Evanescent dark energy and dark matter

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Cosmological dark energy and dark matter are identified with evanescent waves at super-horizon scales. They enter the visible universe across the cosmological horizon $\mathcal{H}$ below a fundamental frequency $\omega_0 = \sqrt{1-q}H$ at a Hubble parameter H and deceleration parameter q. The cosmological vacuum hereby acquires a stress-energy tensor with nonzero trace, $T_{ab} = \left[(1-q)\pi_{ab}^- + q\pi_{ab}^+\right]\rho_c$ of radiation (π_{ab}^+) and evanescent waves (π_{ab}^-), where ρ_c denotes the closure energy density, $\mathrm{tr}\,\pi_{ab}^+ = 0$ and $\mathrm{tr}\,\pi_{ab}^- = -2$. It defines fast expansion with $H'(0) = H_0(3\omega_m - 1) \simeq 0$ with $Q(0) \simeq 2.75 \pm 0.15$, $Q(z) = dq(z)/dz$. Nonlinear model regression applied to $H(z)$ over an intermediate range of redshift z identifies a tension-free estimate $H_0 = 74.9 \pm 2.6\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ consistent with Riess et al. (2016) in surveys of the Local Universe and a recent estimate produced by GW170817.

Introduction. Modern cosmology shows a cosmological distribution of dark energy and dark matter that highlights potentially new physics at the lowest energy scales in the Universe [1]. These anomalies appear about the de Sitter scale of acceleration $a_{dS} = cH$, where c is the velocity of light and H is the Hubble parameter. By Einstein's equivalence principle [2], this represents weak gravitation far below the scales conventionally encountered as Newtonian gravity in the Solar system [3]. While Einstein's theory of gravitation gives a highly successful extension to strong gravity, the limits of which include black holes and gravitational waves from mergers [4], taking it to weak gravity in the face of a_{dS} and the cosmological horizon $\mathcal{H}$ at the Hubble radius $R_H = c/H$ is highly uncertain if the vacuum is entangled [5,6] with the cosmological background of our Friedmann-Robertson-Walker (FRW) Universe [7].

On the largest scales, our three-flat Universe features a cosmological horizon $\mathcal{H}$ at the Hubble radius $R_H = c/H$ (Fig. 1), where c is the velocity of light and $H = \dot{a}/a$ is the Hubble parameter in the FRW line-element in $(t, r, \vartheta, \varphi)$,

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2) \quad (1)$$

with deceleration parameter $q = -\ddot{a}a/\dot{a}^2$, $q(z) = -1 + (1+z)H^{-1}(z)H'(z)$ for $H(z)$ as a function of redshift z. Evolving (1) by general relativity on a classical vacuum, we infer a dark energy density $\Lambda/8\pi > 0$ from the observed sign [8]

$$q_{GR} = \Omega_m - \frac{1}{2}\Omega_\Lambda < 0. \quad (2)$$

In the standard frame work of a constant Λ and cold dark matter (ΛCMD), we asymptotically evolve towards a de Sitter state ($q = -1$) in the distant future ($z = -1$), as H descends to a finite value $\sqrt{\Lambda/3}$. These relations highlight tight relation between dark energy and $\mathcal{H}$ at the Hubble radius $R_H = c/H$, where c is the velocity of light.

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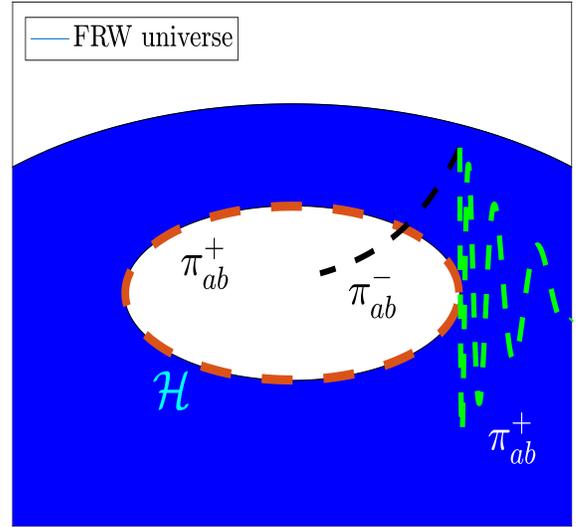
FIG. 1. FRW cosmologies feature a cosmological horizon $\mathcal{H}$, defined in the geometric optics limit as a causal separation surface bounding our visible universe at given cosmic time t. Since $\mathcal{H}$ is compact, it is dispersive, due to a finite eigenfrequency ω_0 defined by the equations of geodesic separation. Below ω_0, super-horizon scale waves are evanescent, and tunnel through $\mathcal{H}$ into our visible universe. Evanescent waves have stress-energy (π_{ab}^-) with nonzero trace, different from radiation (π_{ab}^+) on sub-horizon scales, whose trace is zero. By evanescence, our vacuum is entangled with the cosmological background.

