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# Superinsulator and quantum synchronization

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**Synchronized oscillators are ubiquitous in nature[1], and synchronization plays a key part in various classical and quantum phenomena. Several experiments[2–4] have shown that in thin superconducting films, disorder enforces the droplet-like electronic texture—superconducting islands immersed into a normal matrix—and that tuning disorder drives the system from superconducting to insulating behaviour. In the vicinity of the transition, a distinct state[4] forms: a Cooper-pair insulator, with thermally activated conductivity. It results from synchronization of the phase of the superconducting order parameter at the islands across the whole system[5]. Here we show that at a certain finite temperature, a Cooper-pair insulator undergoes a transition to a superinsulating state with infinite resistance. We present experimental evidence of this transition in titanium nitride films and show that the superinsulating state is dual to the superconducting state: it is destroyed by a sufficiently strong critical magnetic field, and breaks down at some critical voltage that is analogous to the critical current in superconductors.**

We consider an exemplary tunable system for the superconductor-to-insulator transition studies, an array of small superconducting islands, each coupled to its nearest neighbours by Josephson weak links[6] (Fig. 1). The behaviour of the array is quantified by two competing energy scales: $E_J$, the Josephson coupling energy of the two adjacent superconducting islands, and the charging energy $E_c$, the energy cost to transfer a charge $2e$ between the neighbouring islands. Depending on the ratio $E_J/E_c$, the system can either be in the superconducting state, $E_J > E_c$, or in the insulating state, $E_J < E_c$. Near the superconductor-to-insulator transition the dynamic screening decreases the charging energy. From the quantum uncertainty principle, $\tau_0 \Delta \approx RC\Delta \approx \hbar$, where $\tau_0$ is the island charge lifetime, $R$ is the leakage resistance, $C$ is the capacitance, and $\Delta$ is the superconducting gap. By introducing the dimensionless conductance $g = 2\pi\hbar/(e^2R)$, we obtain the Coulomb blockade energy $E_c \approx e^2/C \approx \Delta/g$. In an array of superconducting islands the superconductor-to-insulator transition occurs in a region where $g \gtrsim 1$ (ref. 7), so we conclude that the relevant charging energy $E_c < \Delta$, and thus the charge transfer is mediated by the activated motion of Cooper pairs[8]. Furthermore, because $g \gtrsim 1$, a propagating Cooper pair spreads over several islands to form a charge soliton, an ultimate charge carrier for the thermally activated conductivity.

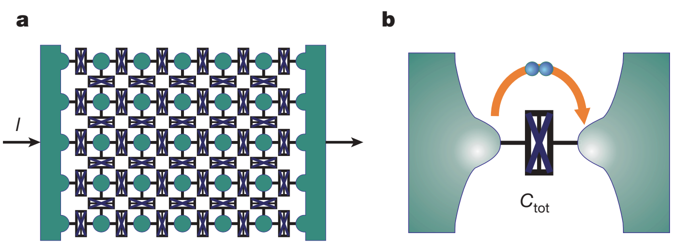


**Figure 1 | Two-dimensional Josephson junction array.** **a**, Sketch of the 2D array. Circles represent superconducting islands and crossed rectangles stand for Josephson weak links connecting these islands. The bias current $I$ is injected to the left electrode and collected from the right electrode of the array. **b**, Phase synchronization allows us to view the 2D Josephson junction array as a single effective junction with the effective capacitance $C_{tot} = C/\ln N$.

In the superconducting state, the array undergoes the Berezinskii–Kosterlitz–Thouless (BKT) transition, separating a superconducting low-temperature phase with the bound vortex–antivortex pairs from a resistive high-temperature phase with free vortices, the transition temperature being $T_{BKT} \cong E_J/k_B$. In the insulating state the charge binding–unbinding transition that is dual to the superconducting BKT transition occurs at temperature $T \cong E_c/k_B$ (refs 9 and 10). Here we show that this transition separates a low-temperature superinsulating phase with practically infinite resistance, and an insulator with the large but finite thermally activated resistance $R \propto \exp[\Delta_c/(k_BT)]$, where the collective Coulomb barrier $\Delta_c = E_c(L/d)$ in one-dimensional arrays and $\Delta_c = E_c\ln(L/d)$ in two-dimensional (2D) arrays ($L$ being the characteristic linear size of the system and $d$ being the size of the elemental cell of the array). These formulae hold provided the screening length in the system that is related to capacitance to the ground exceeds the sample size, which we assume to be the case. Generalizing the technique developed in ref. 5 for temperature interval $E_c/k_B < T < \Delta_c/k_B$ to the low-temperature case ($T < E_c/k_B$; see Methods), we find the low bias ($eV < \Delta_c$) current–voltage characteristic in a superinsulating state to be:

$$I_s \cong I_c \exp\left\{-\frac{(\Delta_c - eV)^2 \exp(E_c/2k_BT)}{E_c\Delta_c}\right\} \qquad (1)$$

with $I_c$ being the Josephson critical current.

To gain an insight into the transition from activated insulator to superinsulator, we notice that the conductivity of the Cooper-pair insulator is proportional to the concentration, $\aleph_s$, of thermally activated charge solitons. We introduce the local charge density, $n_s(\mathbf{r})$, which is normalized to give the soliton energy as $\Delta_c = E_c\int d\mathbf{r}\, n_s^2(\mathbf{r})$. The probability for such a local density to appear at point $\mathbf{r}$ is proportional to $\exp[-n_s^2(\mathbf{r})/(2\langle\delta n^2\rangle)]$, where $\langle\delta n^2\rangle$ is the mean square fluctuation of the local charge density. Thus, the soliton density, which is proportional to the probability that a soliton will appear, is a product of all these local probabilities at all the points of the system: $\aleph_s \propto \Pi_{\mathbf{r}}\exp[-n_s^2(\mathbf{r})/(2\langle\delta n^2\rangle)] = \exp\{-[1/(2\langle\delta n^2\rangle)]\int d\mathbf{r}\, n_s^2(\mathbf{r})\} = \exp\{-[\Delta_c/(2E_c\langle\delta n^2\rangle)]\}$. At temperatures above the charge binding–unbinding superinsulating transition, $T_{SI} \cong E_c/k_B$, the solitons are unbound and the equipartition theorem $\langle\delta n^2\rangle = k_BT/E_c$ gives rise to the thermally activated resistance $R \propto \exp(\Delta_c/k_BT)$. At low temperatures $T < T_{SI}$, the charge solitons and antisolitons are bound, and

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therefore $\langle\delta n^2\rangle^{1/2}$ is the probability of breaking these pairs, that is, $\exp(-E_c/k_BT)$. This yields a double-exponential resistivity in the superinsulating phase:

$$R \propto \exp\left\{\frac{\Delta_c}{E_c}\exp\left(\frac{E_c}{2k_BT}\right)\right\} \qquad (2)$$

This result follows from equation (1) on taking the limit $eV \ll \Delta_c$.

The physics of the charge-soliton-mediated transport stems from the quantum uncertainty principle $\Delta\varphi\Delta n \geq 1$, where $\varphi$ is the Josephson phase difference across the junction, and $n$ is the number of the Cooper pairs transferred through the junction. Thus, charge and phase cannot be specified simultaneously to an arbitrary precision. The precise control of the charge at each junction enforced by the Coulomb blockade causes the corresponding phases $\varphi$ at different junctions to fluctuate almost independently. However, when activated current passes through the system, the phases across the neighbouring junctions tend to synchronize each other to minimize Joule losses. The synchronization of phases of the superconducting order parameter across the system implies that the whole array can be viewed as a single effective junction with total capacitance $C_{tot}$; see Fig. 1b.

The Cooper pair is transferred across this effective junction in a form of a charge soliton spread over the array[9–12]. The one-dimensional array is a series of capacitances and $C_{tot}^{-1} = NC^{-1}$, where $C$ is the capacitance between the adjacent islands and $N = L/d$ is the total number of islands. In the 2D case, taking into account that bias is applied from left to right, we find $C_{tot}^{-1} = 1/(2C)\ln N$, resulting in $\Delta_c = (E_c/2)\ln N$. This gives rise to the activation temperature dependence for the resistance, $R \propto \exp(\Delta_c/k_BT)$, in the temperature interval $E_c/k_B < T < \Delta_c/k_B$, where charge transport is mediated by the gas of free solitons and antisolitons (charge vortices). It yields the double-exponential resistivity of equation (2) at $T < E_c/k_B$, below the charge binding transition. The logarithmic scaling of the activation energy with the sample size agrees very well with the experimental findings[5].

