$

where we recognize the quantity σn_i as the unsaturated small-signal gain coefficient g_0 and we have defined a new dimensionless parameter z, corresponding to the ratio of the roundtrip small-signal gain to roundtrip dissipative loss. Substituting (10b) into (8) and solving for x yields an expression related to the optimum reflectivity which depends only on the dissipative optical loss L and the gain to dissipative loss ratio z, i.e.,

$$x_{\text{opt}} = \ln\left(\frac{1}{R_{\text{opt}}}\right) = L\left[\frac{z - 1 - \ln z}{\ln z}\right] \quad (11a)$$

with the optimum output coupler reflectivity given by

$$R_{\text{opt}} = \exp\left[-x_{\text{opt}}\right]. \quad (11b)$$

Equation (11a) is plotted in Fig. 1.

We can now express all of the important parameters of the optimized Q-switched laser as relatively simple analytic functions of the dimensionless parameter z multiplied by a few easily obtained constants. Substitution of (10b) and (11a) into (7) yields a simple expression for the optimized output energy, i.e.,

$$E_{\max} = \frac{Ah\nu L}{2\sigma\gamma}\left[z - 1 - \ln z\right] \quad (12)$$

which is plotted in Fig. 2. In the limit of large z, the output energy approaches the total useful energy stored in the rod given by

$$E_u = \lim_{z\to\infty} E_{\max} = \frac{Ah\nu L}{2\sigma\gamma} z = \frac{Vh\nu n_i}{\gamma}. \quad (13)$$

Thus, one can define an energy extraction efficiency

$$n_E(z) = \frac{E_{\max}}{E_u} = 1 - \frac{(1 + \ln z)}{z} \quad (14)$$

plotted in Fig. 3. Fig. 3 makes it clear that, beyond the choice of an optimum coupler, laser efficiency can only be improved by increasing z. This can be accomplished either by 1) reducing the dissipative loss L, or 2) increasing and/or further concentrating the pump energy within the active mode volume. It is also interesting to note that the laser is over 80 percent efficient for z values greater than 20 and increases only slowly with the deposition of still more pump energy.

Substituting (11a) in (A.8), we obtain a simple expression for the threshold inversion density in the optimally coupled laser, i.e.,

$$n_t = \frac{L}{2\sigma l}\left[\frac{z - 1}{\ln z}\right]. \quad (15)$$

Using (6), (10), (11), and (15) in (A.13) and (A.16), we can also write analytic equations for the peak extracavity laser power:

$$P_{\max} = \frac{Ah\nu L^2}{2\sigma\gamma t_r}\left[\frac{z - 1 - \ln z}{\ln z}\right] \cdot \left\{z - \left[\frac{z - 1}{\ln z}\right]\left[1 + \ln\left(\frac{z \ln z}{z - 1}\right)\right]\right\} \quad (16)$$

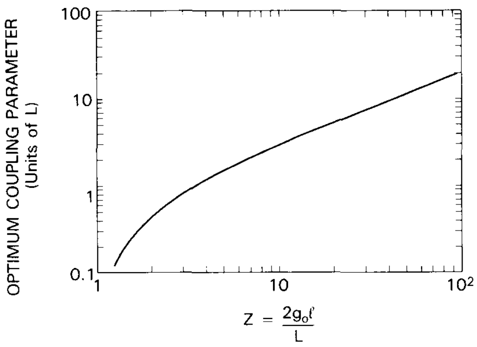

Fig. 1. The optimized laser coupling parameter x_{opt} expressed in units of the two-way dissipative loss L defined by (A.2). The optimum mirror reflectivity is given by $R_{opt} = \exp[-x_{opt}]$ where x_{opt} is the graph value multiplied by L.

