Fractional Wavelet Transform Using an Unbalanced Lifting Structure

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ABSTRACT

In this article, we introduce the concept of fractional wavelet transform. Using a two-channel unbalanced lifting structure it is possible to decompose a given discrete-time signal $x[n]$ sampled with period T into two sub-signals $x_1[n]$ and $x_2[n]$ whose average sampling periods are pT and qT, respectively. Fractions p and q are rational numbers satisfying the condition: $1/p + 1/q = 1$. The low-band sub-signal $x_1[n]$ comes from $[0, \pi/p]$ band and the high-band wavelet signal $x_2[n]$ comes from $(\pi/p, \pi]$ band of the original signal $x[n]$. Filters used in the lifting structure are designed using the Lagrange interpolation formula. It is straightforward to extend the proposed fractional wavelet transform to two or higher dimensions in a separable or non separable manner.

Keywords: Lifting, wavelet transform, multirate signal processing

1. INTRODUCTION

Lifting structures provide computationally efficient implementation of the wavelet transform[1–7] . They found applications in image and video coding and signal and image analysis applications.[8–10] Modified versions of the balanced wavelet structures were also proposed for fractional scalability of the images, and videos[11] . However, lifting structures providing scalable subsignals have not been developed.

In this article, new wavelet lifting structures producing unbalanced decomposition of the input signal are introduced. In standard lifting structures the input signal is first decomposed into even and odd indexed samples using the lazy wavelet transform as shown in Fig. 1. In the proposed unbalanced lifting structure, a structure similar to the lazy filterbank forwards every p^{th} sample of the original signal to the upper-branch and remaining $p-1$ samples out of p samples go to the lower branch. Discrete-time update and prediction filters interconnect the upper and lower branches, respectively. Discrete-time filters operating in the lower sampling rates are designed using the Lagrange interpolation formula which is also used in many filterbank designs including our filterbanks, Smith-Barnwell and Daubechies filterbanks. In this way, two sub-signals with different sampling rates are obtained from the original signal. When the input signal has a sampling rate of T the upper- and lower- branches of the unbalanced lifting filterbank have sampling rates of pT and $Tp/(p-1)$, respectively. In standard balanced lifting the sampling periods of upper and lower branches are the same: $2T$.

The unbalanced lifting decomposition can be easily generalized to other sampling strategies in which the upper-branch has a sampling rate of pT and the lower-branch has a sampling rate of qT with the property that

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1}$$

Perfect reconstruction can be easily achieved by changing the signs of the filters in the reconstruction part of the ordinary balanced lifting structures.

In Section 2, an example filterbank design with $p = 3:1$ and $q = 3:2$ is presented. In Section 3, another filterbank example with $p = 3:2$ and $q = 3:1$ is presented. In Section 4, two-dimensional (2D) filterbank design examples are presented.

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2. UNBALANCED LAZY FILTERBANK AND LIFTING STRUCTURES

An unbalanced lazy filterbank for $p = 3 : 1$ and $q = 3 : 2$ is shown in Figure 2. In the upper-branch a regular downsampling block by a factor of three is used. In the lower-branch the downsampling block for $q = 3 : 2$ is used. In Figure 2, the signal $x[n]$ is fed to the lazy filterbank and outputs of downsampling blocks are shown. Every 3rd sample of the original signal goes to the upper-branch and remaining samples appear in the lower branch.

We describe the update and the prediction filter design for the unbalanced lifting structure in the following two subsections.