The double-exponential current–charge ($I$–$V$) characteristic is derived for a regular array of Josephson junctions with all junction parameters identical. To examine the effect of irregularity in real systems we consider a one-dimensional array with position-dependent capacitances. Writing $C_{tot}^{-1} = \sum_i^N C_i^{-1} = N[(1/N)\sum_i^N C_i^{-1}] \equiv N\langle C^{-1}\rangle$, we obtain all the results for a regular array by substituting $E_c \to \langle E_c\rangle$. A similar consideration applies to 2D arrays. We thus conclude that the results obtained for regular arrays hold for systems with the random parameters, provided that the average $\langle C^{-1}\rangle$ is well defined.

The superinsulating state is experimentally observed in disordered titanium nitride (TiN) films which, to a degree of disorder, are in the vicinity of the superconductor-to-insulator transition[4]. Near the transition the conductance $g$ is of the order of unity, according to experimental data at high magnetic fields in which superconductivity is suppressed and the film behaves in a metallic way[13]. We performed voltage-biased two-probe differential conductance measurements using standard low-frequency lock-in techniques with an alternating voltage of 10 μV. Magnetic fields were applied perpendicularly to the film surface.

Shown in Fig. 2 are plots of the differential conductance versus applied direct voltage and the corresponding $I$–$V$ characteristics obtained by integration of the $dI/dV$ versus $V$ curves. The 70 mK data reveal 'normal' insulator behaviour: the differential conductivity is finite (Fig. 2a), and the $I$–$V$ dependence is linear (Fig. 2b) up to a direct voltage of $10^{-3}$ V, reflecting ohmic conductivity behaviour in the activated insulator state. Lowering the temperature down to 20 mK drives the film into a superinsulating state: the differential conductivity and current remain immeasurably small at low bias voltages. At the depinning threshold voltage $V_T$, $dI/dV$ abruptly jumps up over at least four orders of magnitude. The high- and low-temperature $\log I$–$\log V$ curves nearly coincide at high voltages and dramatically diverge at the low bias, which indicates that the superinsulating transition at $B = 0.9$ T occurs somewhere in between 20 mK and 70 mK.

As we have shown above, the superinsulator critical temperature is $T_{SI} = E_c/(2k_B) \cong \Delta/(gk_B)$. The magnetic field suppresses the superconducting gap $\Delta$, so the critical temperature depends on the magnetic field. This defines a superinsulating critical field $B_{SI}$. Thus the superinsulating transition can be crossed either by varying the temperature or by tuning the magnetic field. The fan-like set of $\log I$ versus $\log V$ curves in Fig. 3b offers unambiguous evidence of the

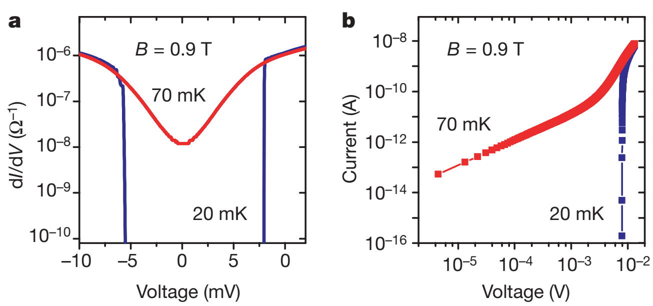


**Figure 2 | Conductivity in normal-insulating and superinsulating states.** **a**, Differential conductance $dI/dV$ versus direct voltage in the insulating (conductive) state at 70 mK and in the superinsulating (zero-conductive) state at $T = 20$ mK measured for magnetic field $B = 0.9$ T. **b**, The corresponding $I$–$V$ characteristics plotted in a log–log scale. In the insulating state the $I$–$V$ curve is linear (ohmic) at small voltages. In the superinsulating state ($T = 20$ mK) the $I$–$V$ curve shows depinning behaviour and an unmeasurably small current below the threshold voltage. The transition from superinsulator to 'normal' insulator occurs somewhere within the temperature interval 20–70 mK.

