On the physical inconsistency of a new statistical scaling symmetry in incompressible Navier-Stokes turbulence

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Abstract

A detailed theoretical investigation is given which demonstrates that a recently proposed statistical scaling symmetry is physically void. Although this scaling is mathematically admitted as a unique symmetry transformation by the underlying statistical equations for incompressible Navier-Stokes turbulence on the level of the functional Hopf equation, by closer inspection, however, it leads to physical inconsistencies and erroneous conclusions in the theory of turbulence.[†]

The new statistical symmetry is thus misleading in so far as it forms within an unmodelled theory an analytical result which at the same time lacks physical consistency. Our investigation will expose this inconsistency on different levels of statistical description, where on each level we will gain new insights for its non-physical transformation behavior. With a view to generate invariant turbulent scaling laws, the consequences will be finally discussed when trying to analytically exploit such a symmetry. In fact, a mismatch between theory and numerical experiment is conclusively quantified.

We ultimately propose a general strategy on how to not only track unphysical statistical symmetries, but also on how to avoid generating such misleading invariance results from the outset. All the more so as this specific study on a physically inconsistent scaling symmetry only serves as a representative example within the broader context of statistical invariance analysis. In this sense our investigation is applicable to all areas of statistical physics in which symmetries get determined in order to either characterize complex dynamical systems, or in order to extract physically useful and meaningful information from the underlying dynamical process itself.

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Contents

1. Introduction

With the aid of today's modern computer algebra systems, the method of symmetry analysis is one of the most prominent and efficient tools to investigate differential equations arising in various sciences (Ovsiannikov, 1982; Stephani, 1989; Olver, 1993; Ibragimov, 1994; Andreev *et al.*, 1998; Bluman & Kumei, 1996; Meleshko, 2005). A considerable number of special techniques for simplifying, reducing, mapping and solving differential equations have been developed and enhanced so far.

The natural language for symmetry transformations is that of a mathematical group, which either can be discrete or continuous. If an invariant transformation group involves one or more parameters which can vary continuously it is called a Lie symmetry group, named after Sophus Lie who first developed the theory of continuous transformation groups at the end of the nineteenth century (Lie, 1893).

In fact, most differential equations of the sciences possess nontrivial Lie symmetry groups. Under favorable conditions these symmetries can be exploited for various purposes, e.g. performing integrability tests and complete integration of ODEs, finding invariant and asymptotic solutions for ODEs and PDEs, constructing conservation laws and dynamical invariants, etc. Not to forget that Lie-groups are also successfully utilized in the 'opposite' direction in modelling dynamical behavior itself, i.e. used for constructing dynamical equations which should admit a certain given set of symmetries. The most impressive results to date were gained by gauge theory for quantum fields (Weinberg, 2000; Penrose, 2005). Hence the existence of symmetries thus has a profound and far-reaching impact on solution properties and modelling of differential equations in general. Their presence very often simplifies our understanding of physical phenomena.

Of particular interest are *scaling* symmetries as they lead to concepts as scale invariance of dynamical laws or self-similarity of solution manifolds. A scaling symmetry of a physical system can either be associated with a finite dimensional Lie-group (global scaling symmetry) in which all group parameters are strict constants, or with an infinite dimensional Lie-group (local scaling symmetry) in which at least one group parameter is not constant, e.g. by showing a space-time coordinate dependence of the considered system.

Most physical processes, however, only admit global scaling symmetries since the requirement for a local scaling symmetry is too restrictive. In fact, a physical process which admits a local scaling symmetry also admits this symmetry globally. For example, for a local scaling symmetry exhibiting space-time dependent group parameters (which essentially forms the cornerstone of every quantum gauge theory) the corresponding global symmetry is then just given by the same symmetry where only the group parameters are identically fixed at every point in space-time. The opposite, in which a global symmetry automatically implies a local symmetry, is, of course, not the rule.

The purpose of this article is to show that in general caution has to be exercised when interpreting and exploiting symmetries if they act in a purely *statistical* manner. Although being mathematically admitted as statistical symmetries by the underlying statistical system of dynamical equations, they nevertheless can lead to physical inconsistencies. Without loss of generality, we will demonstrate this issue at the example of a new and recently proposed global statistical scaling symmetry for the incompressible Navier-Stokes equations. Our study and its conclusion can then be easily transferred to any other statistical symmetry within the Navier-Stokes theory, or, more generally, to any other theory within physics which necessitates a statistical description in the thermodynamical sense.

The current study is organized as follows: Section 2 opens the investigation by introducing the single and only continuous (Lie-point) scaling *symmetry* which the *deterministic* incompressible Navier-Stokes equations can admit. Although being the only true scaling *symmetry*, it is yet not the only scaling transformation which leaves these equations invariant when viewed in a broader context. Regarding the class of all possible invariant Lie-point scaling transformations, a brief outline is given to distinguish between the con-

cept of a *symmetry* transformation and that of an *equivalence* transformation. A careful distinction between these two concepts is surely necessary in to order fully grasp the spirit of this article.