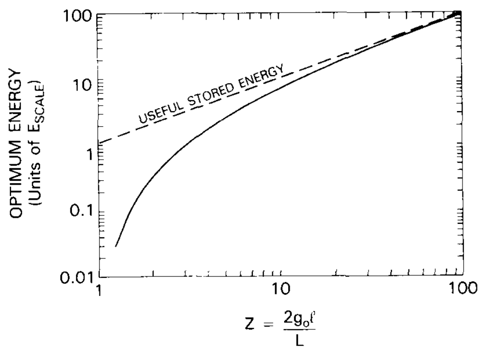

Fig. 2. The output energy of the optimally coupled laser E_{max} expressed in units of $E_{scale} = Ah\nu L/2\sigma\gamma$ where A is the average beam cross-sectional area in the laser media, $h\nu$ is the photon energy, σ is the stimulated emission cross section, γ is the inversion reduction factor, and L is the two-way dissipative optical loss.

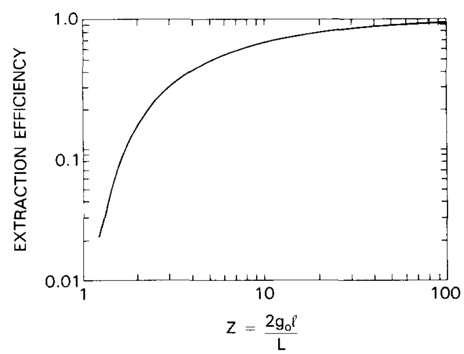

Fig. 3. The extraction efficiency of the optimally coupled laser expressed as a function of the dimensionless parameter z.

and the FWHM pulsewidth:

$$t_p = \frac{t_r}{L}\left\{\left[\frac{\ln z}{z}\right]\left[\frac{1}{1-\left[\frac{z-1}{\ln z}\right]\left[1+\ln\left[\frac{z\ln z}{z-1}\right]\right]}\right]\right\}. \qquad (17)$$

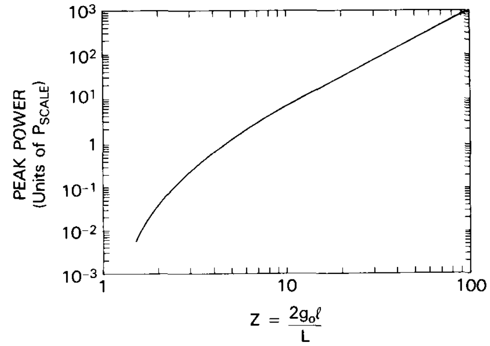

Fig. 4. The peak power P_{max} of the optimally coupled laser expressed in units of $P_{scale} = h\nu AL^2/2\sigma\gamma t_r$ where $h\nu$ is the photon energy, L is the roundtrip dissipative optical loss, σ is the stimulated emission cross section, γ is the inversion reduction factor, and t_r is the roundtrip cavity transit time. This curve can also be used in conjunction with Fig. 1 and (17a) and (17b) to compute the peak two-way internal circulating power at the output mirror and rear reflector.

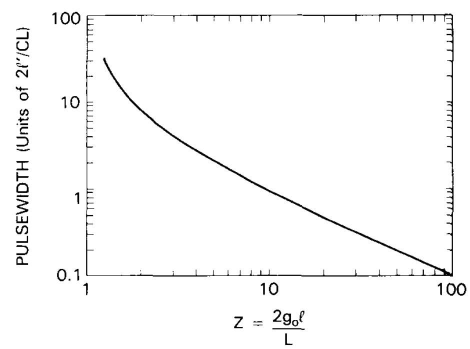

Fig. 5. The FWHM pulsewidth of the optimally coupled laser in units of t_r/L where t_r is the roundtrip cavity transit time and L is the two-way dissipative optical loss.

The latter two equations are plotted in Figs. 4 and 5, respectively. Figs. 1 and 4 can also be used to calculate the intracavity intensities at the output and rear reflectors via (A.17a) and (A.17b), respectively.

