Projector's Weighting for W-MUSIC: An Alternative to RMT

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***Abstract*—In the last decade, modified subspace DoA estimation methods such as G-MUSIC have been proposed, in the context where the number of available snapshots N is of the same order of magnitude than the number of sensors M. In this context, the conventional MUSIC algorithm fails in presence of close sources because the empirical covariance matrix is a poor estimate of the true covariance matrix. The G-MUSIC algorithm is based on Marcenko-Pastur's works about the distribution of the eigenvalues of the empirical covariance matrix. A new modified MUSIC algorithm is proposed. It is based on the correction of the noise projector obtained by complex Wishart distribution of the empirical covariance matrix.**

***Index Terms*—MUSIC, DoA estimation , Performances analysis, Wishart distribution, Random matrices**

I. Introduction

The estimation of the Direction of Arrival (DoA) of plane waves impinging on an array of sensors is an important problem of great interest in radar and radiocommunication context. In presence of Multiple sources, MUSIC [1] is one of the most famous high resolution algorithm. MUSIC is based on a subspace approach with the estimation of the noise operator from the empirical covariance matrix. The algorithm is asymptotically unbiased when the number N of snapshots is larger than the number of sensors. However, when the number N decreases the MUSIC estimation becomes biased with a limited resolution power. These statistical performances have been highly studied during these last three decades [2][3][4][5]. Most of these studies with performances predictions are based on the perturbation analysis of the MUSIC noise projector [6][7] and the Wishart distribution of the empirical covariance matrix [2][8].

In the context where the number of available snapshots N is of the same order of magnitude than the number of sensors M, the previous works show that the MUSIC performances degrades drastically. In this context, the conventional MUSIC algorithm can fail in presence of close sources and/or high correlated sources. Indeed, the empirical covariance matrix is a poor estimate of the true covariance matrix and the perturbation of the noise projector is important. More precisely, it is difficult to separate the eigenvalues associated to the noise subspace and the ones associated to the signal subspace.

In order to improve the MUSIC performances McCloud and Scharf [9] propose a first modified MUSIC algorithm and when the number of antennas M is large and at the same order of the snapshots number N, Mestre et al [10][11] proposed a second modified MUSIC algorithm G-MUSIC. The approach uses the large Random Matrix Theory (RMT) results of Marcenko-Pastur [12] in order to exploit the statistical distribution of the eigen elements of the empirical covariance matrix in presence of i.i.d source and noise signals. According to [10][11] the performances of such new estimator outperforms the traditional DoA subspace estimator.

The purpose of this paper is to give an alternative to G-MUSIC. For that the MUSIC projector is corrected according to the results of the MUSIC performances prediction papers [[2]$\cdots$[5]]. Indeed, our paper proposes a modified MUSIC projector by using the Wishart distribution of the empirical covariance matrix and the perturbation analysis of the noise projector. Our approach uses these mathematical tools in order to obtain the statistical distribution of the MUSIC criterion value and remove its bias of the MUSIC criterion value. The new algorithm (W-MUSIC as Weighted-MUSIC) is then able to resolve unresolved source with a corrected MUSIC approach. A comparison with G-MUSIC and the deterministic Cramer-Rao Bound (CRB) is given in this paper.

II. Signal Modeling, Assumptions and Problems formulation

In presence of K sources, the signal at the output of the array of M sensors is

$$\mathbf{x}(t) = \sum_{k=1}^{K} \mathbf{a}_k \times s_k(t) + \mathbf{n}(t) = \mathbf{A} \times \mathbf{s}(t) + \mathbf{n}(t) \quad (1)$$

where $\mathbf{x}(t)$ is a $M \times 1$ vector, $\mathbf{n}(t)$ is the additional noise, $\mathbf{s}(t) = \begin{bmatrix} s_1(t) & \cdots & s_K(t) \end{bmatrix}^T$, $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_K \end{bmatrix}$ and $\mathbf{a}_k = \mathbf{a}(\Theta_k)$ is the steering vector associated to a source of direction Θ_k and signal $s_k(t)$. The MUSIC algorithm [1] estimates the DoAs Θ_k of impinging sources by exploiting the eigen-structure of the covariance matrix $\mathbf{R}_x = E\left[\mathbf{x}(t)\,\mathbf{x}^H(t)\right]$ where $E[.]$ is the mathematical mean. Assuming that the noise $\mathbf{n}(t)$ is spatially white and independent of the signals $s_k(t)$, the true covariance matrix is

$$\mathbf{R}_x = \mathbf{S} + \sigma^2 \mathbf{I}_M \quad \text{with } \mathbf{S} = \mathbf{A}\mathbf{P}\mathbf{A}^H \quad (2)$$

where $E\left[\mathbf{n}(t)\,\mathbf{n}^H(t)\right] = \sigma^2 \mathbf{I}_N$, σ^2 is the noise level and $\mathbf{P} = E\left[\mathbf{s}(t)\,\mathbf{s}^H(t)\right]$ is the sources covariance matrix. In MUSIC the sources covariance matrix $\mathbf{S}$ is assumed to be full-rank with uncoherent multi-paths. Considering all these assumptions, the

