A Brief Analysis of the Hooke's Joint

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Abstract

The purpose of this paper is to derive the mathematical equations which govern the movements of a Hooke's joint, also known as a universal joint. A Hooke's joint has an input shaft and an output shaft. If the input shaft rotates through an angle IN, then the output shaft will simultaneously rotate through an angle OUT. The functional relationship between input angle and output angle is a function of the angular offsets between the two shafts and of the original state of the Hooke's joint for zero angle IN. The derivation is obtained by considering the individual three dimensional rotations that characterize an arbitrary Hooke's joint state. These rotations are represented by 3x3 rotation matrices. An electro-mechanical magnetic flux-gate compass application is also discussed.

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1 Derivation of Hooke's Joint Equations

An arbitrary orientation of a rigid body in three-dimensional space can be achieved by small number of successive rotations of the body around specific axes from an initial set of axes. Corresponding to these (physical) rotations are *rotation matrices*, mathematical devices relating the Cartesian coordinates of any point or vector components measured in the initial coordinate system to the coordinates measured in the final coordinate system. Successive physical rotations are represented by successive multiplications of rotation matrices.

We will derive the equations for the Hooke's joint step-by-step, considering the individual rotation matrices needed to explain its operation.

Figure 1 below depicts a Hooke's joint with the axles or shafts translated in space. Ideally, a Hooke's joint might consist of rings or gimbals having a common origin around which all rotations would occur.

The purpose of having the axes translated or spread out in space in Figure 1 is to provide a picture which might better illustrate the rotations that are mathematically described as occurring *sequentially*. In reality, the various rotations considered do not occur sequentially; however, the mathematical model of the resultant three dimensional rotation is considered as a product of rotations around individual axes. The angles and rotations considered by individual matrices can be examined singly by referring to the single rotation matrices which account for them.

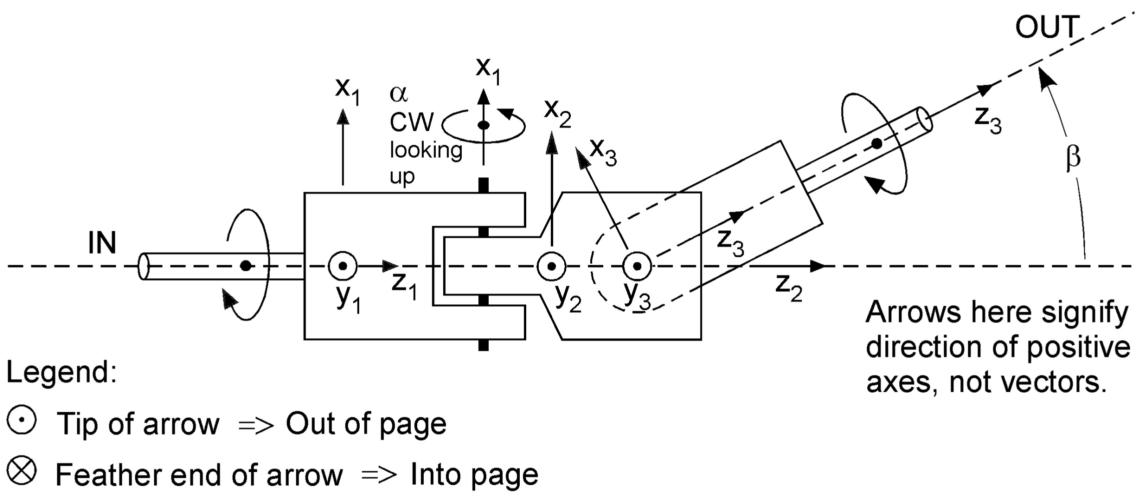

Figure 1: The Hooke Joint.

1.1 Hookejoint Rotations

Sequentially cause the following rotations to the device depicted in Figure 1. The first rotation is at the left of the figure, and the last one is at the right.