Section 3 then changes from the deterministic to the statistical description. By choosing the functional Hopf formulation we are dealing with a formally closed and thus complete statistical approach to turbulence. Instead of the weaker invariant class of equivalence transformations, this enables us to generate true statistical *symmetry* transformations, in particular a new scaling symmetry is considered which first got mentioned in the study of Wacławczyk & Oberlack (2013*a*).

Section 4 is at the heart of the article's line of reasoning. It not only demonstrates that the new Hopf scaling symmetry induces a disguised symmetry, which, on a lower level of statistical description, only acts as an equivalence transformation, but also gives a mathematical proof that both the Hopf symmetry and its induced equivalence transformation are essentially unphysical.

Section 5 presents the consequences when generating statistical scaling laws from such a misleading symmetry transformation. These laws will be matched to DNS data at the example of a zero-pressure-gradient (ZPG) turbulent boundary layer flow for the high Reynolds number case of $Re_\theta = 6000$ (based on the momentum thickness θ of the flow). Best curve fits are generated with the aid of using basic tools from statistical data analysis, as the chi-square method to quantitatively measure the quality of the fits relative to the underlying DNS error. As a result, a mismatch between theory and numerical experiment is clearly quantified.

Section 6 concludes and completes the investigation. Theoretically as well as graphically we will conclude that all recently proposed statistical scaling laws which are based on this new unphysical symmetry have no predictive value and, in our opinion, should be discarded to avoid any further misconceptions in future work when generating turbulent scaling laws according to the invariance method of Lie-groups. In a brief historical outline we finally point out that even if this method of Lie-groups in its full extent is applied and interpreted correctly, it nevertheless faces strong natural limits which prevents the effect of achieving a significant breakthrough in the theory of turbulence.

A large but indispensable part of this investigation has been devoted to the appendix. All appendices stand for their own and can be read independently from the main text. In particular Appendix A & C are written in the form of a compendium to serve as an aid and to accompany the reader through the main text. Their purpose is to mathematically support the criticism we put forward in our first part, the theoretical part of our study from Section 2 to Section 4.

2. The deterministic incompressible Navier-Stokes equations

For reasons of simplicity we will in the following only consider the general solution manifold of the incompressible Navier-Stokes equations in the infinite domain without specifying any initial or boundary conditions (Batchelor, 1967; Pope, 2000; Davidson, 2004).

The corresponding deterministic equations can either be written in local differential form as

$$\left.\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u}, \end{aligned}\right\} \tag{2.1}$$

or equivalently, when using the continuity equation to eliminate the pressure from the momentum equations, in nonlocal integro-differential form as

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \int \frac{\nabla' \cdot \left[(\mathbf{u}' \cdot \nabla')\mathbf{u}'\right]}{4\pi \cdot |\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' + \nu \Delta \mathbf{u}. \tag{2.2}$$

By construction, equation (2.2) has the property that if the initial velocity field $\mathbf{u}$ is solenoidal, i.e. if $\nabla \cdot \mathbf{u}$ is initially zero, then it will be solenoidal for all times.

The single and only continuous (Lie-point) scaling symmetry which the deterministic incompressible Navier-Stokes equations (2.1), or in the form (2.2), can admit is given by (Olver, 1993; Fushchich *et al.*, 1993; Frisch, 1995; Andreev *et al.*, 1998)

$$\mathsf{S}: \quad \tilde{t} = e^{2\varepsilon} t, \quad \tilde{\mathbf{x}} = e^{\varepsilon} \mathbf{x}, \quad \tilde{\mathbf{u}} = e^{-\varepsilon} \mathbf{u}, \quad \tilde{p} = e^{-2\varepsilon} p, \tag{2.3}$$

being just a global scaling symmetry with constant group parameter ε. That (2.3) really acts as a symmetry transformation can be easily verified due its globally uniform structure: By inserting transformation (2.3) into system (2.1), or into (2.2), will leave the equations in each case fully indifferent.

Before we turn in the next section to a complete (fully determined) statistical description of the Navier-Stokes equations, it is essential at this stage to make a careful distinction between two different kinds of invariant transformations. Those being true *symmetry* transformations and those being only *equivalence* transformations (Ovsiannikov, 1982; Ibragimov, 1994, 2004).

Although both types of invariant transformations form a Lie-group, they each have a completely different impact when trying to extract valuable information from a given dynamical system. The knowledge of symmetry transformations is mainly used to construct special or general solutions of differential equations, while equivalence transformations are used to solve the equivalence problem for a certain class of differential equations by group theory, that is, to find general criteria whether two or more different differential equations are connected by a change of variables drawn from a transformation group. Hence, the quest for a symmetry transformation is thus fundamentally different to that for an equivalence transformation. The difference between these two kinds of transformations is defined as:

- A *symmetry* of a differential equation is a transformation which maps every solution of the differential equation to another solution of the *same equation*. As a consequence a symmetry transformation leads to complete form-*indifference* of the equation. It results as an invariant transformation if the considered equation is *closed*.[†]