III. Using the Design Curves: Nd:YAG and Alexandrite Examples

Figs. 1-5 provide useful normalized design plots of the optimum coupling parameter, maximum output energy, extraction efficiency, peak power, and FWHM pulsewidth. A typical design sequence would be to first read the minimum value of z necessary to achieve a certain laser output energy from Fig. 2. The value of the coupling parameter x_{opt} (or, equivalently, the optimum mirror reflectivity) corresponding to that value of z can then be read off the ordinate of Fig. 1. This combination of pump level and mirror reflectivity represents the most efficient means of generating the desired laser energy for a given laser beam cross section A and optical dissipative loss L. The corresponding extraction efficiency can be read from Fig. 3. Clearly, the nonuseful loss L and pumping volume should be reduced to their smallest possible values in order to achieve maximum laser efficiency. The values for peak power and pulsewidth associated with the optimized laser can be read directly from the ordinates of Figs. 4 and 5, respectively. Figs. 1 and 4 can also be used to obtain the peak circulating power inside the laser.

As an illustration of the technique, we consider the most efficient design of a multimode Nd:YAG laser having a modest output energy of $E = 50$ mJ. We first compute $E_{\text{scale}} = Ah\nu L/2\sigma\gamma$. Three of the constants depend on the laser material itself, i.e., the laser photon energy $h\nu = 1.87 \times 10^{-19}$ J, the spectroscopic stimulated emission cross section $\sigma = 6.5 \times 10^{-19}$ cm^2 (from [5]), and $\gamma = 1.3$ (see Appendix B). If we assume a 2 mm radius multimode beam and a two-way dissipative loss of 4 percent, we obtain $AL = \pi(0.2 \text{ cm})^2(0.04) = 0.005$ cm^2 and $E_{\text{scale}} = 0.55$ mJ. We now compute the ratio $E/E_{\text{scale}} = 90.9$ and read the corresponding value of $z = 90$ off the abscissa in Fig. 2. The logarithmic single-pass small-signal gain necessary to achieve this energy is given by $g_0 l = zL/2 = 90(0.04)/2 = 1.8$, corresponding to a power gain $G_0 = \exp(g_0 l) = 6.05$. Reading off the ordinate of Fig. 3, we obtain an energy extraction efficiency of about 91 percent. From Fig. 1, the value of the coupling parameter x_{opt} corresponding to $z = 90$ is $x_{\text{opt}} = 18.7L = 18.7(0.04) = 0.75$, which when substituted into (11), yields an optimum output mirror reflectivity of 47 percent. From Fig. 5, the FWHM laser pulsewidth is $t_p = 0.11(t_r/L) = 2.8t_r = 5.6$ ns if we assume a nominal 30 cm long cavity. From Fig. 4, the peak power in the laser pulse is $P_{\max} = 800\ P_{\text{scale}}$ where $P_{\text{scale}} = h\nu AL^2/2\sigma\gamma t_r = 1.11 \times 10^4$ W. Thus, $P_{\max} = 800(1.11 \times 10^4 \text{ W}) = 8.85$ MW. Using (A.17a) and (A.17b) in Appendix A and the value $R = 0.47$ obtained earlier from Fig. 1, we can also compute the two-way peak circulating powers internal to the laser, i.e., $P_R = 2.77P_{\max} = 24.5$ MW at the output coupler and $P_{R'} = 2.58P_{\max} = 22.9$ MW at the rear reflector. With the beam area assumed at the outset, the average two-way circulating intensity is about 189 MW/cm^2.

We could have computed the peak laser power directly from the original energy and pulsewidth, i.e., $P_{\max} = 0.050 \text{ J}/5.6 \times 10^{-9} \text{ s} = 8.9$ MW without resorting to Fig. 4. However, one might want to design a laser which gives a particular peak power for a nonlinear optics experiment. In this instance, one could work backwards starting with Fig. 4 to determine a minimum z and then reading parameters off the other figures. One can also compute the maximum energy achievable with a particular small-signal gain (pump level). Whatever constraint one applies, an optimally coupled (maximum efficiency) configuration satisfying that constraint results.