Eigen Value Decomposition (EVD) of $\mathbf{R}_x$ and $\mathbf{S}$ are

$$\mathbf{R}_x = \sum_{k=1}^{M} \lambda_k \mathbf{e}_k \mathbf{e}_k^H = \sum_{k=1}^{K} \lambda_k \mathbf{e}_k \mathbf{e}_k^H + \sigma^2 \mathbf{\Pi} \quad (3)$$

$$\mathbf{S} = \sum_{k=1}^{K} \left(\lambda_k - \sigma^2\right) \mathbf{e}_k \mathbf{e}_k^H \quad (4)$$

where $(^H)$ denotes the transconjugate, λ_k ($\lambda_1 \geq \cdots \geq \lambda_M$) is the eigen-value associated to the eigen-vector $\mathbf{e}_k$ and $\mathbf{\Pi} = \sum_{k=K+1}^{M} \mathbf{e}_k \mathbf{e}_k^H$ is the noise projector. According to (1)(3)(4) the steering vectors $\{\mathbf{a}_1 \cdots \mathbf{a}_K\}$ span the signal subspace of $\{\mathbf{e}_1 \cdots \mathbf{e}_K\}$ and are orthogonal to the noise subspace $\{\mathbf{e}_{K+1} \cdots \mathbf{e}_M\}$ such that: $\mathbf{a}_k^H \mathbf{\Pi} \mathbf{a}_k = 0$. Thus, the MUSIC algorithms determines the source DoAs Θ_k as the K minima of the following criterion $\eta(\Theta)$

$$\eta(\Theta) = \mathbf{a}^H(\Theta)\, \mathbf{\Pi}\, \mathbf{a}(\Theta) \quad (5)$$

In practice, the true covariance matrix $\mathbf{R}_x$ is not available. The matrix is then estimated from N i.i.d snapshots $\mathbf{x}(t_n)$, such that

$$\hat{\mathbf{R}}_{x,N} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}(t_n)\, \mathbf{x}^H(t_n) = \sum_{k=1}^{M} \hat{\lambda}_{k,N} \hat{\mathbf{e}}_{k,N} \hat{\mathbf{e}}_{k,N}^H \quad (6)$$

where $\hat{\mathbf{R}}_{x,N}$ is the empirical covariance matrix and where the eigen-elements $\hat{\lambda}_{k,N}$ and $\hat{\mathbf{e}}_{k,N}$ are random variables such that $\hat{\lambda}_{K+i,N}$ are different to the noise level σ^2. The MUSIC criterion of (5) becomes

$$\eta_N(\Theta) = \mathbf{a}^H(\Theta)\, \hat{\mathbf{\Pi}}_N\, \mathbf{a}(\Theta) \text{ with } \hat{\mathbf{\Pi}}_N = \sum_{k=K+1}^{M} \hat{\mathbf{e}}_{k,N} \hat{\mathbf{e}}_{k,N}^H \quad (7)$$

where $\eta_N(\Theta)$ is a random variable for each direction Θ and $\hat{\mathbf{\Pi}}_N$ is the empirical noise projector where $\mathbf{a}_k^H \hat{\mathbf{\Pi}}_N\, \mathbf{a}_k \neq 0$ because the subspace $\{\hat{\mathbf{e}}_{K+1,N} \cdots \hat{\mathbf{e}}_{M,N}\}$ is different to the true one $\{\mathbf{e}_{K+1} \cdots \mathbf{e}_M\}$. This is the reason why the MUSIC performances are limited in this context.

In G-MUSIC[10], the authors modify the criterion $\eta_N(\Theta)$ from the RMT results of Marcenko-Pastur [12] on the distribution of the eigen-elements $\hat{\lambda}_{k,N}$ and $\hat{\mathbf{e}}_{k,N}$ when the number of antennas M is large and at the same order of N. In the background, the MUSIC performance prediction are mainly determined from the Wishart distribution of the empirical covariance matrix $\hat{\mathbf{R}}_{x,N}$ and the perturbation analysis of the noise projector determines the bias $\mathbf{\Pi} - \hat{\mathbf{\Pi}}_N$ with respect to the covariance error $\mathbf{R}_x - \hat{\mathbf{R}}_{x,N}$. In this paper, these previous tools allow us to predict the MUSIC criterion bias $\Delta\eta_N(\Theta) = E[\eta_N(\Theta) - \eta(\Theta)]$ in order to obtain a new algorithm with a unbiased MUSIC criterion.