- An *equivalence transformation* for a differential equation in a given class is a change of variables which only maps the equation to another equation in the *same class*. As a consequence an equivalence transformation *only* leads to a weaker form-*invariance* of the equation. It results as an invariant transformation either if existing parameters of the considered equation get identified as own independent variables, or if the considered equation itself is *unclosed*.[‡]

Hence, although both transformations are *invariant* transformations and both form a *Lie-group*, they yet lead to different implications. Let us illustrate this decisive difference at two simple examples:

Example 1: By considering the viscosity ν in (2.1) not as a parameter, but rather, next to the space-time coordinates, as an own independent variable, a detailed invariance analysis will give the following additional scaling group, which in infinitesimal form reads as (Ünal, 1994, 1995)

$$\mathsf{X}_{\mathsf{E}_1(\mathsf{f})}: \quad f(\nu) \cdot \Big(t\partial_t + x^i \partial_{x^i} + \nu \partial_\nu \Big), \tag{2.4}$$

being an infinite dimensional Lie-group with a group parameter f depending on the viscosity variable ν. Specifying for example $f(\nu) = 1$ will reduce to a finite dimensional

[†]A set of equations is defined as *closed* if the number of equations involved is either equal to or more than the number of dependent variables to be solved for.

[‡]A set of equations is defined as *unclosed* if the number of equations involved is less than the number of unknown dependent variables.

subgroup, for which the non-infinitesimal form can then be explicitly determined to

$$\mathsf{E}_1: \quad \tilde{t} = e^{\varepsilon_1} t, \quad \tilde{\mathbf{x}} = e^{\varepsilon_1} \mathbf{x}, \quad \tilde{\mathbf{u}} = \mathbf{u}, \quad \tilde{p} = p, \quad \tilde{\nu} = e^{\varepsilon_1} \nu. \tag{2.5}$$

Hence, just by considering the viscosity, or alternatively the Reynolds number $Re \sim 1/\nu$, as an own independent variable, we see that next to the global scaling symmetry S (2.3) we gained an additional global scaling invariance E_1 (2.5): The viscosity as well as the space-time coordinates scale in exactly the same manner respective to the constant group parameter ε_1. However, this additional invariant transformation (2.5) does not act as a true symmetry, but only in the weaker sense as an equivalence transformation, in that it only maps the Navier-Stokes equation in the class of different viscosities to another equation in that same class. Indeed, inserting transformation (2.5) into form (2.1), will not leave it form-*indifferent*, but only form-*invariant*

$$\left.\begin{aligned} \tilde{\nabla} \cdot \tilde{\mathbf{u}} &= 0, \\ \partial_{\tilde{t}} \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\mathbf{u}} &= -\tilde{\nabla} \tilde{p} + \tilde{\nu} \tilde{\Delta} \tilde{\mathbf{u}}, \end{aligned}\right\} \tag{2.6}$$

since the parametric value changed to $\tilde{\nu} \neq \nu$. Particularly in this simple case, however, we can alternatively also say that transformation (2.5) actually maps a solution of equation (2.1), with a certain value in viscosity ν, to another solution of the same equation (2.6), but with a different value in viscosity $\tilde{\nu}$. Yet, note that irrespective of the functional choice for the continuous group parameter f, the invariant transformation (2.4) will never reduce to a true symmetry transformation. Every specific functional choice of f will give a different global equivalence scaling transformation.

Example 2: Taking the statistical ensemble average of the deterministic Navier-Stokes equations in the form (2.1), we get, due to the existence of the nonlinear convective term, the following unclosed (underdetermined) set of equations (Pope, 2000; Davidson, 2004)

$$\left.\begin{aligned} \nabla \cdot \langle \mathbf{u} \rangle &= 0, \\ \partial_t \langle \mathbf{u} \rangle + \nabla \cdot \mathbf{T} &= -\nabla \langle p \rangle + \nu \Delta \langle \mathbf{u} \rangle, \end{aligned}\right\} \tag{2.7}$$

where the second rank tensor $\mathbf{T} = \langle \mathbf{u} \otimes \mathbf{u} \rangle$ is the unclosed second velocity moment based on the full instantaneous velocity field $\mathbf{u}$. In the most general case $\mathbf{T}$ is to be identified as an unknown and thus arbitrary functional of the space-time coordinates $(\mathbf{x}, t)$ and of the mean fields of velocity $\langle \mathbf{u} \rangle$ and pressure $\langle p \rangle$ along with its spatiotemporal variations, either in local, nonlocal or mixed form.