H_0 tension. However, $H_0 = H(0)$ measured from surveys of the Local Universe [9] increasingly indicate a value higher than H_0 obtained from ΛCDM analysis of the Cosmic Microwave Background (CMB) [6]. While ΛCMB provides a powerful framework to CMB power spectra, its applicability to low-z evolution of $H(z)$ is challenged by potential dynamical evolution of Λ and associated evolution in the cosmological horizon.

As an apparent horizon surface, $\mathcal{H}$ is defined in the geometric optics limit as a causal separation surface, originally introduced by Penrose as an outermost trapped surface in black hole formation by gravitational collapse

[10]. In the face of particles described as waves in quantum field theory, Hawking radiation demonstrates that $\mathcal{H}$ is dispersive, allowing radiation to leak out by pair creation, effectively photons at super-horizon scales propagating back out to the celestial sphere [11]. However, this picture does not address the converse, how wave mechanics - in quantum mechanics or field theory - gives rise to spacetime curvature as described by general relativity (e.g. [12]). In particular, the vacuum of general relativity is classical, and does not account for the Hawking temperature of black holes or the Gibbons-Hawking temperature of $\mathcal{H}$ [13]. Perhaps the most striking discrepancy is that universal coupling to energy in classical general relativity runs counter to the UV-divergent bare cosmological constant Λ_0 [14].

And yet, spacetime - of black holes and matter alike - is sure to satisfy the third law of thermodynamics. It leads to the Bekenstein-Hawking entropy $S_H = (1/4)k_B N$ of black hole event horizons that, by unitarity, must lead to information retrieval over the time scale of evaporation in detailed Hawking radiation in photon emission one-by-one back to the celestial sphere [5,15]. Here, k_B is Boltzmann's constant and $N = A_H/l_p^2$ denotes black hole surface area measured in Planck sized units l_p^2, $l_p = \sqrt{G\hbar/c^3}$, where G is Newton's constant. The fact that S scales with horizon area rather than black hole volume is significant. As information, though inaccessible during the lifetime of the black hole, it suggests a holographic principle, wherein the dimension of phase space is set by area of a boundary and entanglement entropy encodes a distribution of mass-energy within [5,6]. The vacuum within wave fronts, generally expanding at nonextremal entanglement, hereby acquires a stress-energy tensor with nonzero trace. *This outcome should be the same for the cosmological vacuum associated with $\mathcal{H}$.* In its present form, however, holography leaves open the detailed mechanism of inducing off-shell vacua from entanglement information.

Cosmological vacuum. In this Report, we derive the stress-energy tensor of the cosmological vacuum from evanescent waves across the cosmological horizon as an apparent horizon surface, resulting from tension between dispersive wave mechanics and $\mathcal{H}$ as a classical, compact surface in the geometric optics approximation. Explicit solutions for $H(z)$ in three-flat Friedmann-Robertson-Walker (FRW) cosmologies enable a direct confrontation with current observational data.

The dispersive nature of $\mathcal{H}$ arises from entanglement between the visible universe within and the universe unseen outside. This is described by transition amplitudes defined by the propagator of a particle

$$\langle B|A \rangle = e^{iS}, \tag{3}$$

between spacetime positions A and B as a function of the action $S = [\varphi]_A^B \xi(A,B)$, where $[\varphi]_A^B = \varphi(B) - \varphi(A)$ is the jump in Compton phase between A and B and $\xi(A,B) = \{1, i\}$ represents the local causal structure

defined by light cones in the geometric optics approximation. For time-like separations, $\langle B, A \rangle$ is the Feynman phase factor e^{iS}, whereas for spacelike separations, e^{-S} gives an exponentially small tunneling amplitude *of evanescent waves entering the light cone* [16].

