GENERALIZED LORENTZ TRANSFORMATIONS

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This article presents the generalized Lorentz transformations of time, space, velocity and acceleration which can be applied in any inertial or non-inertial (non-rotating) frame.

Introduction

If we consider a (non-rotating) frame S relative to another inertial frame Σ then the time (t), the position ($\mathbf{r}$), the velocity ($\mathbf{v}$) and the acceleration ($\mathbf{a}$) of a (massive or non-massive) particle relative to the frame Σ are given by:

$$t \; = \int_{0}^{\mathtt{t}} \gamma \, \mathtt{dt} \; + \; \gamma \, \frac{\vec{r} \cdot \boldsymbol{\varphi}}{c^2} \; + \; \mathtt{h}$$

$$\mathbf{r} \; = \; \vec{r} \; + \; \frac{\gamma^2}{\gamma + 1} \, \frac{(\, \vec{r} \cdot \boldsymbol{\varphi} \,) \, \boldsymbol{\varphi}}{c^2} \; + \int_{0}^{\mathtt{t}} \gamma \, \boldsymbol{\varphi} \, \mathtt{dt} \; + \; \mathtt{k}$$

$$\mathbf{v} \; \doteq \; \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} \; \doteq \; \frac{d\mathbf{v}}{dt}$$

where ($\mathtt{t}$, $\vec{r}$) are the time and the position of the particle relative to the frame S ($\boldsymbol{\mu}, \boldsymbol{\varphi}, \boldsymbol{\alpha}$) are the position, the velocity and the acceleration of the origin of the frame S relative to the frame Σ, ($\vec{\mu}$) is the position of the origin of the frame Σ relative to the frame S, ($\mathtt{h}$, $\mathtt{k}$) are constant between the frames Σ & S (c) is the speed of light in vacuum, and $\gamma \; \doteq \; (1 - \boldsymbol{\varphi} \cdot \boldsymbol{\varphi} / c^2)^{-1/2}$

- $\dfrac{\gamma^2}{\gamma+1}\,\dfrac{1}{c^2} \;=\; \dfrac{\gamma-1}{\boldsymbol{\varphi}^2} \qquad (\, \boldsymbol{\varphi}^2 \,\doteq\, \boldsymbol{\varphi}\cdot\boldsymbol{\varphi}\,)$

- $\vec{r} \,+\, \dfrac{\gamma^2}{\gamma+1}\,\dfrac{(\,\vec{r}\cdot\boldsymbol{\varphi}\,)\,\boldsymbol{\varphi}}{c^2} \;=\; \gamma\,\vec{r} \,+\, \dfrac{\gamma^2}{\gamma+1}\,\dfrac{(\,\vec{r}\times\boldsymbol{\varphi}\,)\times\boldsymbol{\varphi}}{c^2}$

- $\vec{\mu} \,+\, \dfrac{\gamma^2}{\gamma+1}\,\dfrac{(\,\vec{\mu}\cdot\boldsymbol{\varphi}\,)\,\boldsymbol{\varphi}}{c^2} \;=\; \gamma\,\vec{\mu} \,+\, \dfrac{\gamma^2}{\gamma+1}\,\dfrac{(\,\vec{\mu}\times\boldsymbol{\varphi}\,)\times\boldsymbol{\varphi}}{c^2}$

- $\boldsymbol{\mu} \;=\; \displaystyle\int_{\mathtt{0}}^{\mathtt{t}} \gamma\,\boldsymbol{\varphi}\,\mathtt{dt} \,+\, \mathtt{k} \;=\; \int_{\mathtt{0}}^{t} \boldsymbol{\varphi}\,dt \,+\, \mathtt{k} \;=\; -\,\vec{\mu} \,-\, \dfrac{\gamma^2}{\gamma+1}\,\dfrac{(\,\vec{\mu}\cdot\boldsymbol{\varphi}\,)\,\boldsymbol{\varphi}}{c^2}$

The frame S is inertial when $(\,\boldsymbol{\alpha} \;=\; 0\,)$

The frame S is non-inertial (rectilinear accelerated motion) when $(\,\boldsymbol{\alpha} \;\neq\; 0\,)$ and $(\,\boldsymbol{\alpha}\times\boldsymbol{\varphi} \;=\; 0\,)$