For reasons of simplicity let us consider $\mathbf{T}$ for the moment as an arbitrary function which only shows an explicit dependence on the space-time coordinates, i.e. $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$. If we now perform an invariance analysis of the underdetermined system (2.7), by extending, next to the mean velocity $\langle \mathbf{u} \rangle$ and the mean pressure $\langle p \rangle$, the list of dependent variables with the unclosed and thus arbitrary function $\mathbf{T}$ as an own dependent variable, we immediately gain the following invariant *statistical* scaling[†]

$$\mathsf{E}_2: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}} = \mathbf{x}, \quad \langle \tilde{\mathbf{u}} \rangle = e^{\varepsilon_2} \langle \mathbf{u} \rangle, \quad \tilde{\mathbf{T}} = e^{\varepsilon_2} \mathbf{T}, \quad \langle \tilde{p} \rangle = e^{\varepsilon_2} \langle p \rangle, \tag{2.8}$$

which globally only scales the system's dependent variables while the coordinates stay invariant. It is clear that this invariant transformation cannot act as a symmetry transformation. It can only act in the weaker sense as an equivalence transformation, since in

[†]Note that in the general case a careful distinction must be made between the transformed expression $\widetilde{\langle \mathbf{u} \rangle}$, which directly refers to the transformed mean velocity field, and the transformed expression $\langle \tilde{\mathbf{u}} \rangle$, which, on the other hand, refers to the transformed instantaneous (fluctuating) velocity field being averaged only after its transformation. However, in the specific and simple case as (2.8) both transformed fields are identical $\widetilde{\langle \mathbf{u} \rangle} = \langle \tilde{\mathbf{u}} \rangle$. The obvious reason is that since transformation E_2 (2.8) only represents a globally uniform scaling with the constant factor e^{ε_2}, it will commute with every averaging operator $\langle , \rangle$ (for a more detailed discussion on this subject, see Appendix D.1).

the considered functional class of arbitrary second moment functions $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ it only maps the unclosed first moment equation (2.7) into another equation of the same class:

$$\left.\begin{aligned} \tilde{\nabla} \cdot \langle \tilde{\mathbf{u}} \rangle = 0, \\ \partial_{\tilde{t}} \langle \tilde{\mathbf{u}} \rangle + \tilde{\nabla} \cdot \tilde{\mathbf{T}} = -\tilde{\nabla} \langle \tilde{p} \rangle + \nu \tilde{\Delta} \langle \tilde{\mathbf{u}} \rangle, \end{aligned}\right\} \tag{2.9}$$

where the unclosed and thus arbitrary function $\mathbf{T}$ itself gets mapped to a new and different, but still unclosed and thus arbitrary function $\tilde{\mathbf{T}} \neq \mathbf{T}$. However, since $\tilde{\mathbf{T}}$ is from the same considered functional class as $\mathbf{T}$, it thus also exhibits an explicit dependence only on the coordinates: $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}(\tilde{\mathbf{x}}, \tilde{t})$.

Again, the invariant transformation (2.8) only represents an equivalence and *not* a symmetry transformation of the unclosed system (2.7), since it turns this system only into an equivalent but *not* identical form. To see this explicitly, imagine we would specify the unclosed moment function $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$, say by

$$\mathbf{T}(\mathbf{x}, t) = \nabla \phi(\mathbf{x}) \otimes \nabla \phi(\mathbf{x}), \quad \text{with} \quad \phi(\mathbf{x}) = e^{-\mathbf{x}^2}. \tag{2.10}$$

Then according to (2.8) the transformed moment is *defined* or given by

$$\left.\begin{aligned} \tilde{\mathbf{T}}(\tilde{\mathbf{x}}, \tilde{t}) &= e^{\varepsilon_2} \mathbf{T}(\mathbf{x}, t) \\ &= e^{\varepsilon_2} \Big(\nabla \phi(\mathbf{x}) \otimes \nabla \phi(\mathbf{x}) \Big) \\ &= e^{\varepsilon_2} \Big(\tilde{\nabla} \phi(\tilde{\mathbf{x}}) \otimes \tilde{\nabla} \phi(\tilde{\mathbf{x}}) \Big), \quad \text{since } \tilde{\mathbf{x}} = \mathbf{x}. \end{aligned}\right\} \tag{2.11}$$

Hence, while system (2.7) turns into the closed form

$$\left.\begin{aligned} \nabla \cdot \langle \mathbf{u} \rangle = 0, \\ \partial_t \langle \mathbf{u} \rangle + \nabla \cdot \big[\nabla \phi(\mathbf{x}) \otimes \nabla \phi(\mathbf{x}) \big] = -\nabla \langle p \rangle + \nu \Delta \langle \mathbf{u} \rangle, \end{aligned}\right\} \tag{2.12}$$

the transformed system (2.9), according to (2.11), will turn into

$$\left.\begin{aligned} \tilde{\nabla} \cdot \langle \tilde{\mathbf{u}} \rangle = 0, \\ \partial_{\tilde{t}} \langle \tilde{\mathbf{u}} \rangle + e^{\varepsilon_2} \tilde{\nabla} \cdot \big[\tilde{\nabla} \phi(\tilde{\mathbf{x}}) \otimes \tilde{\nabla} \phi(\tilde{\mathbf{x}}) \big] = -\tilde{\nabla} \langle \tilde{p} \rangle + \nu \tilde{\Delta} \langle \tilde{\mathbf{u}} \rangle, \end{aligned}\right\} \tag{2.13}$$

which obviously, due to the explicit factor e^{ε_2}, is *not identical* to the corresponding untransformed differential system (2.12). Instead, we can only say that system (2.13) is *equivalent* to system (2.12) in that they originate from the *same* class of functions $\tilde{\mathbf{T}}$ and $\mathbf{T}$ which both only show an explicit dependence on the coordinates.

