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Topological defects in the wake bubble and the Euler characteristic number

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Abstract In this article we study the connection between the topological defects (singularities of the velocity field) inside a region with recirculating viscous flow past an arbitrary blunt body (wake bubble). The topological defects in the wall of the blunt body and inside the bubble form a circuit or net, where the viscous steady dynamic is developed. The primary motivation is to understand the structure of this connection for several known types of two-dimensional incompressible flows.

Keywords Poincare-Hopf · Topological defect · cylinder flow

1 Introduction

This work has two components: one hand, it applies an important theorem (Poincare-Hopf) of differential geometry (Arnold, 1974) to specific types of incompressible and viscous flows in two dimensions, which signifies its applied edge. On the other hand, in its most theoretical aspect, we believe that in two dimensions the stable structures that nature chooses, when no matter is created or sucked, are of the vortex-saddle-vortex configuration (Peixoto, 1962 and Smale, 1967). This configuration has a characteristic Euler number that is equal to one. To some extent this work advances in these two directions and leads us to a more profound reading of stationary dynamics than what we had in mind. Although we know that an incompressible and viscous fluid rotates, it deforms and moves. Whenever it passes through an object it separates the potential part where it only slightly moves and becomes deformed; on the

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other hand, the viscosity part, where it does the three things: rotates, moves and deforms. We know that for naval and aeronautical engineering this region is very important for its role in calculating drag and lift. However, this work tells us that this region is limited by the topological singularities of two types, fractional and whole, and to a certain extent these singularities create a kind of lattice, within which the viscous rotational flow is developed.

This study attempts to analyse those viscous flows in its first phase of low Reynold numbers. That is to say, for dimensionless Reynold numbers in which the viscous force competes with the force of inertia (nonlinear convective operator or convective acceleration versus Laplacian). If we look at this process from the purely mathematical point of view, we are referring to flows with null local acceleration, or stationary fluids. Examples of these flows include cavity flow, steep flow, circular cylinder flow, orthogonal plate flow, two Cylinder flow, shear flow, infinite Shear flow and couette flow, when the first instability or bifurcation does not yet appear and the velocity field is steady.

For example, in the case of the flow around the two-dimensional cylinder (Fornberg, 1980), our interest is in Reynold numbers that are smaller than 50 and greater than 10. It is precisely in that interval where there is the famous recirculating and steady bubble. We have always wondered about the nature of this bubble, which accompanies objects of different geometry that travel or interpose the passage of a viscous fluid.

What is the nature of this bubble?

We believe that this bubble is a result not only of the competition between these two inertial and viscous forces, but of something more ... of a deeper compromise between the differential problem that pertains to the Navier-Stokes equations, and another hidden participant: the topology, with its famous invariant Euler's number. From our point of view, the Euler number establishes a topological limitation that refers to the domain or manifold where the fluid moves. In some way, this process forces the differential (physical) counterpart to respect this limitation. This physical-mathematical process manifests two presuppositions: on one hand, the number of points where the vector field is nullified in the manifold (every point is known as a topological defect), and on the other the behavior of the field at those points. That which is responsible for measuring the behavior of the field in such singularities is a mathematical concept called index, and it basically gives us the number of turns that the field executes in the neighborhoods of the singularity. These integers are summed and the total must match the topological invariant called the Euler number.

The Poincare-Hopf theorem is formulated mathematically in 2. and 2.1. In 3 and 3.1 we discuss the results. Section 4 discusses the conclusions.

2 Fluid dynamic and mathematical foundations

What we are going to do is justify the application of the theorem for the benchmark that we studied in this work. For this purpose, we proceed as

Singularity(R^2)	Symbol	$Ind_x(\Upsilon, p_i)$
Vortice	$\bullet$	+1
Saddle	†	-1
Sink	$\bullet$	+1
Source	$\bullet$	+1
Dipole	$\ominus$	+2
Incomplete saddle (Wall)	$\odot$	$hInd_x(\Upsilon, p_i) \in \frac{1}{2}Z$

Table 1 Index at topological defects, p_i.

follows: First, we identify the observable singularities; that is to say, from the qualitative point of view, we look at those points where the velocity field becomes null; second, we associate with each incomplete singularity at the border a negative fractional value, responding to the theorem of Hopf (1956); and third, we verify that the flow is indeed compatible with the Poincare-Hopf theorem under the assumptions of two previous steps.

Theorem (Hopf): A line field Υ with singularities $p_1, ..., p_n$ on a closed orientable surface M has:

$$\sum_{i=1}^{n} hInd_x(\Upsilon, p_i) = \chi(M)$$

The Hopf index $hInd_x(\Upsilon, p_i) \in \frac{1}{2}Z$ is the number of total rotations made by vectoral field Υ as a simple closed curve around p_i is traversed (Grant et all, 2016).

The author found an interesting equivalent attempt to define the value of the index in the boundaries of the manifold (Ma and Wang 1999, 2001). For the applications under discussion the calculation of the index values at the border of dominion is a result of the previous Hopf theorem for the dynamic configuration that was observed in the flow in question.

Table 1 represents the types of singularities that appear in two dimensions and the symbols that we have adopted from them, as well as the value resulting from the calculation of the index for each of them.

For the cavity flow and for the step flow, we have an isolated singularity of the vortex type ($\bullet$). The flow around the cylinder is richer, and we have seven singularities of the type (2-$\bullet$,1-†,4-$\odot$). The flow around two cylinders side by side is a duplication of the previous case (4-$\bullet$,2-†,8-$\odot$). For a 2-bubble couette flow, we have (6-$\bullet$,2-†,8-$\odot$).