Nd:YAG has a relatively high stimulated emission cross section compared to other solid-state materials such as Nd:glass and alexandrite, and it is worthwhile to illustrate the different results one gets with a lower gain medium. In this example, we will assume the same output energy, beam cross section, and dissipative loss for an alexandrite laser operating at 727 nm—well off the peak of its gain curve at 755 nm. Using $h\nu = 2.73 \times 10^{-19}$ J, an effective stimulated emission cross section $\sigma^* = 1.1 \times 10^{-20}$ cm^2 (from [6]), and an effective inversion reduction factor $\gamma^* = 1.02$, we compute $E_{\text{scale}} = 61$ mJ. The reader is referred to Appendix B for a discussion of effective cross sections and inversion reduction factors as it pertains to alexandrite. Thus, $E/E_{\text{scale}} = 0.82$ and we obtain $z = 2.7$ from Fig. 2. Fig. 3 yields an extraction efficiency of only 26 percent. Fig. 1 yields $x_{\text{opt}} = 0.7L = 0.028$, corresponding to an optimum reflectivity of $R = 97.2$ percent. From Fig. 5, the FWHM pulsewidth is $t_p = 6t_r/L = 300$ ns and the peak power is 0.16 MW from Fig. 4. The internal circulating power is about 11 MW and the circulating intensity is 87 MW/cm^2.

Since the low-gain material has a higher value of E_{scale}, operation at a given energy will occur for a smaller value of z, as can be easily seen from Fig. 2. Similarly, the remaining figures imply that the low-gain medium will be characterized by higher optimum mirror reflectivities, lower extraction efficiencies, longer pulsewidths, and lower internal circulating powers.

IV. Conclusion

In Appendix A, we have rederived the fundamental Q-switch equations and demonstrated the manner in which various device parameters of interest to the laser engineer can be computed from them. In Appendix B, we have examined the dependence of the "inversion reduction factor" on the magnitudes of the various level relaxation and sublevel thermalization rates relative to the Q-switched pulsewidth and discussed the use of effective versus spectroscopic cross sections. In the main body of the paper, we have demonstrated that, in spite of the transcendental nature of the basic equations, simple analytic formulas can be derived for the key parameters of the optimally coupled laser. These equations are functions of a single dimensionless parameter $z = 2\sigma n_i l/L$ multiplied by easily obtained physical constants. Using the design curves presented in this paper and a simple hand calculator, one can develop maximum efficiency Q-switched laser designs and analyze their properties in minutes.

Appendix A
Derivation of the Fundamental Q-Switched Equations

We define $\phi(t_m)$ as the spatially averaged photon density in the laser resonator at time $t_m = mt_r$ where $t_r = 2l'/c$ is the roundtrip transit time of light in the resonator. The photon density is actually the sum of two longitudinally varying components traveling in opposite directions within the resonator. The photon density at time $t_{m+1} = (m+1)t_r$ after an additional roundtrip through the resonator is given by

$$\phi(t_{m+1}) = \phi(t_m)e^{2\sigma n(t_m)l}e^{-2\alpha l}RR'\prod_i T_i^2 \quad \text{(A.1)}$$

where R is the output mirror reflectivity, R' is the reflectivity of the rear mirror, T_i is the one-way optical transmission of the ith internal element, σ is the stimulated emission cross section, $n(t_m)$ is the instantaneous population inversion density, l is the laser rod length, and α is the total loss coefficient in the laser rod. The latter coefficient is the sum of linear absorption and scattering components.