This of course stands in strong contrast to any given symmetry transformation of a closed system. For example, the scaling symmetry S (2.3) of the deterministic Navier-Stokes equations (2.1), which, if we would specify a certain solution $\mathbf{u} = \mathbf{u}_0$ and $p = p_0$, it will be mapped according to S (2.3) to another solution $\mathbf{u}_0 \to \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0$ and $p_0 \to \tilde{p} = \tilde{p}_0$ of the *same* and thus to (2.1) identical equation:

$$\left.\begin{aligned} \tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0, \\ \partial_{\tilde{t}} \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla}) \tilde{\mathbf{u}} = -\tilde{\nabla} \tilde{p} + \nu \tilde{\Delta} \tilde{\mathbf{u}}. \end{aligned}\right\} \tag{2.14}$$

Furthermore, the statistical symmetry

$$\mathsf{S}: \quad \tilde{t} = e^{2\varepsilon} t, \quad \tilde{\mathbf{x}} = e^{\varepsilon} \mathbf{x}, \quad \langle \tilde{\mathbf{u}} \rangle = e^{-\varepsilon} \langle \mathbf{u} \rangle, \quad \tilde{\mathbf{T}} = e^{-2\varepsilon} \mathbf{T}, \quad \langle \tilde{p} \rangle = e^{-2\varepsilon} \langle p \rangle, \tag{2.15}$$

which corresponds to S (2.3) when reformulated for the mean fields up to the second velocity moment, leaves only the unclosed system (2.7) invariant, but *not* the specified closed system (2.12). That means that the specification (2.10) is not compatible with the statistical symmetry S (2.15), thus showing that the specific functional choice (2.10) on the

averaged level is inconsistent to the underlying deterministic (fluctuating) level (2.1). In strong contrast to the statistical equivalence transformation E_2 (2.8) which is compatible to both the *unspecified* system (2.7) and the *specified* system (2.12).

This explicit demonstration clearly shows that a Lie *symmetry* transformation induces a far more stronger invariance than a Lie *equivalence* transformation. Hence, the consequences which can be drawn from a symmetry transformation are by far more richer than for any equivalence transformation.

Three things should be noted here. Firstly, since the transformation (2.8) only scales the system's dependent variables by keeping the coordinates invariant, it is a typical scaling invariance which only linear systems can admit. Indeed, due to the identification of the unclosed function $\mathbf{T}$ as an own dependent variable, we turned the underdetermined statistical system (2.7) formally into a linear set of equations. As we will discuss in more detail in the next sections, such an identification is misleading, since it is hiding essential information about the underlying deterministic theory. In other words, although transformation (2.8) correctly acts as a mathematical equivalence transformation for the statistical system (2.7), we will demonstrate that it nevertheless leads to a physical inconsistency.

Secondly, the type and particular structure of an equivalence transformation strongly depends on the explicit variable dependence of $\mathbf{T}$ itself. Allowing for various different functional dependencies, as e.g. for $\mathbf{T} = \mathbf{T}(\mathbf{x}, t; \mathbf{u})$, or more generally for $\mathbf{T} = \mathbf{T}(\mathbf{x}, t; \mathbf{u}_{\{n\}})$ where $\mathbf{u}_{\{n\}}$ denotes the collection of functions $\mathbf{u}$ together with all their derivatives up to order n, can cause different equivalence groups in each case (Meleshko, 1996; Ibragimov, 2004; Bila, 2011; Chirkunov, 2012).

Thirdly, the equivalence transformation (2.8) given in this example has a much weaker impact when trying to extract information from the solution manifold of its underlying dynamical set of equations than the equivalence transformation (2.5) given in the previous example. In contrast to E_1 (2.5), which at least could map between specific solutions of different viscosity, the equivalence transformation E_2 (2.8) is completely unable to map between specific solutions. The reason is that the considered system of equations is unclosed and thus underdetermined, however not arbitrarily, but in the specified sense that the unclosed term $\mathbf{T} = \langle \mathbf{u} \otimes \mathbf{u} \rangle$ can be physically and uniquely determined from the underlying but analytically non-accessible deterministic velocity field $\mathbf{u}$. In other words, this circumstance, in having an underlying theory from which the unclosed term $\mathbf{T}$ physically emerges, opens the high possibility that physical solutions get mapped into unphysical ones when employing an equivalence transformation as E_2 (2.8). This problem will be discussed next.

2.1. *The concept of an invariant solution*

In order to understand and recognize the subtle difference between a symmetry and an equivalence transformation in its full spectrum, we will discuss this difference again, however, from a different perspective, from the perspective of generating invariant solutions.

First of all, one should recognize that the Lie algorithm to generate invariant transformations for differential equations can be equally applied in the same manner without any restrictions to *under-*, *fully-* as well as *overdetermined* systems of equations (Ovsiannikov, 1982; Stephani, 1989; Olver, 1993; Ibragimov, 1994; Andreev *et al.*, 1998; Bluman & Kumei, 1996; Meleshko, 2005), even if the considered system is infinite dimensional (Frewer, 2015*a*,*b*). However, only for *fully* or *overdetermined* systems these invariant Lie transformations are called and have the effect of *symmetry* transformations, while for *underdetermined* systems these invariant Lie transformations are called and have the effect of *equivalence* transformations.