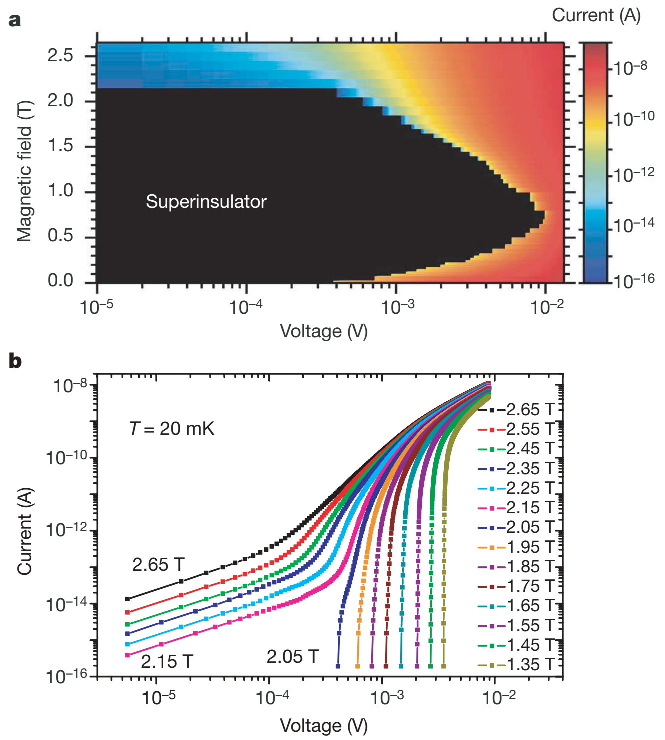


**Figure 3 | Magnetic-field-tuned transition to superinsulating state.** **a**, The two-dimensional colour map of the current values in the $B$–$V$ plane. The colour scale on the right-hand side represents current. The black domain in the map corresponds to the superinsulating state. The border between the black and coloured areas represents the field dependence of the threshold voltage on the magnetic field. **b**, Fan-like $\log I$–$\log V$ curves at $T = 20$ mK. The critical field $B_{SI}$ is crossed on decreasing the magnetic field. Two diverging families of $\log I$–$\log V$ curves represent the 'normal' insulator at $B \geq 2.15$ T and the superinsulator at $B \leq 2.05$ T, and the conductivity shows linear $I$ versus $V$ dependence at low biases in the 'normal' insulator state, and sharp depinning at the voltage threshold in the superinsulator.

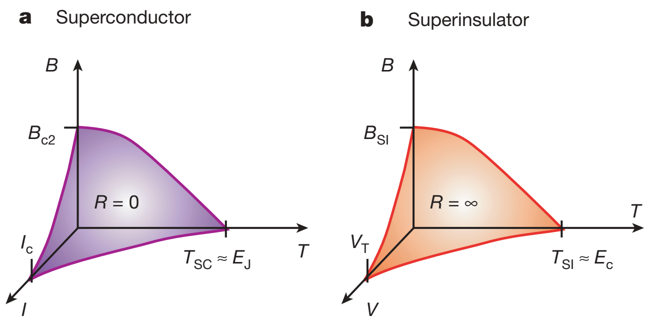


**Figure 4 | Sketch of dual-phase diagrams for a superconductor and a superinsulator. a,** Magnetic-field–temperature–current ($B$–$T$–$I$) superconductor phase diagram. **b,** A dual-phase diagram for a superinsulator is obtained from the superconductor phase diagram by interchanging the $I$ and $V$ axes. The threshold voltage $V_T$, the maximal voltage at which a superinsulator can retain a zero-conductivity state, corresponds to the critical current of a superconductor. The critical temperature for a superconductor $T_{SC}$ maps onto the critical temperature of the 'normal' insulator–superinsulator transition $T_{SI}$.

superinsulating transition, placing $B_{SI}$ at $T = 20$ mK between 2.05 and 2.15 T. Figure 3a displays current $I$ versus $B$ and $V$ on a colour map. The border enclosing the infinitely resistant superinsulating domain (in black) visualizes the dependence of the threshold voltage upon the magnetic field.

Figure 4 summarizes our findings and presents a sketch of the dual superconducting (coordinates $B$, $T$, $I$), and superinsulating (coordinates $B$, $T$, $V$) phase diagrams. It shows a mirror-like symmetry between the superconducting and superinsulating phases: both collective states occupy the low-magnitude corners of their respective phase diagrams. In both cases the relevant variables are the magnetic field and temperature, the current for a superconductor and the voltage for a superinsulator. The temperature dependence of the critical field $B_{SI}$ of the superinsulator shown in Fig. 4b follows the temperature behaviour of the upper critical field $B_{c2}$ of the superconductor (Fig. 4a), given that $T_{SI} \propto \Delta/k_B \propto T_c$.