III. Background and Tools

In this section the G-MUSIC[10] algorithm, the deterministic Cramer Rao Bound[13], the Wishart distribution and the perturbation analysis results of the noise projector are presented.

A. G-MUSIC algorithm [10]

When the magnitudes of N and M are close, the empirical noise eigenvector $\hat{\mathbf{e}}_{k+i,N}$ is not perfectly included into the true noise subspace $\{\mathbf{e}_{K+1} \cdots \mathbf{e}_M\}$ and is not perfectly orthogonal to the true signal subspace $\{\mathbf{e}_1 \cdots \mathbf{e}_K\}$. It is then a linear combination of all the vectors $\{\mathbf{e}_1 \cdots \mathbf{e}_M\}$. This is the reason why G-MUSIC algorithm modifies the MUSIC noise projector in order to obtain a best separation between the noise and signal subspace. More precisely, the G-MUSIC projector $\hat{\mathbf{\Pi}}_N^G$ depends on all the eigen-elements of the empirical covariance matrix $\hat{\mathbf{R}}_{x,N}$ such that the projector $\hat{\mathbf{\Pi}}_N^G$ is close to $\hat{\mathbf{\Pi}}_N$ when N is larger than the antenna number M of the array. Thus, the G-MUSIC algorithms determines the source DoAs Θ_k as the K minima of the following criterion $\eta_{G,N}(\Theta)$

$$\eta_{G,N}(\Theta) = \mathbf{a}^H(\Theta)\; \hat{\mathbf{\Pi}}_N^G\; \mathbf{a}(\Theta) \quad (8)$$

$$\hat{\mathbf{\Pi}}_N^G = \sum_{m=1}^{M} \Phi(m)\, \hat{\mathbf{e}}_{m,N} \hat{\mathbf{e}}_{m,N}^H \quad (9)$$

where

$$\Phi(m) = \begin{cases} 1 + \sum_{i=1}^{K} \delta\Phi_{mi} & m > K \\ -\sum_{i=K+1}^{M} \delta\Phi_{mi} & m \leq K \end{cases} \quad (10)$$

$$\delta\Phi_{mi} = \left(\frac{\hat{\lambda}_{i,N}}{\hat{\lambda}_{m,N} - \hat{\lambda}_{i,N}}\right) - \left(\frac{\nu_i}{\hat{\lambda}_{m,N} - \nu_i}\right)$$

where $\nu_1 \geq \cdots \geq \nu_M$ are the eigenvalues of $diag\left(\hat{\boldsymbol{\lambda}}_N\right) - \left(\hat{\boldsymbol{\lambda}}_N \hat{\boldsymbol{\lambda}}_N^T\right)/N$ with $\hat{\boldsymbol{\lambda}}_N^T = \left[\hat{\lambda}_{1,N} \cdots \hat{\lambda}_{M,N}\right]$. The coefficients $\delta\Phi_{mi}$ can be seen as correction of the MUSIC noise projector.

B. Deterministic Cramer Rao Bound (CRB)

The Root Mean Square (RMS) error of DoA estimation is limited by the CRB such that $\sqrt{E[\Delta\Theta_k^2]} > CRB(\Theta_k)$ where $\Delta\Theta_k = \hat{\Theta}_k - \Theta_k$ is the DoA estimation error . In presence of Gaussian noise, the expression of the CRB of the $k-th$ source is

$$CRB(\Theta_k) = \sqrt{\frac{\mathbf{H}_{[k][k]}^{-1}}{2N}} \text{ with } \mathbf{H} = \Re\left(\left(\dot{\mathbf{A}}^H \mathbf{\Pi} \dot{\mathbf{A}}\right) \odot \hat{\mathbf{S}}_N\right) \quad (11)$$

where $\mathbf{W}_{[i][j]}$ is the ij^{th} element of the matrix $\mathbf{W}$, $\odot$ is the Hadamard product, $\dot{\mathbf{A}} = \left[\; \dot{\mathbf{a}}(\Theta_1) \;\; \cdots \;\; \dot{\mathbf{a}}(\Theta_K) \;\right]$, $\dot{\mathbf{a}}(\Theta)$ is the first order derivative of $\mathbf{a}(\Theta)$ with respect to Θ, $\hat{\mathbf{S}}_N = \frac{1}{N}\sum_{n=1}^{N} \mathbf{s}(t_n)\, \mathbf{s}^H(t_n)$ is the source empirical covariance matrix and $\mathbf{\Pi}$ is the true MUSIC noise projector where

$$\mathbf{\Pi} = \mathbf{I}_M - \mathbf{A}\left(\mathbf{A}^H \mathbf{A}\right)^{-1} \mathbf{A}^H \quad (12)$$