By unitarity of the propagator, the probabilities $P_\pm$ of finding a particle outside or inside satisfy $P_- + P_+ \equiv 1$. As an identity, the exponentially small tunneling probability has an associated information I_1 on the light cone that encodes the presence of a particle inside [17]. For a particle of mass m at the centre of a spherical wave front of radius r, $I_1 = 2\pi\varphi_C$, where $\varphi_C = kr$, where $k = 2\pi/\lambda_C$ is the wave number of the Compton wave length $\lambda_C = m\hbar/c$, where $\hbar$ is Planck's constant.

Entanglement entropy I_1 is a two-point correlation, visible as the Einstein area A_E in strong gravitational lensing in the appearance of double images of point sources. For a lens of mass m half-way between the source and the observer, Einstein ring in the source plane observed at a distance $2r$ has an area $A_E = \pi (2r\theta_E)^2 = 4I_1 l_p^2$. Generally, two images are observed at angles $\theta_{1,2}$, $\theta_1\theta_2 = \theta_E^2$, where θ_1 is inside and θ_2 is outside the Einstein ring seen at the Einstein angle $\theta_E = \sqrt{2R_s/r}$, $R_S = 2Gm/c^2$ [18]. The sphere of radius r about m hereby leaves a remaining phase space (measured by area) $A' = A - A_E = A\left(1 - \theta_E^2\right)$. Taken to a minimum at $r = R_S$, the result is a black hole event horizon for which A_E saturates $A_H = 4\pi R_S^2$. In the limit $A_E = A_H$, the wave front is trapped with vanishing expansion by extremal entanglement entropy. For $r > R_S$, A_E from I_1 leads to wave front regression

$$r\frac{dI_1}{dt} = I_1 - k_B^{-1}S_H \geq 0 \tag{4}$$

by wave regression at group velocity $dr/dt = 1 - A_E/A = 1 - k_B I_1/S_H$ (a perturbation of Huygen's principle by θ_E^2) of null-geodesics in the Schwarzschild line-element in $(t, r, \vartheta, \varphi)$,

$$ds^2 = -(1-\theta_E^2)dt^2 + \frac{dr^2}{1-\theta_E^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \tag{5}$$

wherein m within appears in Gauss' law

$$aA = 4\pi Gm, \tag{6}$$

from the associated surface gravity a.

Vacuum stress-energy tensor. Here, we show that the cosmological horizon induces dark energy and mass from dispersion by tunneling, described by a stress-energy tensor with nonzero trace derived from a fundamental frequency $\omega_0 = \sqrt{1-q}H$ defined by the equations of geodesic separation and surface gravity implied by the Gauss-Bonnet theorem [19]. Applied to a null-vector $k^b = (k^t, 0, k^\theta, 0)$ using local eigentime τ as an affine parameter, an azimuthal separation $u = r\delta\varphi$ satisfies

$$\ddot{u} + \omega^2 u = 0 \tag{7}$$

with $\omega^2 = R_{\vartheta\varphi\vartheta}{}^\varphi r^{-2} = 2m/r^3$, where the right hand side of the second is specific to (5). In spherical symmetry,

ω is the fundamental eigenfrequency of the wave front as a compact surface. According to (7), ω^2 with twice the Gauss curvature of the wave front at a fixed time τ, giving a dynamical surface gravity [20], here of the form

$$a = \frac{1}{2}\omega^2 r, \qquad (8)$$

where A, V refer to the area and volume of the wave front. As a measures Gauss-curvature, (8) represents the Gauss-Bonnet theorem, inferring it from from internal geometry, here measured by geodesic separation.

By the above, super-horizon scale evanescent waves give rise to a stress-energy of the cosmological vacuum from time-like, off-shell behaviour $\mathcal{H}$ as a compact apparent horizon surface. It represents a mass-energy ccording to the quantum mechanical definition as a time rate-of-change of total phase, $d\Phi/dt = mc^2/\hbar$. By (8) and Gauss' law (6), it has a dark energy density

$$e = (1-q)\rho_c, \qquad (9)$$

satisfying $\rho_T V_H = a_H A_H/4\pi$, where $V_H = (4\pi/3)R_H^3$ is the Hubble volume associated with the Hubble area $A_H = 4\pi R_H^2$ and $\rho_c = 3c^2 H^2/8\pi G$ is the closure energy density with corresponding mass density $\rho_c/c^2 \simeq 10^{-29}\mathrm{g\,cm}^{-3}$ at present.