Taking the natural log of (A.1) and defining the dissipative (nonuseful) optical loss by

$$L \equiv 2\alpha l + \ln\left[\frac{1}{R' \prod_i T_i^2}\right] \tag{A.2}$$

we may write

$$\begin{aligned}\Delta \ln \phi &\equiv \ln \phi(t_{m+1}) - \ln \phi(t_m) \\ &= 2\sigma n(t_m) l - \left[\ln\left(\frac{1}{R}\right) + L\right].\end{aligned} \tag{A.3}$$

Since the integer m is actually a time scale normalized to the cavity roundtrip transit time, i.e., $m = t/t_r$, the time rate of change of the photon density is given by

$$\begin{aligned}\frac{d\phi}{dt} &= \phi \frac{d \ln \phi}{dt} \approx \frac{\phi}{t_r}\frac{\Delta \ln \phi}{\Delta m} \\ &= \frac{\phi}{t_r}\left\{2\sigma n l - \left[\ln\left(\frac{1}{R}\right) + L\right]\right\} \\ &= \frac{2\sigma n l}{t_r}\phi - \frac{\phi}{t_c}\end{aligned} \tag{A.4}$$

where we have defined a photon decay time given by [3]

$$t_c = \frac{t_r}{\left[\ln\left(\frac{1}{R}\right) + L\right]}. \tag{A.5}$$

The equation for the time rate of change of the population inversion density is now assumed to be of the form

$$\frac{dn}{dt} = -\gamma \sigma c \phi n \tag{A.6}$$

where $0 \le \gamma \le 2$ has been called a "degeneracy factor" by Koechner [3], but is best described as an "inversion reduction factor" since it really corresponds to the net reduction in the population inversion resulting from the stimulated emission of a single photon. As discussed in Appendix B, the value of γ depends not only on the level degeneracies, but also on the speed of the various relaxation mechanisms in the laser medium relative to the buildup time of the Q-switched laser pulse.

We eliminate time in the usual way [3] by taking the quotient of (A.5) and (A.6) to obtain the differential equation

$$\frac{d\phi}{dn} = -\frac{l}{l'\gamma}\left(1 - \frac{n_t}{n}\right). \tag{A.7}$$

The threshold inversion n_t is obtained by setting the left-hand side of (A.4) equal to zero, i.e.,

$$n_t = \frac{1}{2\sigma l}\left[\ln\left(\frac{1}{R}\right) + L\right]. \tag{A.8}$$

Equation (A.7) has the solution

$$\phi(t) = \frac{l}{l'\gamma}\left\{n_i - n(t) - n_t \ln\left[\frac{n_i}{n(t)}\right]\right\}. \tag{A.9}$$

From (A.9), the two-way photon density in the resonator reaches a peak value of

$$\phi_{\max} = \frac{l}{l'\gamma}\left\{n_i - n_t\left[1 + \ln\left(\frac{n_i}{n_t}\right)\right]\right\} \tag{A.10}$$

when the population inversion density reaches its threshold value ($n(t) = n_t$).

At the end of the pulse, the photon density is again zero. Thus, setting the left-hand side of (A.9) equal to zero yields the usual transcendental equation which relates the initial and final population inversion densities n_i and n_f, i.e. [3],

$$n_i - n_f = n_t \ln\left(\frac{n_i}{n_f}\right) \tag{A.11}$$

which must be solved numerically. From (A.4), we obtain the instantaneous power coupled from the cavity by the output mirror

$$P(t) = -h\nu A l' \left.\frac{d\phi}{dt}\right|_R = \frac{h\nu A l' \ln\left(\frac{1}{R}\right)}{t_r}\phi(t) \tag{A.12}$$

where $h\nu$ is the photon energy and Al' is the resonator volume occupied by the photons.