C. Wishart distribution

The matrix error $\Delta\mathbf{R}_x = \hat{\mathbf{R}}_{x,N} - \mathbf{R}_x$ is a Random Matrix where $N\hat{\mathbf{R}}_{x,N}$ follows Wishart distribution with parameter $\mathbf{R}_x$. According to the central limit theorem and [8], the elements of $\Delta\mathbf{R}_x$ are asymptotically Gaussian distributed such that

$$E[\Delta\mathbf{R}_x] = \mathbf{0} \quad (13)$$

$$E\left[\Delta\mathbf{R}_{x[i][j]} \Delta\mathbf{R}_{x[m][n]}\right] = \frac{\mathbf{R}_{x[m][j]}\, \mathbf{R}_{x[i][n]}}{N} \quad (14)$$

According to [2] and (13)(14), the Random Matrix is verifying

$$P_1: \ E\left[Tr\left(\Delta\mathbf{R}_x\mathbf{A}\right)Tr\left(\Delta\mathbf{R}_x\mathbf{B}\right)\right] = \frac{Tr(\mathbf{R}_x\mathbf{A}\mathbf{R}_x\mathbf{B})}{N} \quad (15)$$

$$P_2: \ E\left[\Delta\mathbf{R}_x\mathbf{A}\Delta\mathbf{R}_x\right] = \frac{Tr(\mathbf{R}_x\mathbf{A})\mathbf{R}_x}{N} \quad (16)$$

$$P_3: \ E\left[\Delta\mathbf{R}_x\mathbf{A}\Delta\mathbf{R}_x^T\right] = \frac{\mathbf{R}_x\mathbf{A}^T\mathbf{R}_x}{N} \quad (17)$$

where $Tr\left(.\right)$ denotes the trace of a matrix.

D. Perturbation analysis of the noise projector

The noise projector error $\hat{\mathbf{\Pi}}_N - \mathbf{\Pi}$ is depending on the covariance error $\Delta\mathbf{R}_x$ such that $\hat{\mathbf{\Pi}}_N = \mathbf{\Pi}$ for $\Delta\mathbf{R}_x = 0$. According to [2], the second order Taylor expansion of $\Delta\hat{\mathbf{\Pi}}$ with respect to $\Delta\mathbf{R}_x$ gives

$$\hat{\mathbf{\Pi}}_N = \mathbf{\Pi} + \boldsymbol{\delta}\mathbf{\Pi} + \boldsymbol{\delta}^2\mathbf{\Pi} + ... \quad (18)$$

where

$$\boldsymbol{\delta}\mathbf{\Pi} = -\mathbf{U}_0 - \mathbf{U}_0^H \quad (19)$$

$$\mathbf{U}_0 = \mathbf{\Pi}\Delta\mathbf{R}_x\mathbf{S}^{\#} \quad (20)$$

$$\mathbf{S} = \mathbf{R}_x - \sigma^2\mathbf{I}_M \quad (21)$$

where $\mathbf{S}^{\#}$ is the Moore-Penrose pseudo-inverse of $\mathbf{S}$ such that

$$\mathbf{S}^{\#} = \sum_{k=1}^{K}\frac{\mathbf{e}_k\mathbf{e}_k^H}{\lambda_k - \sigma^2} = \mathbf{A}^{\#H}\mathbf{P}^{-1}\mathbf{A}^{\#} \quad (22)$$

and where the second order term is

$$\boldsymbol{\delta}^2\mathbf{\Pi} = -\mathbf{U}_0\mathbf{U}_0^H + \mathbf{U}_0^H\mathbf{U}_0 + \mathbf{V}_0 + \mathbf{V}_0^H \quad (23)$$

$$\mathbf{V}_0 = \mathbf{\Pi}\left(\Delta\mathbf{R}_x\mathbf{S}^{\#}\right)^2 \quad (24)$$

IV. MUSIC CRITERION STATISTIC TOWARD W-MUSIC

A. Bias determination of MUSIC Criterion

The MUSIC criterion error value $\Delta\eta_N\left(\Theta\right) = \eta_N\left(\Theta\right) - \eta\left(\Theta\right)$ is depending on the direction parameter Θ. According to (5)(7)(18), its bias is

$$E\left[\Delta\eta_N\left(\Theta\right)\right] = \mathbf{a}^H\left(\Theta\right)\Delta\mathbf{\Pi}_N\ \mathbf{a}\left(\Theta\right) \quad (25)$$

$$\Delta\mathbf{\Pi}_N = E\left[\hat{\mathbf{\Pi}}_N - \mathbf{\Pi}\right] \approx E\left[\boldsymbol{\delta}\mathbf{\Pi}\right] + E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right] \quad (26)$$

The noise projector bias is

$$\Delta\mathbf{\Pi}_N \approx \frac{\sigma^2}{N}\left(-Tr\left(\mathbf{R}_x\mathbf{S}^{2\#}\right)\mathbf{\Pi} + \left(M-K\right)\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}\right) \quad (27)$$

where (27) is proven in subsection VII-A according to the results of subsections III-C and III-D. Let us note that

$$\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#} = \mathbf{S}^{\#} + \sigma^2\mathbf{S}^{2\#} \quad (28)$$