2.1 Update Filter

We first review with the regular balanced lifting filterbank to illustrate the framework of the unbalanced lifting. Consider the lazy filterbank shown in Figure 1.
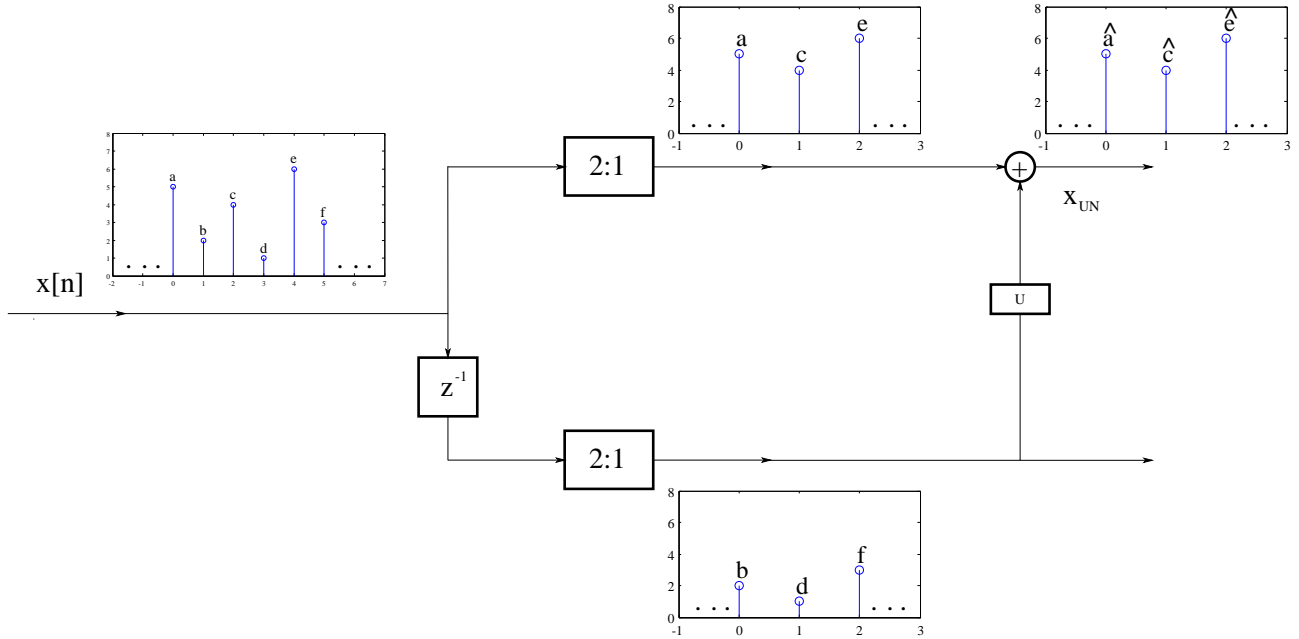

Figure 1. Lazy filterbank based regular lifting structure.

The signal $x[n]$ is divided into two sub-signals containing even and odd indexed samples of the original signal. The upper branch should be aliasing free as much as possible. The update filter tries to cancel aliasing in Figure 1. In many two-channel perfect reconstruction filter banks the Lagrange filter $h_l[n] = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ is used for smoothing the signal[4] . For example, in Figure 1, the even indexed sample c is linearly estimated using the odd indexed samples b and d by the update filter, and it is replaced by the value $(\frac{b+d}{2} + c)/2$ or

$$x_u[1] = \hat{c} = \frac{b}{4} + \frac{c}{2} + \frac{d}{4} \tag{2}$$

after updating. All the other samples of the upper-branch sub-signal $x_u[n]$ are updated in a similar manner using the linear Lagrange filter.