On the scale of Planck sized surface elements, ω_0 is a fundamental mode by which super-horizon scale electromagnetic and gravitational radiation become dispersive, by an effective mass ω_0^2, satisfying the dispersion relation

$$\omega = \sqrt{c^2 k^2 + \omega_0^2} \qquad (10)$$

for frequencies ω at wave numbers k. (As a wave front, $\mathcal{H}$ itself is hereby chromatic.) The cosmological vacuum assumes a dynamical dark energy

$$\Lambda = \omega_0^2, \qquad (11)$$

given the common coupling of the electromagnetic vector field A_a and the Riemann-Cartan connections $\omega_{a\mu\nu}$ to the Ricci tensor R_{ab} [21]. A three-flat cosmology hereby acquires dark energy and matter densities $\Omega_\Lambda = (1-q)/3$, $\Omega_m = (2+q)/3$, the latter accompanied by a pressure $q = 3\Omega_p$ in satisfying the Friedmann energy and momentum equations. Crucially, this gives

$$q = 2q_{GR}, \qquad (12)$$

showing a cosmological vacuum with nonzero trace stress-energy tensor

$$T_{ab} = (\rho_m + p)u_a u_b + p g_{ab} + \Lambda g_{ab}, \qquad (13)$$

where $\rho_m = (1/3)(2+q)\rho_c$ with $u^b = (\partial_t)^b$.

Originating in evanescent wave propagation through $\mathcal{H}$, we identify T_{ab} with on- and off-shell radiation tensors $\pi_{ab}^\pm$ of sub- and super-horizon scale waves, satisfying $\mathrm{tr}\,\pi_{ab}^+ = 0$, $\mathrm{tr}\,\pi_{ab}^- = -2$ for the equations of state $p = \pm(\gamma - 1)e$, $\gamma = 4/3$. Positive and negative pressures explicitly derive from (10) in the form $ck = \pm\sqrt{\omega^2 - \omega_0^2}$

Table. 1. Canonical cosmological states.

q	TYPE	$\rho_c^{-1}T_{ab}$
1	Radiation dominated	π_{ab}^+
$\frac{1}{2}$	Matter dominated	$u_a u_b$
0	Zero Hubble flow	π_{ab}^-
-1	de Sitter	$-g_{ab}$

from $\omega > \omega_0$ and, respectively, $\omega < \omega_0$, whereby the trace of the stress-energy tensor of radiation and evanescent waves is zero, respectively, nonzero. Thus, we write

$$T_{ab} = \rho_c \left[(1-q)\pi_{ab}^- + q\,\pi_{ab}^+\right]. \qquad (14)$$

It supports the four canonical states listed in Table 1, where $u^b = (\partial_t)^b$.

Hubble parameter H_0. An explicit solution

$$H(z) = H_0 \frac{\sqrt{1 + \omega_m(6z + 12z^2 + 12z^3 + 6z^4 + (6/5)z^5)}}{1+z} \qquad (15)$$

from the FRW equations of motion with (14) shows a relatively fast evolution in the Hubble parameter $H(z)$ around redshift zero: $H'(0) = H_0(3\omega_m - 1) \simeq 0$, distinct from $H'(0) = \frac{3}{2}\omega_m \simeq 0.5$ in ΛCDM [19]. Nonlinear regression over $H_0 = H(0)$ and present-day matter content $\omega_m = \Omega_m(0)$ against data $H(z)$ over an intermediate range of redshifts z (see [22]) obtains

$$H_0 = 74.9 \pm 2.6\,\mathrm{km\,s}^{-1}\mathrm{Mpc}^{-1} \qquad (16)$$

with $\omega_m = 0.2719 \pm 0.028$, in tight agreement with $H_0 = 74.4 \pm 4.9\,\mathrm{km\,s}^{-1}\,\mathrm{Mpc}^{-1}$ obtained by a model-independent cubic polynomial fit, $H(z) = H_0(1 + (1 + q_0)z + \frac{1}{2}(Q_0 + q_0(1+q_0))z^2 + b_3 z^3 + \cdots)$ to the same data.[19] This result agrees with H_0 from surveys of the local Universe [18]. In contrast, analysis of the same data for $H(z)$ in ΛCDM gives $H_0 = 66.8 \pm 1.9\,\mathrm{km\,s}^{-1}\,\mathrm{Mpc}^{-1}$, $\omega_m = 0.3330 \pm 0.040$. These results can be conveniently presented in a qQ-diagram (Fig. 2), where $q(z) = -1 + (1+z)H^{-1}(z)H'(z)$, $Q(z) = dq(z)/dz$ [23]. Models (14) and ΛCDM are distinct with gap between $Q_0 \gtrsim 2.5$ for the first and $Q_0 \lesssim 1$ for the second. Results for (14) and aforementioned cubic fit give consistent results $Q_0 = 2.37 \pm 0.073$ and, respectively, $Q_0 = 2.49 \pm 0.55$. Table 2 lists model fits to analytic solutions $H(z)$ parameterized by (H_0, ω_m) for (14), ΛCDM with $H(z) = H_0\sqrt{1 - \omega_m + \omega_m(1+z)^3}$ and a recent proposal for scale-free cosmology with $H(z) = H_0\left(\omega_m(1+z)^{9/4} + (1-\omega_m)(1+z)^{3/4}\right)^{2/3}$ [24], along with model-independent analysis based on cubic and quartic polynomials.