Thus, the noise projector (27) error is proportional to $1/N$ and $\mathbf{S}^{\#}$ such that it is low for large value of N, high sources level and well conditioned matrix $\mathbf{S}$ assumed for non quasi-coherent and well separated sources. More precisely and according to section VII-B, the criterion bias for the source direction Θ_k is

$$E\left[\Delta\eta_N\left(\Theta_k\right)\right] \approx \frac{\sigma^2\left(M-K\right)\left(1+\sigma^2\mathbf{P}_{[k][k]}\mathbf{Q}_{[k][k]}\right)}{N\mathbf{P}_{[k][k]}} \quad (29)$$

where $\mathbf{Q} = \mathbf{P}^{-1}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{P}^{-1}$ and $\mathbf{P}_{[k][k]}/\sigma^2$ is the signal noise ratio of the k^{th} source. The element $\mathbf{Q}_{[k][k]}$ is then large when the matrices $\mathbf{P}$ and $\mathbf{A}^H\mathbf{A}$ are almost full rank. Finally, the bias of the MUSIC criterion value is large in presence of correlated and/or close sources.

B. W-MUSIC Algorithm

The purpose of W-MUSIC algorithm is to modify the criterion value with respect to the direction Θ. For that, the bias criterion value is removed in order to improve the sources resolution power. Thus, the W-MUSIC algorithms estimates the source DoAs $\hat{\Theta}_k$ as the K minima of the following criterion $\eta_W^{True}\left(\Theta\right)$

$$\eta_W^{True}\left(\Theta\right) = \eta_N\left(\Theta\right) - E\left[\Delta\eta_N\left(\Theta\right)\right] \quad (30)$$

where

$$\eta_W^{True}\left(\Theta\right) = \mathbf{a}^H\left(\Theta\right)\ \hat{\mathbf{\Pi}}_N^{W-True}\ \mathbf{a}\left(\Theta\right) \quad (31)$$

$$\hat{\mathbf{\Pi}}_N^{W-True} = \hat{\mathbf{\Pi}}_N - \Delta\mathbf{\Pi}_N \quad (32)$$

where $\hat{\mathbf{\Pi}}_N^{W-True}$ is a pseudo-projector. However the matrix $\Delta\mathbf{\Pi}_N$ is depending on the unknown true covariance matrix according to (27). In practice $\mathbf{R}_x$ is replaced by its corresponding sample estimate where

$$\Delta\hat{\mathbf{\Pi}}_N = \frac{\hat{\sigma}^2}{N}\left(-Tr\left(\hat{\mathbf{R}}_{x,N}\hat{\mathbf{S}}^{2\#}\right)\hat{\mathbf{\Pi}}_N + \left(M-K\right)\hat{\mathbf{S}}^{\#}\hat{\mathbf{R}}_{x,N}\hat{\mathbf{S}}^{\#}\right) \quad (33)$$

$\hat{\mathbf{S}} = \hat{\mathbf{R}}_x - \hat{\sigma}^2\mathbf{I}_M$ and $\hat{\sigma}^2 = \frac{\sum_{i=K+1}^{M}\hat{\lambda}_{i,N}}{N-K}$

Finally and after some calculations, the modified pseudo-projector of W-MUSIC is

$$\hat{\mathbf{\Pi}}_N^W = \hat{\mathbf{\Pi}}_N - \Delta\hat{\mathbf{\Pi}}_N = \alpha_W\ \hat{\mathbf{\Pi}}_N + \beta_W\ \hat{\mathbf{S}}^{\#}\hat{\mathbf{R}}_{x,N}\hat{\mathbf{S}}^{\#} \quad (34)$$

$$\alpha_W = 1 + \frac{\hat{\sigma}^2}{N}Tr\left(\hat{\mathbf{R}}_{x,N}\hat{\mathbf{S}}^{2\#}\right) \quad \beta_W = -\frac{\hat{\sigma}^2}{N}\left(M-K\right) \quad (35)$$