The frame S is non-inertial (uniform circular motion) when $(\,\boldsymbol{\alpha} \;\neq\; 0\,)$ and $(\,\boldsymbol{\alpha}\cdot\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is inertial then the observer S must use a fixed origin O such that $(\,\vec{\mu}\times\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is non-inertial (rectilinear accelerated motion) then the observer S must use a fixed origin O such that $(\,\vec{\mu}\times\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is non-inertial (uniform circular motion) then the observer S must use a fixed origin O such that $(\,\vec{\mu}\cdot\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is inertial then $(\,\boldsymbol{\alpha} \;=\; 0\,)$, $(\,\boldsymbol{\varphi} \;=\; \text{constant}\,)$, $(\,\gamma \;=\; \text{constant}\,)$ $(\,\int_{\mathtt{0}}^{\mathtt{t}}\gamma\,\mathtt{dt} = \gamma\,\mathtt{t}\,)$, $(\,\boldsymbol{\mu} \;=\; \gamma\,\boldsymbol{\varphi}\,\mathtt{t} + \mathtt{k}\,)$ and $(\,\vec{\mu}\times\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is non-inertial (rectilinear accelerated motion) then $(\,\boldsymbol{\alpha} \;\neq\; 0\,)$ $(\,\boldsymbol{\alpha}\times\boldsymbol{\varphi} \;=\; 0\,)$ and $(\,\vec{\mu}\times\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is non-inertial (uniform circular motion) then $(\,\boldsymbol{\alpha} \;\neq\; 0\,)$ $(\,\boldsymbol{\alpha}\cdot\boldsymbol{\varphi} \;=\; 0\,)$, $(\,\gamma \;=\; \text{constant}\,)$, $(\,\int_{\mathtt{0}}^{\mathtt{t}}\gamma\,\mathtt{dt} = \gamma\,\mathtt{t}\,)$ and $(\,\vec{\mu}\cdot\boldsymbol{\varphi} \;=\; 0\,)$

If the frame S is inertial or non-inertial (non-rotating) then the observer S can use test particles such that $(\,\vec{r}\cdot\boldsymbol{\varphi} \;=\; 0\,)$ or $(\,\vec{r}\times\boldsymbol{\varphi} \;=\; 0\,)$

General Observations

It is known that in inertial frames the local geometry is Euclidean and that in non-inertial frames the local geometry is in general non-Euclidean.

According to this article, the local line element of the frame S must be obtained from the local line element of the frame Σ.

Therefore, the local line element (in rectilinear coordinates) in the frame Σ and the local line element in the frame S are given by:

$$ds^2 \;=\; c^2 dt^2 - d\mathbf{r}^2$$

$$ds^2 \;=\; \left[\, \left(1 + \frac{\mathbf{w} \cdot \vec{r}}{c^2}\right)^2 - \left(\frac{\boldsymbol{\phi} \times \vec{r}}{c}\right)^2 \,\right] c^2\, \mathtt{dt}^2 \;-\; 2 \left(\boldsymbol{\phi} \times \vec{r}\right) d\vec{r}\, \mathtt{dt} \;-\; d\vec{r}^{\,2}$$

$$\mathbf{w} \;\doteq\; \gamma^2 \left(\boldsymbol{\alpha} + \frac{\gamma^2}{\gamma + 1}\, \frac{(\boldsymbol{\alpha} \cdot \boldsymbol{\varphi})\, \boldsymbol{\varphi}}{c^2}\right) \qquad , \qquad \boldsymbol{\phi} \;\doteq\; \gamma^1 \left(\frac{\gamma^2}{\gamma + 1}\, \frac{(\boldsymbol{\varphi} \times \boldsymbol{\alpha})}{c^2}\right)$$

According to this article, the kinematic quantities ($t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) are the proper kinematic quantities of the frame Σ.

Therefore, the kinematic quantity (t) is a tensor of rank 0 and the kinematic quantities ($\mathbf{r}, \mathbf{v}, \mathbf{a}$) are tensors of rank 1.

Finally, the velocity of light in vacuum is ($\mathbf{c}$) in the frame Σ and ($\vec{c}$) in the frame S and ($\mathbf{c} \cdot \mathbf{c}$) & ($\vec{c} \cdot \vec{c}$) are constant in the frames Σ & S.

Bibliography

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