- Step 1. Rotate coordinate system S_1 around x_1 axis by angle α :

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = X(\alpha) \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \ \alpha \in [0, \ \pi/2];$$

- Step 2. Rotate coordinate system S_2 around axis y_2 by angle β :

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = Y(\beta) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix};$$

Let $H(\alpha,\beta)$ be the Hooke's joint matrix defined by

$$H(\alpha,\beta) = Y(\beta)\,X(\alpha) \qquad (1)$$

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = H(\alpha,\beta) \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad (2)$$

$$H(\alpha,\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\boxed{H(\alpha,\beta) = \begin{pmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix}} \qquad (3)$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = R \begin{pmatrix} \sin\phi_1\cos\theta_1 \\ \sin\phi_1\sin\theta_1 \\ \cos\phi_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = R \begin{pmatrix} \sin\phi_3\cos\theta_3 \\ \sin\phi_3\sin\theta_3 \\ \cos\phi_3 \end{pmatrix}$$

$$\begin{pmatrix} \sin\phi_3\cos\theta_3 \\ \sin\phi_3\sin\theta_3 \\ \cos\phi_3 \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \sin\phi_1\cos\theta_1 \\ \sin\phi_1\sin\theta_1 \\ \cos\phi_1 \end{pmatrix}$$

But $\phi_3 = \phi_1 = 90^\circ$, since there are no vector components along the z_1 or z_3 axes,

$$\begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\beta\cos\theta_1 + \sin\alpha\sin\beta\sin\theta_1 \\ \cos\alpha\sin\theta_1 \\ \cos\theta_1\sin\beta - \cos\beta\sin\alpha\sin\theta_1 \end{pmatrix}$$

$$\begin{cases} \cos\theta_3 = \cos\beta\cos\theta_1 + \sin\alpha\sin\beta\sin\theta_1 & (4) \\ \sin\theta_3 = \cos\alpha\sin\theta_1 & (5) \\ 0 = \cos\theta_1\sin\beta - \cos\beta\sin\alpha\sin\theta_1 & (6) \end{cases}$$

(4) $\times \sin\beta \quad \Rightarrow \quad \cos\theta_3\sin\beta = \sin\beta\cos\beta\cos\theta_1 + \sin\alpha\sin^2\beta\sin\theta_1$

(6) $\times \cos\beta \quad \Rightarrow \quad 0 = \cos\theta_1\sin\beta\cos\beta - \cos^2\beta\sin\alpha\sin\theta_1$

Subtracting, the last equation from the previous equation,

$\Rightarrow \quad \cos\theta_3\sin\beta = \sin\alpha\sin\theta_1$

But from (6), $\sin\alpha\sin\theta_1 = \dfrac{\cos\theta_1\sin\beta}{\cos\beta}$, so $\cos\theta_3 = \dfrac{\cos\theta_1}{\cos\beta}$

Dividing (5) by this equation $\Rightarrow$

$$\tan\theta_3 = \cos\alpha\cos\beta\tan\theta_1 \quad (7)$$

In general, let $\mathbf{R}$ be a vector $\mathbf{R} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z = (\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}\,)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, where $\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}$ are unit vectors.

$\mathbf{k}_3 \cdot \mathbf{k}_1 = z_3/z_1 = \cos\phi$, where ϕ is the angle between the z_3 axis and the z_1 axis.

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_3$ are the components of the unit vectors along the z_1 and z_3 axes respectively.