The W-MUSIC DoA estimation criterion is then

$$\begin{aligned}\eta_W\left(\Theta\right) &= \mathbf{a}^H\left(\Theta\right)\ \hat{\mathbf{\Pi}}_N^W\ \mathbf{a}\left(\Theta\right) \quad (36)\\ &= \alpha_W\ \eta_N\left(\Theta\right) + \beta_W\ \eta_C\left(\Theta\right)\\ \eta_C\left(\Theta\right) &= \mathbf{a}^H\left(\Theta\right)\hat{\mathbf{S}}^{\#}\hat{\mathbf{R}}_{\mathbf{x}}\hat{\mathbf{S}}^{\#}\mathbf{a}\left(\Theta\right)\end{aligned}$$

where the K minima $\hat{\Theta}_k$ of $\eta_W\left(\Theta\right)$ are the sources DoAs given by W-MUSIC algorithm. Let us note that $1/\eta_C\left(\Theta\right)$ is close to Capon criterion $1/(\mathbf{a}^H\left(\Theta\right)\hat{\mathbf{R}}_{\mathbf{x}}^{-1}\mathbf{a}\left(\Theta\right))$ that determines the power in each direction Θ. According to (36), the modified SSMUSIC[9] criterion is $\eta_N\left(\Theta\right)\eta_S\left(\Theta\right)$ with $\eta_S\left(\Theta\right) = 1/(\mathbf{a}^H\left(\Theta\right)\hat{\mathbf{S}}^{\#}\mathbf{a}\left(\Theta\right))$ close to Capon criterion. The W-MUSIC pseudo-projector $\hat{\mathbf{\Pi}}_N^W$ can be rewritten similarly to the G-MUSIC one (9) with

$$\hat{\mathbf{\Pi}}_N^W = \sum_{m=1}^{M}\Phi_W\left(m\right)\hat{\mathbf{e}}_{m,N}\hat{\mathbf{e}}_{m,N}^H \quad (37)$$

$$\Phi_W\left(m\right) = \begin{cases}\alpha_W & m > K\\ \frac{\hat{\lambda}_{m,N}\times\beta_W}{\left(\hat{\lambda}_{m,N}-\hat{\sigma}^2\right)^2} & m \leq K\end{cases} \quad (38)$$

where $\Phi_W\left(m\right)$ and $\Phi\left(m\right)$ are close to 1 for $m > K$ and close to 0 for $m \leq K$ and large value of N.

V. Simulations

Two correlated sources of directions $\Theta_1 = 0°$ and $\Theta_2 = 30°$ with same Signal Noise Ratio (SNR) are considered in simulation. More precisely $\mathbf{P}_{[1][1]} = \mathbf{P}_{[2][2]}$, the noise power is $\sigma^2 = 1$ and the correlation rate is $r_{12} = 0.99 = \mathbf{P}_{[1][2]}/\mathbf{P}_{[1][1]}$. The sources arrive on an Uniform Circular Array (UCA) of N sensors and radius 0.5λ. The performances are given according to the empirical Roots Mean Square (RMS) error of the DoA estimation with $nb = 500$ realizations. The empirical RMS error RMS_k of the $k-th$ source verifies $(RMS_k)^2 = \sum_{i=1}^{nb}\left(\hat{\Theta}_k(i) - \Theta_k\right)^2/nb$ where $\hat{\Theta}_k(i)$ is the estimation of the k^{th} source direction at i^{th} realization. The signals $s_k(t_n)$ and noise $\mathbf{n}(t_n)$ have the same statistical distribution. The performance of MUSIC, G-MUSIC and W-MUSIC are compared to the deterministic CRB and the true W-MUSIC of criterion $\eta_W^{True}(\Theta)$(5) where the MUSIC criteria value is perfectly unbiased. The results are presented with respect to the SNR $10log_{10}\left(\mathbf{P}_{[1][1]}\right)$. For $N = M = 10$ the pseudo-projector weighting $\Phi(m)$ and $\Phi_W(m)$ of G-MUSIC and W-MUSIC are represented with respect to the index m in Fig.1 for $10dB$ of SNR and with respect to the SNR in Fig.2. The weighting of the G-MUSIC(10) and W-MUSIC(38) are close and for high SNR reach 1 and 0 for noise and signal subspace respectively. The Fig.3 and Fig.4 for Gaussian and Uniform distribution respectively give the RMS error of the 1^{st} source with respect to SNR for $N = M = 10$. The close performances show that the influence of distribution is minor for performance. In Fig.3, Fig.5 and Fig.6 this RMS error is given with respect to SNR for $N = M = 10, 20$ and 5 for Gaussian distribution. The performance improvement of G-MUSIC and W-MUSIC when N increase is more important than MUSIC. The performances of G-MUSIC is between the true W-MUSIC and W-MUSIC where the MUSIC criterion value is corrected with the true bias and the empirical one respectively. For large value of N the G-MUSIC algorithm reach the true W-MUSIC and for small value reach W-MUSIC.

VI. Conclusion

The W-MUSIC algorithm corrected the MUSIC criterion value by removing its approximative bias. The ideal not operational true W-MUSIC give the best performances. The simulations show that the performances of G-MUSIC and W-MUSIC are close. These results are confirm by the analysis of the pseudo-projector weighting of G-MUSIC and W-MUSIC similar when the algorithms are able to separate the sources directions. Thus, W-MUSIC explain in part the behavior of G-MUSIC with the MUSIC criterion value bias removing. The W-MUSIC approach can be improved by using more inside the Wishart distribution. This is an ongoing work.