In other words, although both a symmetry as well as an equivalence transformation form a Lie-group which by construction leave the considered equations invariant, the action and the consequence of each transformation is absolutely different. While a *symmetry* transformation always maps a solution to another solution of the *same equation*, an equivalence transformation, in contrast, generally only maps a *possible* solution of one

underdetermined equation to a *possible* solution of *another underdetermined equation*, where in the latter case we assume of course that a solution of an underdetermined equation can be somehow constructed or is somehow given beforehand.

Now, it is clear that *if* for an unclosed and thus underdetermined equation, or a set of equations, the unclosed terms are *not* correlated to an existing underlying theory, then the construction of an invariant solution will only be a particular and non-privileged solution within an infinite set of other possible and equally privileged solutions. But if, on the other hand, the unclosed terms are in fact correlated to an underlying theory, either in that they underly a specific but analytically non-accessible process or in that they show some existing but unknown substructure, then the construction of an invariant *solution* is misleading and essentially ill-defined, in particular if no prior modelling assumptions for the unclosed terms are made. To follow this conclusion in more detail we refer to Appendix A for an extensive discussion on this subject.

Hence, for an unclosed and thus underdetermined system of equations either infinitely many and equally privileged solutions (including all possible invariant solutions) can be constructed, or, depending on whether the unclosed terms are correlated to an underlying but analytically non-accessible theory as turbulence, no true solutions and thus also no true invariant solutions can be determined as long as no prior modelling procedure is invoked to close the system of equations. Therefore, since *closed* systems do not face this problem, the construction of invariant solutions from symmetry transformations is well-defined, while for equivalence transformations, which are admitted by *unclosed* systems, the construction of invariant solutions is misleading and can be even ill-defined as in the statistical theory of turbulence. Thus using for example the equivalence transformation (2.8) to generate a *privileged* statistical invariant solution for the unclosed system (2.7) is basically ill-defined, if no prior modelling assumptions for the underlying substructure of $\mathbf{T}$ is made to close the equations (see first part of Appendix A.2).

However, if nevertheless within the theory of turbulence such invariant results are generated, they must be carefully interpreted as only being functional relations or functional complexes which stay invariant under the derived equivalence group, and not as being privileged *solutions* of the associated underdetermined system, as done, for example, in Oberlack & Günther (2003); Khujadze & Oberlack (2004); Günther & Oberlack (2005); Oberlack *et al.* (2006); Oberlack & Rosteck (2010); She *et al.* (2011); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014) and Wacławczyk *et al.* (2014). In all these studies the underlying statistical system of dynamical equations is unclosed and thus underdetermined, however, not arbitrarily underdetermined, but underdetermined in the sense that all unclosed terms can be physically and uniquely determined from the underlying but analytically non-accessible instantaneous (fluctuating) velocity field. In particular the system considered in Oberlack & Rosteck (2010), although formally infinite in dimension, reveals itself by closer inspection as such an underdetermined system, for which, as was already said before, the determination of invariant solutions is ill-defined (see last part of Appendix A.2). This study of Oberlack & Rosteck (2010), which serves as a key study for the recent results made in Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014) and Wacławczyk *et al.* (2014), will be analyzed in more detail in the next sections.

Important to note is that up to now only in the specific case of homogeneous isotropic turbulence (Davidson, 2004; Sagaut & Cambon, 2008) all those invariant functional complexes which are gained from *equivalence* scaling groups can be further used to yield more valuable results, in particular the explicit values for the decay rates (Oberlack, 2002), since one has exclusive access to additional nonlocal invariants such as the Birkhoff-Saffman or the Loitsyansky integral. However, for *wall-bounded* flows it is not clear yet how to use or exploit such invariant functional complexes in a meaningful way, since up to now no additional nonlocal invariants are known.[†]

[†]This aspect also needs to be addressed in Oberlack's earlier work (Oberlack, 1999, 2001), where also only equivalence transformations were obtained, but which, in addition, were specifically obtained as a result of an incorrect conclusion (Frewer *et al.*, 2014*b*).

Finally it is worthwhile to mention that for example the work of Khabirov & Ünal (2002*a*,*b*) clearly shows in how *equivalence* transformations within the theory of turbulence can be exploited in a correct manner, which stands in strong contrast to the misleading approach of Oberlack et al. The major difference to the Oberlack et al. approach is that in Khabirov & Ünal (2002*a*,*b*) the invariant functions for the unclosed term (which are generated within different optimal Lie subalgebras for all possible Lie-point equivalence transformations of the unclosed Kármán-Howarth equation) are *not* identified as *true solutions* of the underlying unclosed equation itself, but, instead, are identified as *possible model terms* which then in each case consequently leads to a closed model equation. This is done in Khabirov & Ünal (2002*a*), while in Khabirov & Ünal (2002*b*) these *closed* Kármán-Howarth model equations are then solved in each case by the *now* well-defined technique of *invariant solutions*, which Khabirov & Ünal (2002*a*,*b*) then call *physical* invariant solutions. Of course, in how far these solutions then describe reality must be checked in each case by experiment or DNS. But that's a different problem!