In conclusion, we note two things. First, the origin of the duality between a superinsulator and a superconductor lies in the conjugation of superconducting phase $\varphi$ and condensate charge $Q = 2en$ connected by the uncertainty principle $\Delta\varphi\Delta n \geq 1$, where $n$ is the number of Cooper pairs involved in the elemental charge transfer process. The collective phase characterizing a superconductor maps to the collective charge of a superinsulator. As a result, the duality between these two macroscopic quantum phenomena manifests itself via the mapping of all the characteristic parameters: $E_c \leftrightarrow E_J$, $I \leftrightarrow V$, and resistivity↔conductivity. Further, the duality manifests itself in the mirror symmetry of the phase diagram of both states: the upper critical field $B_{c2}$ of a superconductor has its counterpart in the critical field $B_{SI}$ for a superinsulator. The latter depends on temperature similarly to $B_{c2}$, while the temperature and field dependencies of the superconducting critical current are mirrored by those of the threshold voltage for depinning. This dual similarity extends even further. The Joule loss $P = IV$, which is exactly zero in the superconducting state, is also exactly zero in the superinsulator. Whereas the absence of Joule loss in a superconductor is the result of the nondissipative flow of the current and thus the lack of the voltage drop $V = 0$, the zero Joule loss in a superinsulator is ensured by the absence of the current at $V < V_T$, where $V_T$ is the threshold voltage.

Second, our theoretical results were derived for a regular array of Josephson junctions. However, the experiments revealing a superinsulating state were carried out on homogeneously disordered films rather than on the artificially designed Josephson junction patterns. Our understanding of the origin of the superinsulating state in the films relies on the formation of the network of superconducting droplets within the normal matrix. This network of superconducting droplets is precisely the array of superconducting weakly coupled islands considered above, provided this network maintains a relatively regular structure. Although the analytical theory of the droplet state is unknown, we conjecture that the droplet state is an inherent property of the critical region of the superconductor-to-insulator transition in the films and that a regular droplet array may emerge analogously to nucleation of the superconducting vortex lattice on the other side of the transition.

## METHODS SUMMARY

Our analytical derivation of the $I$–$V$ characteristic is based on the general linear-response theory, giving the Josephson current across a system confined between the two superconducting leads as $I_s = \langle \partial H/\partial\varphi \rangle$, where $\varphi$ is the phase difference between the electrodes, $H$ is the hamiltonian of the system of interest, and the angle brackets denote quantum mechanical and thermodynamic averaging. Considering a Josephson junction array in the insulating phase, where $E_c > E_J$, and making use of the perturbation theory with respect to small $E_J/E_c$, we express the d.c. Josephson current as a Fourier transform of the time-dependent Josephson phase correlation function $K(t)$ across the array, where the applied bias voltage $V$ plays the role of the Fourier parameter. The $I$–$V$ characteristic of the Josephson junction array is determined by the instanton phase configuration in which superconducting phases at all Josephson junctions evolve in a synchronized manner.

**Full Methods** and any associated references are available in the online version of the paper at www.nature.com/nature.

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## METHODS

Consider $N \times M$ superconducting islands comprising a one- ($M = 1$) or two-dimensional Josephson junction array placed between the two superconducting (left and right, see Fig. 1) leads and closed by a small (as compared to the quantum resistance for the Cooper pairs $R_{CP} = h/4e^2 \approx 6.45\,\mathrm{k\Omega}$) external resistance $R_{ext}$. Assigning the superconducting phase $\chi_{ij}(t)$ to the island in the $i$th column and the $j$th row, we write the hamiltonian of the array[5]:

$$H = H_0 + H_{int} + \frac{\hbar^2}{8E_c}\sum_{j=1}^{M}\left(\dot\chi_{1j}(t) + \dot\chi_{Nj}(t)\right)^2 - 2E_J\sum_{j=1}^{M}\cos\left[\frac{\chi_{1j}(t)+\chi_{Nj}(t)}{2}\right]\times \times\cos\left[\frac{2eVt/\hbar + \psi(t) + \chi_{1j}(t) - \chi_{Nj}(t)}{2}\right] \quad (3)$$