The peak power external to the laser is easily obtained by substituting (A.10) into (A.12), i.e.,

$$P_{\max} = \frac{Alh\nu}{\gamma t_r}\ln\left(\frac{1}{R}\right)\left\{n_i - n_t\left[1 + \ln\left(\frac{n_i}{n_t}\right)\right]\right\}. \tag{A.13}$$

To calculate the laser output energy, we first integrate (A.12) over time from zero (Q-switch transition) to infinity, change the variable of integration from time to population inversion density through the use of (A.6), and perform the simple integral to obtain

$$\begin{aligned}E &= \int_0^\infty dt\, P(t) = \frac{h\nu A l' \ln\left(\frac{1}{R}\right)}{t_r}\int_0^\infty dt\, \phi(t) \\ &= \frac{h\nu A l' \ln\left(\frac{1}{R}\right)}{\gamma\sigma c t_r}\int_{n_f}^{n_i}\frac{dn}{n} = \frac{h\nu A}{2\sigma\gamma}\ln\left(\frac{1}{R}\right)\ln\left(\frac{n_i}{n_f}\right).\end{aligned} \tag{A.14}$$

Using (A.8) and (A.11), one can convert the latter expression to the form given (but not derived) by Koechner [3], i.e.,

$$E = \frac{1}{\gamma} Vh\nu(n_i - n_f) \frac{\ln\left(\frac{1}{R}\right)}{\ln\left(\frac{1}{R}\right) + L} \tag{A.15}$$

where $V = Al$ is the effective mode volume in the laser rod.

If we assume that the shape of the output pulse can be reasonably approximated by an asymmetric triangle of height $P_{\max}$, baseline width t_b, and area E, an approximate expression for the FWHM pulsewidth is

$$t_p = \frac{t_b}{2} = \frac{E}{P_{\max}} = t_c \frac{(n_i - n_f)}{n_i - n_t\left[1 + \ln\left(\frac{n_i}{n_t}\right)\right]}. \tag{A.16}$$

Because of its relevance to the optical damage problem, the peak two-way circulating intensity within the resonator is also of great interest to the laser engineer. The peak two-way power at the output mirror is clearly

$$P_R = \frac{P_{\max}}{1 - R} + \frac{RP_{\max}}{1 - R} = \left(\frac{1 + R}{1 - R}\right) P_{\max} \tag{A.17a}$$

where $P_{\max}$ is the peak power external to the cavity given by (A.13). Since the net gain in the resonator is zero at the point of maximum power, the circulating power experiences a one-way amplification of $1/\sqrt{R}$. Therefore, if we assume that the reflectivity of the rear mirror is near unity, it can be readily shown that the peak two-way power at the rear reflector is given approximately by

$$P_{R'} = \frac{(1 + R')\sqrt{R}}{1 - R} P_{\max} \approx \frac{2\sqrt{R}}{1 - R} P_{\max}. \tag{A.17b}$$

Appendix B
Inversion Reduction Factor

The population inversion density is given by

$$n = n_a - n_b \tag{B.1}$$

where, in a fairly general laser medium, the rate equations for the population densities of the upper and lower laser levels can be represented by [7]

$$\frac{\partial n_a}{\partial t} = \Lambda_a - \gamma_a n_a - \gamma_u(n_a - f_a n_u) - \sigma c(n_a - n_b)\phi \tag{B.2}$$

and

$$\frac{\partial n_b}{\partial t} = \Lambda_b - \gamma_b n_b - \gamma_l(n_b - f_b n_l) + \sigma c(n_a - n_b)\phi, \tag{B.3}$$

respectively. In the latter equations, Λ_a and Λ_b are the pump rates, γ_a and γ_b are the relaxation rates for the upper and lower multiplet states, γ_u and γ_l are the thermalization rates within the upper and lower multiplet states, n_u and n_l are the instantaneous population densities in the upper and lower multiplets, $f_a n_u$ and $f_b n_l$ are the Boltzmann equilibrium population densities of the upper and lower laser levels, and the final terms depending on ϕ represent the stimulated emission terms.