A curiosity is how the index or the topological charge (Arnold, 1974) varies in the proximity of the wall for flows of a different nature in intermediate Reynold numbers. For example, of the cylinder flow for each singular point on the wall: $Ind_x(\odot) = \frac{1}{2} = 0.5$ or $Ind_x(\odot) = -1$, for the Side by Side Cylinder flow: $Ind_x(\odot) = \frac{1}{2} = 0.5$ or $Ind_x(\odot) = -\frac{1}{2} = -0.5$ or $Ind_x(\odot) = -1$, for the 2-bubble couette flow: $Ind_x(\odot) = -\frac{1}{2} = -0.5$, and for 4-bubble couette flow: $Ind_x(\odot) = -\frac{1}{2} = -0.5$.

2.1 Poincare-Hopf

The Poincare-Hopf theorem is well known in differential topology and was proved by Hopf (1926). This theorem relates the Euler characteristic (topological concept) with the vector field index (analytical concept).

Theorem: *(Poincare-Hopf Theorem or Index Theorem).* If $\Upsilon \in C^0(TM)$ is a vector field on M be a $C^r(r \geq 1)$, with only finitely many singular points $p_i \in M(1 \leq i \leq I)$, then the following formula holds true:

$$\sum_{x \in \Upsilon^{-1}(0)} Ind_x(\Upsilon, p_i) = \chi(M),$$

where $\chi(M)$ is the Euler characteristics of M.

Basically, this theorem tells us that, given a vector field with a finite number of zeros, the sum of the value of the calculation of the index, in each zero, is equal to the Euler characteristic of the manifold. The theorem is also telling us that the sum of the indexes of vector fields with many zeros does not depend on the vector field.

3 Results

It is worth repeating, that in this study, we have dealt only with the applications of the theorem of the two-dimensional case.

3.1 Applying the theorem

Table 2 summarises the case studies. In the first column, we note the types of domains through which the fluid develops according to the names that the community has called them over the last few years; that is, the best known benchmarks for the Reynolds intermediary regime under study. The second column provides the result of calculating the characteristic number for the given domain. The third column details the number of singularity points that the flow has for the given benchmark and, finally, the detailed calculation of the index in each type of point. It is trivial to say that, in order for the Poincare-Hopf theorem to be satisfied, the result of the sum in the last column must coincide with the integer value of the second.

The simplest cylinder flow when two recirculating bubbles appear can be summarised in the following geometric scheme:

$$CylinderFlow \leftrightarrow \begin{bmatrix} & & \bullet & \\ & \odot & & \\ \odot & \odot & & \dagger \\ & \odot & & \\ & & \bullet & \end{bmatrix}$$

Here we are considering the region of interest for the location of the singularities in the flow. The simplest 2-bubble couette flow, when two recirculating

$Manifold(Domain)$	$\chi(\Omega)$	$n : Zeros$	$\sum_{i=1}^{n} Ind_x(\Upsilon, p_i)$
Cavity	1	1	1
Cylinder Flow	0	7	$+\frac{1}{2}+\frac{1}{2}-1-1-1+1+1$
Step flow	1	1	1
Side by side Cylinder Flow	-1	14	$-4(\frac{2}{4})+2(\frac{1}{2})-1-1+4-1-1$
2-Bubble Couette flow	0	16	$-1-1+1+1+1+1+1+1+1-8(\frac{1}{2})$
4-Bubble Couette flow	0	32	$4+8-4-16(\frac{1}{2})$

Table 2 Principal values of Euler characteristic $\chi(\Omega)$ at which recirculating bubbles occur and topological defects, p_i, when vectorial field $\Upsilon(p_i) = 0$

bubbles appear and the exterior wall rotates at constant speed and the inner wall is static, can be summarised in the following geometric scheme:

$$2 - Bubble - CouetteFlow \leftrightarrow \begin{bmatrix} & \bullet & \\ \odot\odot\bullet\dagger\bullet\odot\odot & & \odot\odot\bullet\dagger\bullet\odot\odot \\ & \bullet & \end{bmatrix}$$

It is easy to perceive the correspondence between these schemes and Table 2.

4 Conclusions

We can say that the stationary dynamics for the types of streams that are studied is divided into two types of singularities or particles: those of integer topological charge and those of fractional charge that are close to the boundary. In our view, these particles are connected in a kind of lattice. For now we do not know how to profit from such a network to calculate, for example, drag or lift force, but it seems to us that somehow these topological defects or particles can pave the way to a new approach to studying fluids.

Another interesting question relates to the evolution that this topological network suffers as the Reynold number increases. We know that after a critical Reynolds arrived, the lattice is destroyed and the emission of vortexes, or some kind of Vortex Karman array, appears. We believe that each cyclic region of the array is a union of singularities of the type $\bullet, \dagger, \bullet$.

We can conclude that the deeper study of the Poincare-Hopf theorem for a three-dimensional case and other two-dimensional benchmarks, which are not addressed in this work, can lead to a better understanding of the behavior of a fluid.

If we venture to speculate on the three-dimensional case, we would say that, in this situation it is true that: $R^3 \bigcup \infty \longleftrightarrow S^3$ and $\chi(S^3) = 0$. From there it follows that in three dimensions the vector fields are not canceled anywhere. What implications does this result have for the behavior of fluids in three dimensions? However, there is something more captivating that arises from this result in the form of a question. Perhaps self-similarity and turbulence in the high Reynolds number flows, is an differential-topological attempt to preserve the validity of the Poincare-Hopf theorem.

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