Here

$$H_0 = \sum_{\langle ij,\,kl\rangle}\left[\frac{\hbar^2}{4E_c}\left(\dot\chi_{ij} - \dot\chi_{kl}\right)^2 - E_J\cos\left(\chi_{ij} - \chi_{kl}\right)\right] + \sum_{ij}\frac{\hbar^2}{4E_{c0}}\dot\chi_{ij}^2 \quad (4)$$

and the brackets $\langle ij, kl\rangle$ denote summation over the pairs of adjacent junctions, and the last term in equation (4) represents the self-charge energies of superconducting islands. The $H_{int}$ term in equation (3) describes the coupling of phases on the leads to the thermal heat bath[14]. The charging energy $E_{c0}$ is related to the each island's capacitance to the ground $C_0$. The phases of the left and right leads, $\chi_L(t)$ and $\chi_R(t)$, are fixed by the d.c. voltage $V$ across the array:

$$\chi_R - \chi_L = 2eVt/\hbar + \psi(t) \quad (5)$$

where $\psi(t)$ describes thermal fluctuations in the leads. In the hamiltonian we have singled out the leftmost ($i = 1$) and rightmost ($i = N$) columns of islands directly coupled (adjacent) to the left and right leads respectively. The d.c. Josephson current through the Josephson junction array is given by the standard expression[11]:

$$I_s(V) = \left\langle \frac{\partial H}{\partial[\chi_L - \chi_R]}\right\rangle \quad (6)$$

and acquires the form:

$$I_s(V) = I_c \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\sum_{j=1}^{M}\left\langle \cos\left[\frac{\chi_{1j}(t)+\chi_{Nj}(t)}{2}\right]\sin\left[\frac{2eVt/\hbar + \psi(t) + \chi_{1j}(t) - \chi_{Nj}(t)}{2}\right]\right\rangle \quad (7)$$

where the angle brackets $\langle\ldots\rangle$ stand for averaging over thermal fluctuations in the leads, $\psi(t)$, over quantum mechanical averaging over phases of internal junctions, $\chi_{ij}(t)$, and over the variable $\phi_j = (\chi_{1j} + \chi_{Nj})/2$. In the insulator regime that we address here, both $E_c$ and $E_{c0} \gg E_J$, and we can calculate the quantum-mechanical average $\langle\cos\phi_j\rangle$ in the first-order perturbation theory with respect to small $E_J/E_c$. In most experimental situations $C \gg C_0$, and thus $E_c \ll E_{c0}$, so we can safely neglect the last term in equation (4) and obtain[15]:

$$\left\langle\cos\phi_j\right\rangle = \frac{E_J}{2E_c}\cos\left[\frac{2eVt/\hbar + \psi(t) + \chi_1(t) - \chi_N(t)}{2}\right]\sum_{n=-\infty}^{\infty}\frac{\exp(-E_c n^2/(4k_BT))}{n^2 - 1/4} \cong \frac{E_J}{2E_c}\cos\left[\frac{2eVt/\hbar + \psi(t) + \chi_1(t) - \chi_N(t)}{2}\right] \quad (8)$$

Using the time-dependent part of the hamiltonian as a perturbation we calculate the d.c. current in the framework of the linear response theory[16]:

$$I_s(V) = 2MI_c\frac{E_J}{\hbar}\frac{E_J^2}{E_c^2}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\int_0^t ds\left\langle\left[\hat F(t-s),\,\hat F(s)\right]\right\rangle_{H_0,\psi} \quad (9)$$

where

$$F(t) = \sin\left[\frac{2eV}{\hbar}t + \psi(t) + \chi_{1j}(t) - \chi_{Nj}(t)\right] \quad (10)$$

and $\left[\hat F(t-s), \hat F(s)\right]$ is the commutator of the corresponding Heisenberg operators. The notation $\langle\ldots\rangle_{H_0,\psi}$, means that the quantum mechanical averaging is carried out with respect to the fluctuational field $\psi$ and with the unperturbed hamiltonian $H = H_0$. Assuming gaussian current noise (Nyquist noise) in the leads, we obtain the final expression for the d.c. current:

$$I_s(v) = 4MI_c\frac{E_J}{\hbar}\frac{E_J^2}{E_c^2}\Im m\int_0^\infty dt\exp[-t\delta/\hbar + i(2eVt/\hbar)]K(t) \quad (11)$$

where $\delta = 4e^2R_{ext}k_BT$ (see refs 14 and 17) and the correlation function $K(t)$ of the internal phases is defined as:

$$K(t) = \left\langle \exp i\left[\chi_{1j}(t) - \chi_{1j}(0) - \chi_{Nj}(t) + \chi_{Nj}(0)\right]\right\rangle_{H_0} \quad (12)$$