At super-horizon scales, radiation is found to leak into the visible universe as evanescent waves, giving a nonzero trace to the cosmological vacuum observed as a combination of a cosmological distribution of dark energy and dark matter. This theory recovers the four canonical states of cosmological evolution (Table 1) and produces a Hubble parameter H_0 from cosmological data $H(z)$ in agreement with H_0 measured in local surveys (Table 2).

Table 2. Results for $(H_0, q_0, Q_0, \omega_m)$ with 1σ uncertainties by nonlinear model regression applied to polynomials and models and current data on H_0. H_0 is expressed in units of $\mathrm{km\,s^{-1}Mpc^{-1}}$.

MODEL	H_0	q_0	Q_0	ω_m	$h'(0)$
Cubic	74.4 ± 4.9	-1.17 ± 0.34	2.49 ± 0.55	-	-0.17
Quartic	74.5 ± 7.3	-1.18 ± 0.67	2.54 ± 1.99		-0.18
$\Lambda = \omega_0^2$	74.9 ± 2.6	-1.18 ± 0.084	2.75 ± 0.15	0.2719 ± 0.028	-0.18
ΛCDM	66.8 ± 1.9	-0.50 ± 0.060	1.00 ± 0.060	0.3330 ± 0.040	0.50
Scale-invariant	62.0 ± 1.6	-0.16 ± 0.056	0.33 ± 0.02	0.3404 ± 0.056	0.84

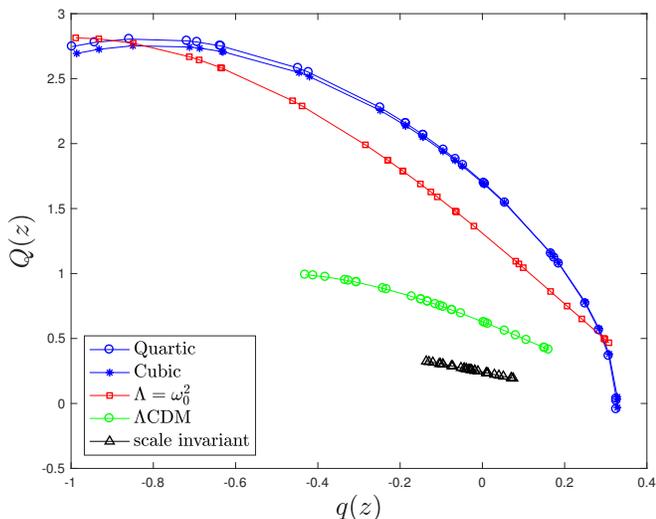| OBSERVATIONS		REFERENCE
Local surveys	73.06 ± 1.76	Anderson & Riess (2017)
GW180817	$75.5 \pm {}^{11.6}_{9.6}$	Guidorzi et al. (2017)
CMB	66.93 ± 0.62	Planck Collaboration et al. (2016)

FIG. 2. qQ-diagram showing results of nonlinear model regression of model-independent polynomials and analytic solutions $H(z)$ for $\Lambda = \omega_0^2$, ΛCDM and scale-invariant cosmologies. The graphs shown are restricted to redshifts $z_{min} \leq z_k \lesssim 1$ within the radius of convergence of the Taylor series expansions about $z = 0$ of these three models, where z_k are the ordinates with $z_{min} = 0.07$ in recently tabulated data [22].

In the qQ-diagram, the results are consistent with those obtained from model-independent analysis using polynomial fits, while ΛCDM is ruled out by $2.7\,\sigma$. The predicted relatively large value of Q_0 is characteristic for $H'(0) \simeq 0$ indicates an unstable de Sitter limit ($q = -1$), driven by entanglement of our visible universe with the global FRW spacetime.

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