The above update filter is based on linear interpolation. Other higher order Lagrange filters can be also used to construct the update filter[6, 12, 13] . For example, the Lagrange filter $h_l[n] = \{\frac{-1}{32}, 0, \frac{9}{32}, \frac{1}{2}, \frac{9}{32}, 0, \frac{-1}{32}\}$ based on higher order polynomial interpolation is widely used in filter bank design[1–7, 14–17] . In Figure 1, this means that the sample c is estimated by the update filter $c' = \frac{9}{32}(b+d) - \frac{1}{32}(f + x[-1])$ and the corresponding upper branch signal is given by $\hat{c} = \frac{c+c'}{2}$.
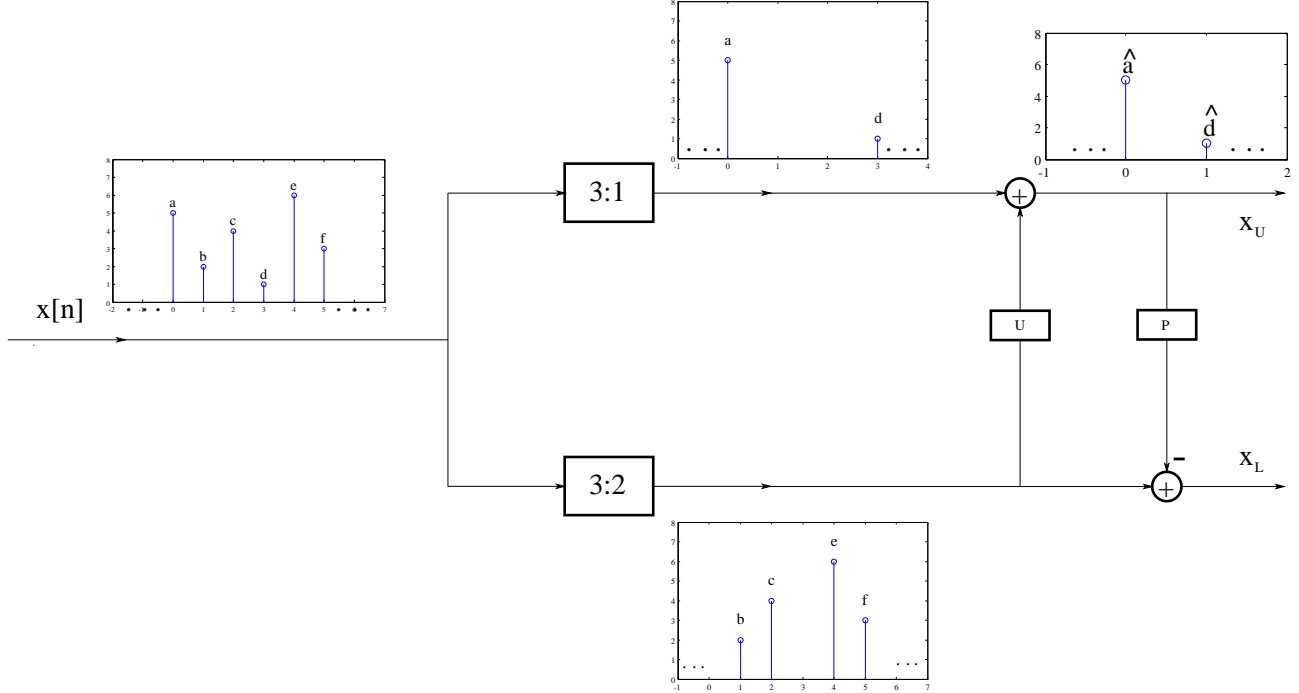
Figure 2. Unbalanced lifting scheme with $p = 3 : 1$ and $q = 3 : 2$ downsampling ratios.

In Figure 2 an unbalanced lifting structure is shown with downsampling ratios $3 : 1$ and $3 : 2$ in the upper and lower branches, respectively. Similar to the regular balanced filterbank case, the upper branch sample d can be estimated using the neighboring lower branch samples b, c, e, and f and an estimate of the sample d is given as follows:

$$d' = \frac{b + 2c + 2e + f}{6}, \tag{3}$$

which is the output of the update filter linking the lower branch to the upper branch. Since samples c and e are closer to the sample d compared to f and b more weight is given to the samples c and e. The sample d and the output of the filter is linearly combined and the updated sample is obtained as follows

$$\hat{d} = \frac{2d'}{3} + \frac{d}{3} \tag{4}$$

Since the downsampling is by a factor of three in the upper-branch original signal must be filtered with a low-pass filter with a cut-off frequency of $\pi/3$. We could have used $\hat{d} = \frac{d'}{2} + \frac{d}{2}$ but this does not produce as good frequency response as (4). Therefore, the upper branch output sample is given by

$$\hat{d} = \frac{2b}{18} + \frac{4c}{18} + \frac{6d}{18} + \frac{4e}{18} + \frac{2f}{18} \tag{5}$$

The corresponding filter coefficients are given as follows

$$h_l[n] = \{\frac{2}{18}, \frac{4}{18}, \frac{6}{18}, \frac{4}{18}, \frac{2}{18}\} \tag{6}$$

The frequency response of this filter is plotted in Figure 3. As it can be observed from this figure the cut-off frequency of this filter is $\pi/3$.