For the two-junction system (a single Cooper pair transistor), $\chi_{1j} \equiv \chi_{Nj}$, thus $K(t) \equiv 1$, and we obtain the results of refs 18 and 19.

In the zero approximation over $E_J/E_c$ we neglect the Josephson coupling inside the array, and $K(t)$ can be found in a closed form as an analytical continuation of $K(\tau)$, where $\tau$ is imaginary time[5,20]:

$$K(\tau) = \int D\left[\chi_{ij}\right]\exp\left\{i\left[\chi_{1j}(\tau) - \chi_{1j}(0) - \chi_{Nj}(\tau) + \chi_{Nj}(0)\right]\right\}\times \times\exp\left(-\frac{\hbar}{4}\int_0^{\hbar/(k_BT)} d\tilde\tau\left[\sum_{\langle ij,\,kl\rangle}\frac{\left(\dot\chi_{ij}(\tilde\tau) - \dot\chi_{kl}(\tilde\tau)\right)^2}{E_c} + \sum_{ij}\frac{\left(\dot\chi_{ij}(\tilde\tau)\right)^2}{E_{c0}}\right]\right) \quad (13)$$

To calculate $K(t)$, we note that the phases $\chi_{ij}(\tau)$ are the periodic functions of $\tau$ with the period $\hbar/(k_BT)$ and thus can be presented as $\chi_{ij} + 2\pi M_{ij}(k_BT\tau/\hbar)$, where $\chi_{ij}$ is now defined for the interval $0 < \tau < \hbar/k_BT$ and $M_{ij}$ are the so-called (integer) winding numbers[20]. At low temperatures and under the condition $E_c \ll E_{c0}$, so that the last term in the exponent can be dropped, the functional integral (13) is determined by the instanton configuration where all $\chi_{ij}(\tau)$ evolve collectively, reflecting phase synchronization across the sample. By Fourier transformation, the functional integration is reduced to calculation of gaussian integrals, and the correlation function can be conveniently presented as an interpolation:

$$K(t) = \exp\left(-\frac{\Delta_c E_c \xi t^2}{\hbar^2} - i\frac{2\Delta_c t}{\hbar}\right) \quad (14)$$

Plugging formula (14) into the general expression (11) for the current, we arrive at the $I$–$V$ characteristic. The interpolation function $\xi(T) = 1$ in the temperature interval $E_c < k_BT < \Delta_c$, where all the non-zero winding numbers can be neglected[5]. At ultralow temperatures $k_BT < E_c$, where the phase quantization becomes essential, the non-zero winding numbers should be taken into account. In the one-dimensional case we can write the correlation function $K(t)$:

$$K(t) = \exp\left(\frac{iNE_ct}{2}\right)\left[Z^{-1}\sum_n\exp\left(iE_cnt - n^2E_c/2k_BT\right)\right]^N \quad (15)$$

where $n^2$ appear as the quantum numbers of the free rotator and the normalization factor:

$$Z = \sum_n\exp\left(-\frac{n^2E_c}{2k_BT}\right) \quad (16)$$

At $k_BT < E_c$ the main contribution comes from the winding numbers $n = 0$, $-1$, $+1$, and we obtain the correlation function as equation (14) with $\xi(T) = \exp(-E_c/k_BT)$ and the collective Coulomb barrier $\Delta_c = E_c\,(L/d)$, where $L$ is the linear size of the array and $d$ is the size of a single junction. The 2D situation is more involved, and we need to adopt the technique developed in ref. 21 for the superconducting BKT transition. We can then derive the expression (14) for the correlation function with the same exponential form of $\xi$ and $\Delta_c = E_c\ln(L/d)$.


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