In a solid-state laser such as Nd:YAG, γ_a and γ_b would represent the relaxation rates of the $^4F_{3/2}$ and $^4I_{11/2}$ multiplets, respectively, whereas γ_u and γ_l are the thermalization rates within these multiplets. In our Nd:YAG example, the fluorescence lifetime of the upper laser multiplet is 230 μs, the relaxation time of the lower laser multiplet is about 300 ns, and the thermalization times within the multiplets are on the order of 12 ns [7].

In a molecular gas laser such as CO_2, γ_a is the relaxation rate of the 001 vibrational state, while γ_b is the relaxation rate of the 100 or 020 vibrational states for the 10.6 and 9.6 μm transitions, respectively. Cheo [8] has shown that thermalization rates (thermalization times) among the rotational sublevels vary approximately linearly (inversely) with operating pressure according to 10^7 $p/\text{s} \cdot \text{torr}$ (10^{-7} s $\cdot$ torr$/p$) where p is in torr. The rates also vary somewhat with temperature and gas mixture.

We will now investigate, for a general laser medium, the conditions under which the equation for the time rate of change of the population takes the requisite form in (A.6) in Appendix A.

If one makes the common textbook assumption [9] that the spontaneous emission, pump, relaxation and thermalization rates are all slow relative to the Q-switch buildup time, we obtain

$$\frac{\partial n}{\partial t} \approx -2\sigma c n \phi \tag{B.4}$$

for the time rate of change of the population inversion density. The value $\gamma = 2$ in (B.4) reflects the fact that the population inversion is reduced by two for each stimulated emission of a photon because, in the absence of rapid relaxation mechanisms, the emitting atom remains trapped in the lower laser level during the remainder of the pulse buildup. (In Nd:YAG, this value of γ would only be valid for pulses much shorter than 12 ns.)

If, on the other hand, the lower multiplet (or single level) relaxes very rapidly relative to the Q-switched pulsewidth ($t_b << t_p$), the instantaneous population of the lower laser level is negligible. We then have

$$\frac{\partial n}{\partial t} \approx \frac{\partial n_a}{\partial t} = -\sigma c n \phi \tag{B.5}$$

or $\gamma = 1$ since the population inversion density is only reduced by one for each photon emitted.

If lower multiplet relaxation is slow but thermalization times within the upper and lower multiplet states are rapid relative to the Q-switch pulsewidth ($t_t << t_p << t_b$), the lower state bottleneck is partially removed. Furthermore, the upper state population is refreshed by the thermalization process. Under these conditions, the total population of the upper and lower multiplets is conserved, but the instantaneous population densities of individual levels follow Maxwell–Boltzmann statistics. Thus, we have

$$\frac{\partial n}{\partial t} = f_a \frac{\partial n_u}{\partial t} - f_b \frac{\partial n_l}{\partial t} = (f_a + f_b)\frac{\partial n_u}{\partial t} = -(f_a + f_b)\sigma c n \phi \quad \text{(B.6)}$$

or $\gamma = (f_a + f_b)$ where f_a and f_b are the Maxwell–Boltzmann probabilities for the upper and lower laser levels, respectively.

In Nd:YAG at 300 K, $f_a = 0.41$ and $f_b = 0.19$, implying $\gamma = 0.6$ for Q-switched pulsewidths satisfying the condition 12 ns $<< t_p <<$ 300 ns. A value of γ less than unity reflects the fact that the upper state population is refreshed by the thermalization process. For pulsewidths on the order of the 12 ns thermalization time, an intermediate value is probably appropriate, i.e., $0.6 < \gamma < 2$.