Other samples of the upper branch of the filterbank are smoothed in a similar manner. In Equation (5), each sample of the upper branch sub-signal is updated using four neighboring samples of the lower branch. It is possible to smooth the samples of the upper branch further by using more samples from the lower branch without
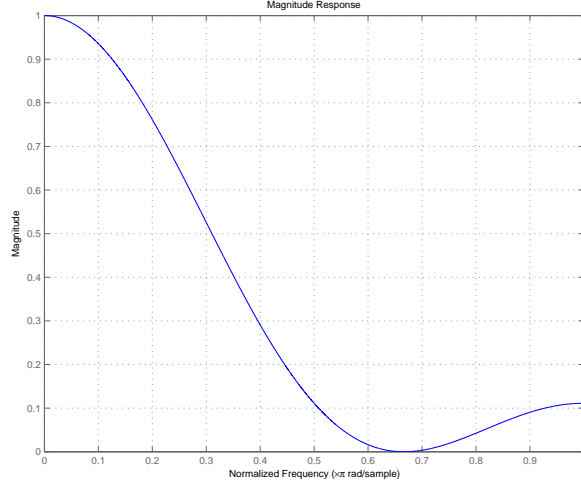
Figure 3. Frequency response of the filter in Equation (6).

effecting the perfect reconstruction capability of the filterbank. In this case, Lagrange interpolation formula or other interpolation methods can be used to determine the update filter coefficients. In general,

$$x_U[n] = \frac{1}{9}x[3n-2] + \frac{2}{9}x[3n-1] + \frac{3}{9}x[3n] + \frac{2}{9}x[3n+1] + \frac{1}{9}x[3n+2] \tag{7}$$

where $x[n]$ is the original input signal to the filterbank.

2.2 Prediction Filter

Samples of the lower branch are estimated from the upper branch and difference is transmitted to the receiver. Lower branch samples can be predicted from the upper branch using the updated samples $\hat{a}, \hat{d}, \hat{g}, ...$ (see Fig.2). The prediction filter can be as simple as the identity operator selecting the nearest upper branch sample as an estimate of the lower branch sample:

$$
\begin{aligned}
x_L[0] &= b - \hat{a}, \\
x_L[1] &= c - \hat{d}, \\
x_L[2] &= e - \hat{d}, \\
x_L[3] &= f - \hat{g}, \\
&\quad \vdots \qquad \vdots
\end{aligned}
\tag{8}
$$

Other samples of $x_L[n]$ can be determined in a similar manner. Although the above prediction strategy is very simple and computationally efficient, the above predictor is not a good estimator. we can use Lagrange interpolation and obtain:

$$
\begin{aligned}
x_L[0] &= b - (2\hat{a} + \hat{d})/3 \\
x_L[1] &= c - (2\hat{d} + \hat{a})/3 \\
x_L[2] &= e - (2\hat{d} + \hat{g})/3 \\
x_L[3] &= f - (2\hat{g} + \hat{d})/3 \\
&\quad \vdots \qquad\qquad \vdots
\end{aligned}
\tag{9}
$$

In general,

$$x_L(n) = \begin{cases} x\left[\frac{3n+2}{2}\right] - \frac{2x_U\left[\frac{n}{2}\right]+x_U\left[\frac{n}{2}+1\right]}{3}, & \text{n is even} \\ x\left[\frac{3n+1}{2}\right] - \frac{2x_U\left[\frac{n+1}{2}\right]+x_U\left[\frac{n-1}{2}\right]}{3}, & \text{n is odd} \end{cases} \tag{10}$$

where $x[n]$ is the original input signal to the filterbank.

To determine $x_L[0]$ the sample b is estimated using the two nearest upper branch samples $\hat{a}$ and $\hat{d}$. Let $P(t)$ be the Lagrange interpolator based on the samples $\hat{a}$ and $\hat{d}$:

$$P(t) = \hat{a}\,\ell_o(t) + \hat{d}\,\ell_1(t) \tag{11}$$

where $\ell_i(t)$ are the Lagrange basis polynomials. The function $P(t)$ is constructed using the input data pairs $(\hat{a}, 0)$ and $(\hat{d}, 3T)$. Since the sample b occurs at $t = T$, $\ell_o(T) = 2/3$ and $\ell_1(T) = 1/3$. Therefore the Lagrange interpolation gives more weight to the sample $\hat{a}$ because it is nearer to the sample b compared to $\hat{d}$ to determine $x_L[0]$ in (9). Similarly, more weight is given to the sample $\hat{d}$ because it is nearer to the sample c compared to $\hat{a}$ to determine $x_L[1]$ etc. As it can be seen from the above equations the predictor is a time varying filter. The predictor can even be an adaptive LMS-type filter as described in references[4, 14, 18]. The adaptive prediction tries to remove as much information as possible from the lower-branch using $x_L[n]$ samples. In this case the computational complexity is higher than the predictors in (8) and (9). Other samples of $x_U[n]$ can be also used by the predictor filter. In this case higher order Lagrange interpolation formula needs to be used[4, 13].