VII. Appendix

A. Noise projector Bias

According to (18), the noise projector bias is

$$E\left[\hat{\mathbf{\Pi}}\right] \approx \mathbf{\Pi} + E\left[\boldsymbol{\delta}\mathbf{\Pi}\right] + E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right] \quad (39)$$

According to (13)(19)(20), we obtain

$$E\left[\boldsymbol{\delta}\mathbf{\Pi}\right] = \mathbf{0} \quad (40)$$

According to (20)(23)(24), $E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right]$ is

$$E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right] = -\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_3^H \quad (41)$$

$$\begin{cases} \mathbf{E}_1 = E\left[\mathbf{U}_0\mathbf{U}_0^H\right] & \mathbf{E}_2 = E\left[\mathbf{U}_0^H\mathbf{U}_0\right] \\ \mathbf{E}_3 = E\left[\mathbf{V}_0\right] \end{cases} \quad (42)$$

The matrix $\mathbf{E}_1$ is according to (20) as following

$$\mathbf{E}_1 = \mathbf{\Pi}E\left[\Delta\mathbf{R}_x\mathbf{S}^{\#}\mathbf{S}^{\#H}\Delta\mathbf{R}_x\right]\mathbf{\Pi}$$

because the Hermitian matrix $\Delta\mathbf{R}_x$ verifies $\Delta\mathbf{R}_x^H$=$\Delta\mathbf{R}_x$. According to (16), the matrix $\mathbf{E}_1$ is

$$\mathbf{E}_1 = \frac{Tr\left(\mathbf{R}_x\mathbf{S}^{\#}\mathbf{S}^{\#H}\right)\mathbf{\Pi}\mathbf{R}_x\mathbf{\Pi}}{N} = \sigma^2\frac{Tr\left(\mathbf{R}_x\mathbf{S}^{2\#}\right)\mathbf{\Pi}}{N}$$

The matrix $\mathbf{E}_2$ verifies $\mathbf{E}_2 = \mathbf{S}^{\#}E\left[\Delta\mathbf{R}_x\mathbf{\Pi}\Delta\mathbf{R}_x\right]\mathbf{S}^{\#}$ according to (20) and according to (16) becomes

$$\mathbf{E}_2 = \frac{Tr(\mathbf{R}_x\mathbf{\Pi})\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}}{N} = \frac{\sigma^2(M-K)\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}}{N}$$

According to (16)(24), the matrix $\mathbf{E}_3$ is

$$\mathbf{E}_3 = \mathbf{\Pi}\left[\Delta\mathbf{R}_x\mathbf{S}^{\#}\Delta\mathbf{R}_x\right]\mathbf{S}^{\#} = \frac{Tr\left(\mathbf{R}_x\mathbf{S}^{\#}\right)\mathbf{\Pi}\mathbf{R}_x\mathbf{S}^{\#}}{K} = 0$$

because $\mathbf{\Pi}\mathbf{S}^{\#}$=0 and $\mathbf{\Pi}\mathbf{R}_x$=$\sigma^2\mathbf{I}_M$. Thus according to (41), the matrix $E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right]$ verifies

$$E\left[\boldsymbol{\delta}^2\mathbf{\Pi}\right] = \frac{\sigma^2}{N}\left(-Tr\left(\mathbf{R}_x\mathbf{S}^{2\#}\right)\mathbf{\Pi} + (M-K)\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}\right) \quad (43)$$

According to (39)(40)(41) and (43), the expression (27) of the noise projector bias is proven

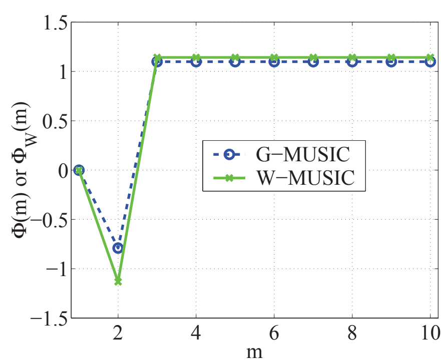

Fig. 1. Weighting $\Phi(m)$ and $\Phi_W(m)$ of the pseudo projector of G-MUSIC and W-MUSIC respectively for $N = M = 10$ and $SNR = 14dB$.