Computing the scalar product of unit vectors, we obtain

$$\mathbf{k}_3 \cdot \mathbf{k}_1 = (0,0,1)_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_3 = (0,0,1)_1 \; H(\alpha,\beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1$$

$$H(\alpha,\beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 = \begin{pmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 = \begin{pmatrix} -\cos\alpha\sin\beta \\ \sin\alpha \\ \cos\alpha\cos\beta \end{pmatrix};$$

Therefore, $(0,0,1)_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_3 = (0,0,1)_1 \begin{pmatrix} -\cos\alpha\sin\beta \\ \sin\alpha \\ \cos\alpha\cos\beta \end{pmatrix}_1 = \cos\alpha\cos\beta.$

$$\boxed{\therefore \quad \cos\alpha\cos\beta = \cos\phi \quad (8)}$$

Substituting (8) into (7), we obtain the equation of the Hooke's joint usually encountered in texts.

$$\boxed{\tan\theta_3 = \tan\theta_1\cos\phi \quad (9)}$$

1.2 An Application

Consider an electro-mechanical compass of the following description. In the past, such compasses had been found to be useful, for example, on oceanographic data buoys and in bore-hole surveying. Modern day compasses dispense with this electro-mechanical servo system described here and use strap-down sensors instead.

A magnetic flux-gate is a species of magnetic sensor or magnetic flux detector (MFD) quite suitable for measuring the geomagnetic field intensity on the surface of the earth.

A pendulous single-axis magnetic flux-gate hangs suspended from the lower Hooke's joint. If the compass tilts, the flux-gate remains in a horizontal position with the lower shaft of the Hooke's joint vertical. The upper shaft of the Hooke's joint, aligned with the longitudinal axis of the compass case, is coupled to a small instrument servo-motor and a potentiometer (POT), which device is also fixed to the case. The potentiometer is the output transducer and provides a D.C. output voltage proportional to the angle through which the upper shaft turns. The flux-gate is provided with an electrical excitation circuit, synchronous demodulator and motor drive amplifier, circuits typically employed for magnetic flux-gate operation. The flux-gate, together with this electronic circuitry, constitute a magnetometer of sorts. The maximum output voltage occurs when the sensitive axis of the flux-gate is facing magnetic North. When it is facing East and West, the output voltage will be zero. This output voltage drives the servo-motor to rotate the Hooke's joint until the output voltage is zero and the sensitive axis of the MFD is oriented at right angles to magnetic North.

In typical deployment, the compass will (inadvertently) experience different degrees of tilt ϕ, and yet be expected to provide correct co-azimuths. The tilt angle ϕ (aka Slant Angle SA) must be input by the user or by a tilt sensor into the compass system to obtain correct co-azimuth. The angles θ_1 and θ_3 are measured in standard mathematical format. However, we are usually interested in azimuth, which in navigational format, is measured in the opposite direction. We may simply use Azimuth = 360$^{\cdot}-$ Co-azimuth or reverse the excitation voltage connections to P_1 and P_2 on the POT (Reference: Figure 2). A typical POT excitation voltage might be, for example, regulated 3.60 VDC, for a sensitivity of 10mV/degree.

If the compass case is rotated through a co-azimuth of, say θ_1, the motor will rotate the Hooke's joint until the lower shaft rotates by an angle $-\theta_1$, keeping the MFD always pointing in the same direction, toward magnetic North. However, the upper shaft and POT wiper will rotate through a different angle $-\theta_3$, relative to the POT and compass case, which angle must be corrected for any "Hookejoint error". The POT voltage can be measured with a digital-to-analog converter (DAC) connected to a micro-controller programmed with equation (9) for this correction. Figure 2 represents an electro-mechanical compass for use on ocean data buoys. The longitudinal axis of the compass case is z_3. A forward marker arrow FWD is attached to the top of the compass case.

The same configuration might be employed in a geophysical survey probe, augmented by an angle ψ through which the survey probe might be rotated. The angle ψ is measured in standard mathematical format. Alternatively, tool face angle $TF = 360° - \psi$, measured in navigational format, may be employed.

The angle θ_3 (measured by the POT) must be compensated for tool face angle to give measured azimuth, $\theta_{meas} = \theta_3 - TF$, and then corrected for Hooke's joint error.