It is trivial to design the reconstruction filterbank as in regular lifting structures. At the reconstruction stage signs of filters U and P are changed and sub-signal samples are realigned to obtain the original signal $x[n]$.

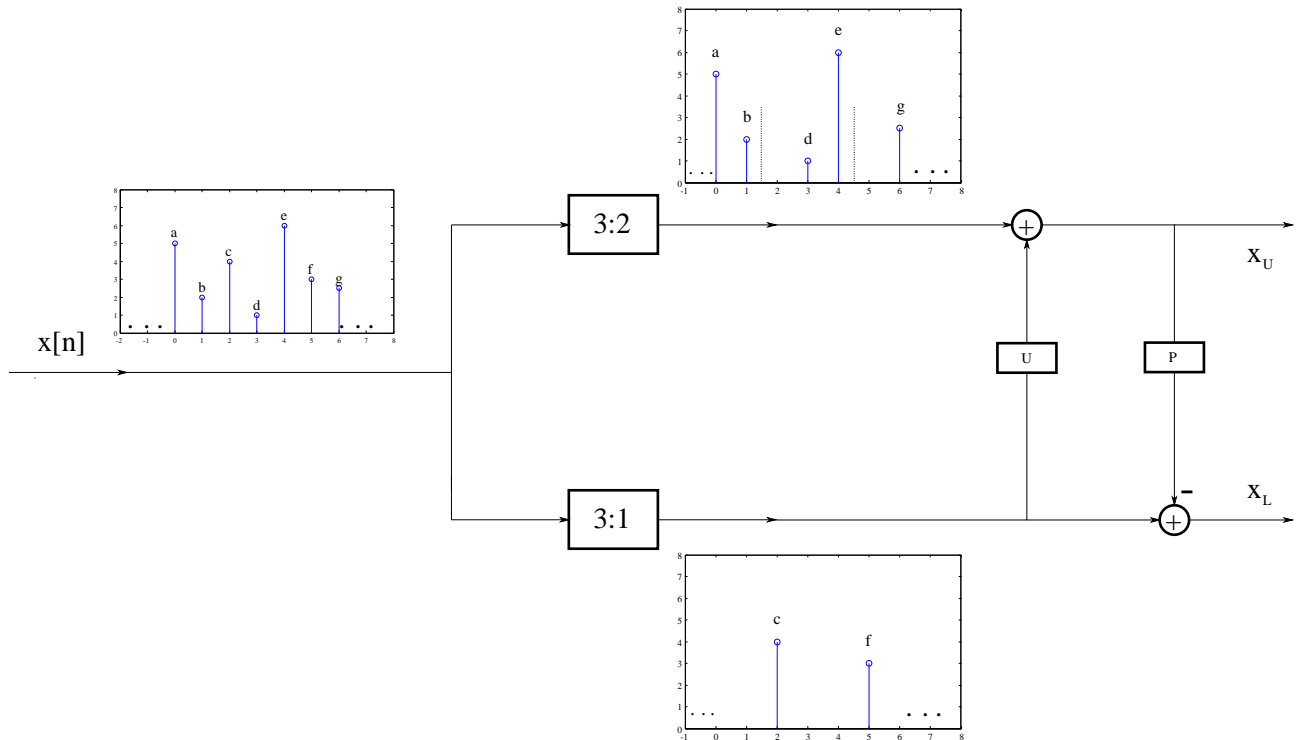## 3. UNBALANCED FILTERBANK WITH $p = 3 : 2$ AND $q = 3 : 1$

Figure 4. Unbalanced lifting structure with p=3:2 and q=3:1.

In Figure 4, the upper-branch has a downsampling ratio of $3 : 2$ and the lower-breach has a downsampling ration of $3 : 1$. In this case, samples of the upper-branch are downsampled in a non-uniform manner. In the

upper branch the original shape of the signal has to be preserved as much as possible in practical applications including image and video processing. Therefore, the samples of the upper branch should be transformed to a uniformly sampled grid.