B. MUSIC Criterion Value in Θ_k

According to (25)(27), the MUSIC Criterion bias for the source direction Θ_k is

$$E\left[\Delta\eta_N(\Theta_k)\right] = \sigma^2\left(\frac{M-K}{N}\right)\mathbf{a}^H(\Theta_k)\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}\mathbf{a}(\Theta_k) \quad (44)$$

because $\mathbf{a}^H(\Theta_k)\mathbf{\Pi}\mathbf{a}(\Theta_k)$=0 for the direction Θ_k. According to (28)

$$\mathbf{a}^H(\Theta_k)\mathbf{S}^{\#}\mathbf{R}_x\mathbf{S}^{\#}\mathbf{a}(\Theta_k) = \mathbf{a}^H(\Theta_k)\mathbf{S}^{\#}\mathbf{a}(\Theta_k) + \sigma^2\mathbf{a}^H(\Theta_k)\mathbf{S}^{2\#}\mathbf{a}(\Theta_k) \quad (45)$$

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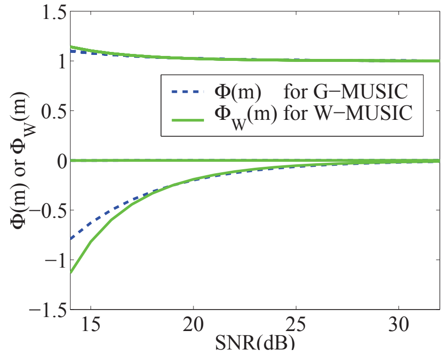

Fig. 2. Weighting $\Phi\,(m)$ and $\Phi_W\,(m)$ of the pseudo projector of G-MUSIC and W-MUSIC respectively for $N = M = 10$.

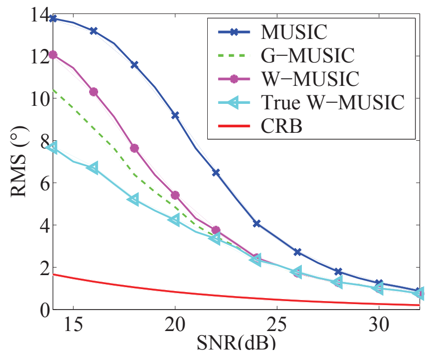

Fig. 3. RMS error of the 1^{st} source for $N = M = 10$ and Gaussian distribution

where $\mathbf{S}^{\#}\mathbf{S}^{\#}$=$\mathbf{A}^{\#H}\mathbf{P}^{-1}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{P}^{-1}\mathbf{A}^{\#}$ and $\mathbf{S}^{\#}$=$\mathbf{A}^{\#H}\mathbf{P}^{-1}\mathbf{A}^{\#}$. Let us note that $\mathbf{A}^{\#}\mathbf{a}\left(\Theta_k\right)$=$\mathbf{1}_k$ where $\mathbf{1}_k$ is a $K \times 1$ vector whose the k^{th} component value is 1 and the others are null. Thus, $\mathbf{a}^H\left(\Theta_k\right)\mathbf{S}^{\#}\mathbf{a}\left(\Theta_k\right)$=$1/\mathbf{P}_{[k][k]}$ and $\mathbf{a}^H\left(\Theta_k\right)\mathbf{S}^{2\#}\mathbf{a}\left(\Theta_k\right)$=$\mathbf{Q}_{[k][k]}$ with $\mathbf{Q}$=$\mathbf{P}^{-1}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{P}^{-1}$. The equation (29) is then proven according to (44)(45).

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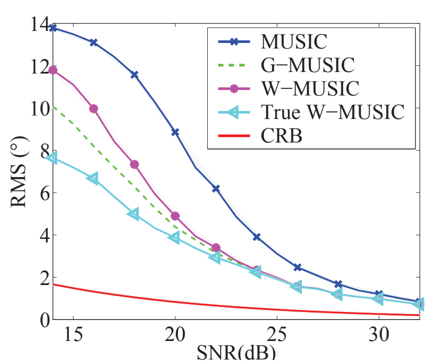

Fig. 4. RMS error of the 1^{st} source for $N = M = 10$ and Uniform distribution

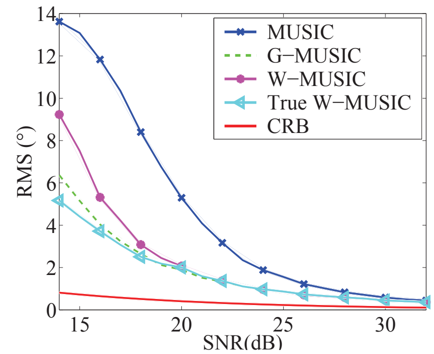

Fig. 5. RMS error of the 1^{st} source for $N = M = 20$ and Gaussian distribution

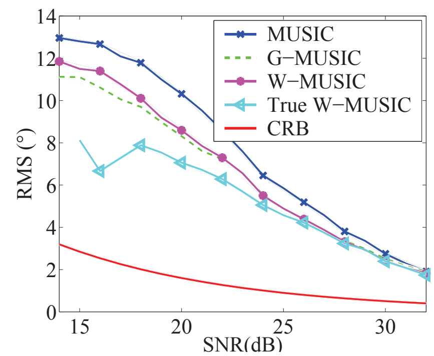

Fig. 6. RMS error of the 1^{st} source for $N = M = 5$ and Gaussian distribution

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