We want to close this section by giving a citation from Khabirov & Ünal (2002*b*) which exactly describes the behavior and effect of an equivalence transformation when trying to exploit it in order to gain insight into the solution manifold of an unclosed and thus underdetermined equation: "Equivalence transformations may affect the behavior of solutions in physical sense. In other words, they may transform *physical* solutions into *unphysical* ones. But *inverse* equivalence transformations may act better in physical sense. These properties of the equivalence transformations will be made use of in the sequel."

3. A complete statistical description: The Hopf equation

In order to determine new statistical *symmetry transformations*, and not *equivalence* transformations, we have to operate within a framework which offers a complete and fully determined statistical description of Navier-Stokes turbulence. Any statistical description which is not formally closed, that is, every statistical description which from the outset would involve unclosed and thus arbitrary functionals, is not suited for this purpose. As was shown in the previous section (Example 2), every invariance analysis would then only generate very weak equivalence transformations.

Currently there are only two statistical approaches to incompressible and spatially unbounded Navier-Stokes turbulence which independently offer a complete and fully determined statistical description. Both approaches *formally* circumvent the explicit closure problem of turbulence in that they not only overcome the local differential framework in favor of a consistent nonlocal integral framework, but also in that they operate on a higher statistical level which goes beyond the level of the statistical moments. In each case the consequence is a linearly infinite but *formally* closed statistical approach.

These two approaches are the Lundgren-Monin-Novikov chain of equations (Lundgren, 1967; Monin, 1967; Friedrich *et al.*, 2012) and the Hopf equation (Hopf, 1952; McComb, 1990; Shen & Wray, 1991). While the former operates on the high statistical level of the *probability density functions* for the n-point velocity moments

$$\mathbf{H}_n = \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle, \quad n \geq 1, \tag{3.1}$$

the Hopf equation operates on the even higher level of the *probability density functionals* for these moments (3.1). As shown in Monin (1967), the Lundgren-Monin-Novikov system is just the discrete version of the functional Hopf equation. The former is iteratively given as an infinite but fully determined hierarchy of linearly coupled equations, while the latter is given as a single fully determined linear functional equation of infinite dimension. Since in both cases no arbitrary functions are involved, they both can be *formally* identified as closed systems.

To note is that a third statistical approach exists, which also leads to a linearly infinite hierarchy of equations, the so-called Friedmann-Keller chain of equations (Monin &

Yaglom, 1971), which, in contrast to the other two approaches, operates directly on the lower level of the n-point velocity moments (3.1). This chain can either be formulated in the local differential framework, as presented in Oberlack & Rosteck (2010) and also recently in Wacławczyk *et al.* (2014), or in the nonlocal integral framework as presented in Fursikov (1999) and re-derived in Appendix B.

However, in contrast to the Lundgren-Monin-Novikov chain or the Hopf equation, the Friedmann-Keller chain is not closed, not even in a formal sense. This matter is extensively discussed in Appendix C. Irrespective of the analytical framework and in the sense as explained in detail in Appendix C, the Friedmann-Keller chain always involves more unknown functions than determining equations. For both the integral framework as presented in Appendix B, as well as for the differential framework as presented in Oberlack & Rosteck (2010) and Wacławczyk *et al.* (2014), this can be easily confirmed by just explicitly counting the number of equations versus the number of functions to be determined. In this sense the Friedmann-Keller chain, although infinite in dimension, does not serve as a fully determined statistical description of Navier-Stokes turbulence. Any invariance analysis performed upon this chain will only generate the weaker class of equivalence transformations, simply because the chain is always permanently underdetermined and thus involving arbitrary functions.

Now, in order to prove our statement that a new statistical scaling symmetry is physically inconsistent with the underlying deterministic Navier-Stokes equations (2.2), either the Lundgren-Monin-Novikov chain or Hopf equation can be used. They are equivalent in so far as they both lead to the same conclusion. However, to prove this statement in the next section as efficiently as possible, we will only choose the functional Hopf-approach.

The functional Hopf equation (HEq)

$$\frac{\partial \Phi}{\partial t} = \int d^3\mathbf{x}\, \alpha_k \left(i \frac{\partial}{\partial x_l} \frac{\delta^2}{\delta \alpha_k \delta \alpha_l} + \nu \Delta \frac{\delta}{\delta \alpha_k} \right) \Phi, \tag{3.2}$$

describes the dynamical evolution of the characteristic or moment-generating functional

$$\Phi[\boldsymbol{\alpha}(\mathbf{x}); t] = \int P[\mathbf{u}(\mathbf{x}); t]\, e^{i \int d^3\mathbf{x}\, \boldsymbol{\alpha}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})} \mathcal{D}\mathbf{u}(\mathbf{x}), \tag{3.3}$$