For example, in Figure 4, we need to estimate $x[1.5]$, $x[4.5]$, ... using the lower-breach samples and the nearest current samples of the upper-branch, which are $x[1]$, $x[4]$, ..., respectively.

We can use linear interpolation and obtain the following updated values:

$$x_U[1] = \hat{x}[1.5] = \frac{b+c}{2}, \quad x_U[3] = \hat{x}[4.5] = \frac{e+f}{2}... \tag{12}$$

One can also use other lower-branch samples as well during estimation without affecting perfect reconstruction.

Samples of $x_U[2n]$ can be set as $x[3n]$ or it is also possible to update other samples of the upper-breach using the lower branch:

$$x_U[0] = a, \quad x_U[2] = \frac{1}{3}c + \frac{2}{3}d, \ x_U[4] = \frac{1}{3}f + \frac{2}{3}g, ... \tag{13}$$

In general,

$$x_U(n) = \begin{cases} \frac{x[1.5n-1]+2x[1.5n]}{3}, & \text{n is even and n} \neq 0 \\ \frac{x[1.5(n-1)+1]+x[1.5(n+1)-1]}{2}, & \text{n is odd} \end{cases} \tag{14}$$

where $x[n]$ is the original input signal to the filterbank.

As usual, the prediction filter tries to remove as much information from the lower-breach as possible. For example, the samples c can be predicted as follows, $x_L[1] = c - x_U[1]$, and, similarly $x_L[2] = f - x_U[3]$, etc...

Similar to the structure in Section 2, other samples of the upper-branch output $x_U[n]$ can be also used by a Lagrange interpolator for prediction purposes without effecting perfect reconstruction resulting a better quality sub-signal $x_L[n]$ as follows:

$$x_L[1] = c - (\frac{2}{3}x_U[1] + \frac{1}{3}x_U[2]) \tag{15}$$

where the predictor uses the same weights as in Equation (8) because the sample $x_U[1]$ occurs at $t = 1.5T$ and $x_U[2]$ occurs at $t = 3T$, and the sample c occurs at $t = 2T$ as shown in Fig. 4. As a result the Lagrange interpolator gives more weight to $x_U[1]$ compared to $x_U[2]$. Similarly, the next sample is predicted as follows

$$x_L[2] = f - (\frac{2}{3}x_U[3] + \frac{1}{3}x_U[4]) \tag{16}$$

In general,

$$x_L[n] = x[3n-1] - (\frac{2}{3}x_U[2n-1] + \frac{1}{3}x_U[2n]) \tag{17}$$

where $x[n]$ is the original input signal to the filterbank.

It is straightforward to design the reconstruction filterbank as in regular lifting structures. At the reconstruction stage signs of filters U and P are changed and sub-signal samples are realigned to obtain the original signal $x[n]$.

4. EXTENSION FOR TWO-DIMENSIONAL SIGNALS

A N-dimensional filter $T(x_1, x_2...x_N)$ is seperable if it can be represented as

$$T(x_1, x_2...x_N) = T_1(x_1)T_2(x_2)...T_N(x_N) \tag{18}$$

Assume that the two-dimensional (2D) filter $f(x, y)$ is seperable. Therefore it can be represented as follows

$$T(x, y) = T_1(x)T_2(y) \tag{19}$$

For a 2D signal or an image, Equation (19) corresponds to first filtering the signal in horizontal direction by $T_1(x)$ and then filtering the output signal in vertical direction by $T_2(y)$.