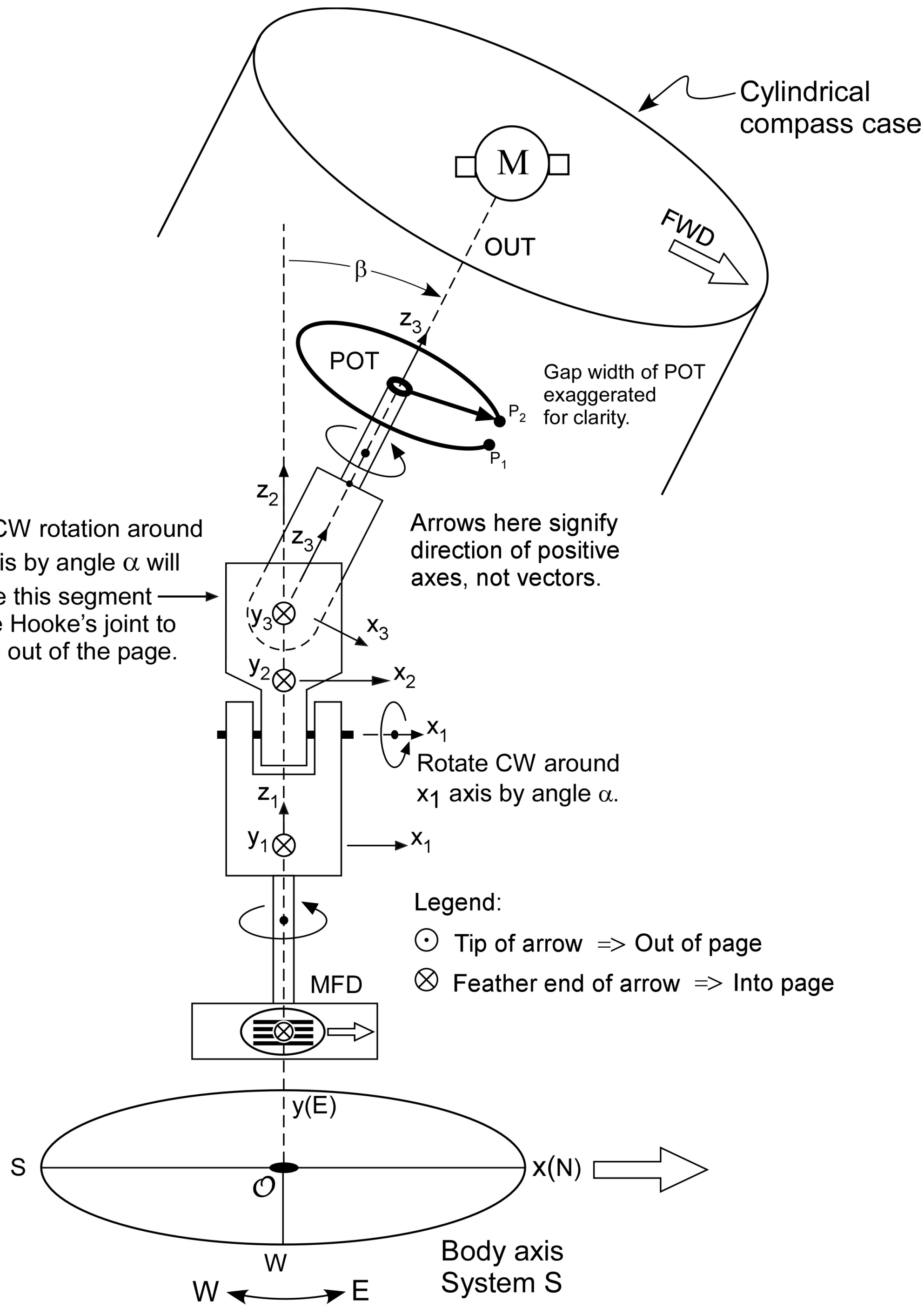

Figure 2: An Electro-Mechanical Compass.

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