One often defines an "effective cross section" as the product of the upper level Boltzmann factor and the spectroscopic cross section, i.e., $\sigma^* = f_a \sigma$. For example, in Nd:YAG, the spectroscopic cross section for the individual transition between Stark sublevels has been recently measured [5] to be $\sigma(R_2 - Y_3) = 6.5 \times 10^{-19}$ cm^2. At a temperature of 295 K, the Maxwell–Boltzmann fraction in the upper Stark sublevel is 0.427, implying an effective cross section for Nd:YAG of $\sigma^*(^4F_{3/3} - {}^4I_{11/2}) = 2.8 \times 10^{-19}$ cm^2. When the effective, rather than the spectroscopic, cross section is used, an effective excited state population density n^* given by

$$n^* = \frac{n}{f_a} = \left(n_u - \frac{f_b}{f_a} n_l\right) \quad \text{(B.7)}$$

must be substituted for n in the fundamental Q-switch equations and in computing the small-signal gain coefficient or the dimensionless parameter z. For Nd:YAG, n_u and n_l are the total population densities in the $^4F_{3/2}$ and $^4I_{11/2}$ manifolds, respectively.

The concept of an effective cross section is especially important in complex vibronic laser media such as alexandrite where individual Stark splittings are difficult or impossible to resolve. Walling *et al.* [6] have applied the McCumber theory of phonon-terminated lasers [10] to the alexandrite medium to obtain an expression for the laser gain given by

$$g_\lambda(\hat{k}, E) = \sigma^*_{e\lambda}(\hat{k}, E)\left[N^* - (N - N^*) \cdot \exp\left[(E - E^*)/k_B T\right]\right] \quad \text{(B.8)}$$

where $\sigma^*_{e\lambda}(\hat{k}, E)$ is the effective cross section for radiation with energy E, unit wave vector $\hat{k}$, and polarization λ, N^* is the total excited state population density, N is the chromium ion concentration, k_B is Boltzmann's constant and E^* is a slightly temperature-dependent effective zero-phonon level which has a numerical value of 14701 cm^{-1} for $T = 300$ K. The quantity $(N - N^*)$ is the total population density in the ground state manifold, and it is easily shown from (B.7) and (B.8) that an "effective inversion reduction factor" for alexandrite is given by

$$\gamma^* = 1 + \frac{f_b}{f_a} = 1 + \exp\left[(E - E^*)/k_B T\right]. \quad \text{(B.9)}$$

References

[1] W. G. Wagner and B. A. Lengyel, "Evolution of the giant pulse in a laser," *J. Appl. Phys.*, vol. 34, pp. 2040–2046, 1963.

[2] R. B. Kay and G. S. Waldman, "Complete solutions to the rate equations describing Q-spoiled and PTM laser operation," *J. Appl. Phys.*, vol. 36, pp. 1319–1323, 1965.

[3] W. Koechner, *Solid State Laser Engineering*. New York: Springer-Verlag, 1976, ch. 8.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume I*. New York: Interscience, 1953, ch. 4.

[5] W. F. Krupke, M. D. Shinn, J. E. Marion, J. A. Caird, and S. E. Stokowski, "Spectroscopic, optical and thermomechanical properties of neodymium and chromium doped gadolinium scandium gallium garnet," *J. Opt. Soc. Amer.*, vol. B3, pp. 102–113, 1986.

[6] J. C. Walling, O. G. Peterson, H. P. Jenssen, R. C. Morris, and E. W. O'Dell, "Tunable alexandrite lasers," *IEEE J. Quantum Electron.*, vol. QE-16, pp. 1302–1315, 1980.

[7] J. J. Degnan, "Physical processes affecting the performance of high power, frequency-doubled, short pulse laser systems: Analysis, simulation, and experiment," Ph.D. dissertation, Univ. Maryland, College Park, May 1979 (available from University Microfilms, Ann Arbor, MI).

[8] P. K. Cheo, in *Lasers—A Series of Advances: Vol. 3*, A. K. Levine and A. J. DeMaria, Ed. New York: Marcel Dekker, 1970.

[9] A. Yariv, *Introduction to Optical Electronics*. New York: Holt, Rinehart, and Winston, 1971, ch. 6.

[10] D. E. McCumber, "Theory of phonon-terminated optical masers," *Phys. Rev.*, vol. 134, pp. A299–A306, 1964.

John J. Degnan, photograph and biography not available at the time of publication.