In Figure 5, the seperable filterbank structure for the unbalanced lifting wavelet transform with $p = 3 : 2$ and $q = 3 : 1$ is given. The input image is first downsampled in the horizontal direction. Then the update and prediction filters are applied to the downsampled images. The intermediate output signals X_U and X_L are then downsampled in the vertical direction and the update and prediction filters are applied again. As an example,
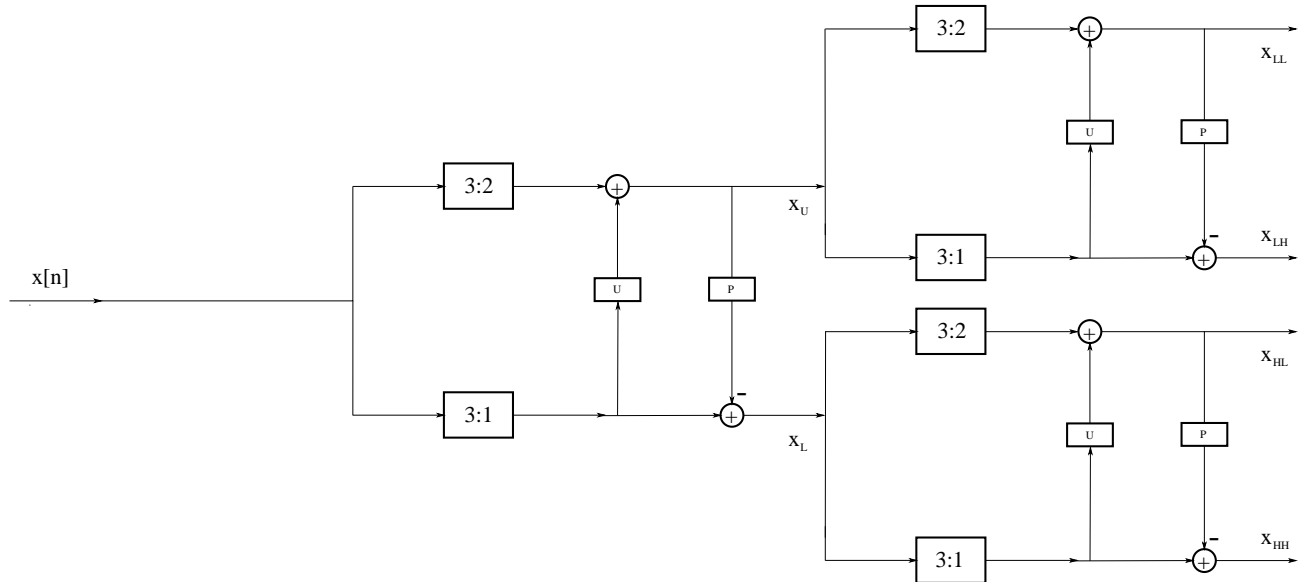

Figure 5. Extension of the unbalanced lifting structure with p=3:2 and q=3:1 to 2D.

the image given at Figure 6 is fractional wavelet transformed using the scheme presented in Figure 5. The transformed image is given at Figure 7. As another example, the image given at Figure 8 is wavelet transformed in Figure 2 using the fractional wavelet transform with $p = 3 : 1$ and $q = 3 : 2$ which is a 2D extended version of the filterbank described in Section 2.

It is also possible to develop non-seperable fractional filterbanks.

CONCLUSION

In this article, fractional wavelet transform (FWT) methods are developed using two-channel unbalanced lifting structures. Sub-signals have different average sampling rates.

The discrete time signal $x[n]$ sampled with period T can be decomposed into two sub-signals with average sampling periods of pT and qT, respectively. Example lifting structures for $p = 3 : 1$ and $q = 3 : 2$ and $p = 3 : 2$ and $q = 3 : 1$ cases are presented.

The FWT method can be easily extended to two or higher dimensions. Image transformation examples are presented.
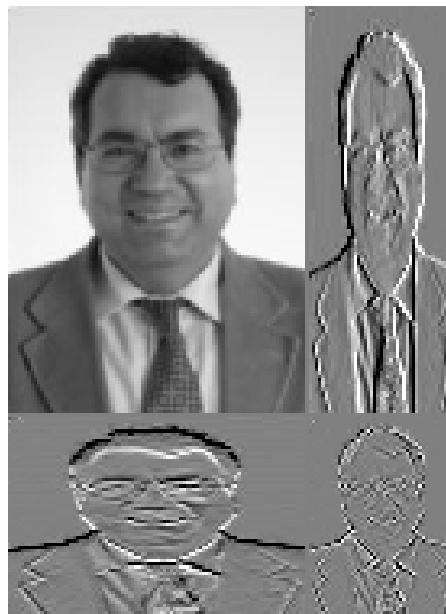
ACKNOWLEDGMENTS

Figure 6. Example image.

Figure 7. 2D unbalanced lifting wavelet transformed image with p=3:2 and q=3:1. Highband subimages are amplified by a factor of three.

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Figure 8. 2D unbalanced lifting wavelet transformed image with p=3:1 and q=3:2. Highband subimages are amplified by a factor of three.

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