which is the functional Fourier transform (Klauder, 2010; Kleinert, 2013) of the probability density functional $P[\mathbf{u}(\mathbf{x}); t]$ for the velocity field $\mathbf{u}$ sampled for each time step in infinitely non-denumerable (continuum) number of points $\mathbf{x}$, which itself plays the role of a continuous index inside the functionals Φ and P, but nonetheless still to be interpreted next to the coordinate t and the field $\boldsymbol{\alpha}(\mathbf{x})$ as an own independent and active variable in the underlying dynamical equation (3.2). In other words, both functionals Φ and P do not explicitly depend on $\mathbf{x}$, i.e. in equation (3.2) the variable $\mathbf{x}$ only appears implicitly in the dependent variable Φ upon which the coordinate operators can then act on. The functional variable $\boldsymbol{\alpha}(\mathbf{x})$, however, is an arbitrary but real, integrable and time-independent solenoidal external source function with vanishing normal component at the (infinite far) boundary. In order to guarantee for physical consistency, a mathematical solution of the Hopf equation (3.2) is only admitted if *for all times* the following conditions are fulfilled

$$\Phi^*[\boldsymbol{\alpha}(\mathbf{x}); t] = \Phi[-\boldsymbol{\alpha}(\mathbf{x}); t], \quad \Phi[0; t] = 1, \quad \left|\Phi[\boldsymbol{\alpha}(\mathbf{x}); t]\right| \leq 1, \tag{3.4}$$

which stem from the fact that the probability density functional is real, non-negative, and normalized to one in sample space, i.e. $\int P[\mathbf{u}(\mathbf{x}); t] \mathcal{D}\mathbf{u}(\mathbf{x}) = 1$, with $P[\mathbf{u}(\mathbf{x}); t] \geq 0$. This then defines the (infinite) physical dimension of the probability density functional as $[P] = [1/\mathcal{D}\mathbf{u}]$ with $\mathcal{D}\mathbf{u} = \prod_{\mathbf{x}} (\sqrt{d^3\mathbf{x}/2\pi})^3 \cdot d^3\mathbf{u}(\mathbf{x})$, while the characteristic functional Φ is dimensionless.

Finally note that the above-presented functional integration element $\int \mathcal{D}\mathbf{u}(\mathbf{x})$ for the Fourier transform (3.3) is symmetrically defined as the following infinite product of one-dimensional integrals over $\mathbf{u}(\mathbf{x_n}) = (u^i(\mathbf{x_n})) \in \mathbb{R}^3$ at every point $\mathbf{x_n} = (x^i)_\mathbf{n} \in \mathbb{R}^3$ for

every component and coordinate $i = 1, 2, 3$ (see e.g. Kleinert (2013), Chapter 13):

$$\int \mathcal{D}\mathbf{u}(\mathbf{x}) = \prod_{i=1}^{3} \int \mathcal{D}u^i(\mathbf{x}) = \prod_{i=1}^{3} \prod_{(n_1,n_2,n_3)=-\infty}^{\infty} \int \frac{du^i(\epsilon \cdot \mathbf{n})}{\sqrt{2\pi/\epsilon^3}}. \tag{3.5}$$

Hereby, space-time is grated into a fine *equidistant* lattice, where for every coordinate x^i the following discrete lattice points were introduced

$$\mathbf{x} \to \mathbf{x_n} = \epsilon \cdot \mathbf{n} = \epsilon \cdot (n_1, n_2, n_3) \in \mathbb{R}^3, \quad n_i \in \mathbb{Z}, \tag{3.6}$$

with a very small lattice spacing ϵ.

4. The new statistical scaling symmetry and its inconsistency

It is straightforward to recognize that the linear Hopf equation (3.2) admits the following (functional) Lie-point scaling symmetry

$$\mathsf{Q}: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}} = \mathbf{x}, \quad \tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}, \quad \tilde{\Phi} = e^q (\Phi - 1) + 1, \tag{4.1}$$

where q is a globally constant and real group parameter. This invariance first got mentioned in Wacławczyk & Oberlack (2013*a*). Note that symmetry Q is only compatible with the first two physical constraints given in (3.4). The third, non-holonomic constraint in (3.4) gets violated if the characteristic functional is generally transformed as (4.1).

But, if the values of the group constant q are restricted to $q \leq 0$ then symmetry Q (4.1) is fully compatible with all three physical constraints (3.4). However, by restricting the values of q to $q \leq 0$, the symmetry (4.1) turns into a semi-group since no inverse element can be defined or constructed anymore. In other words, the third constraint in (3.4) breaks the Lie-group structure of the symmetry (4.1) down to a semi-group.

The connection of the moment generating functional Φ (3.3) to the multi-point velocity correlation functions $\mathbf{H}_n$ (3.1) is given as (Hopf, 1952; McComb, 1990; Shen & Wray, 1991)

$$\mathbf{H}_n = \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle = (-i)^n \frac{\delta^n \Phi[\boldsymbol{\alpha}(\mathbf{x}); t]}{\delta\boldsymbol{\alpha}(\mathbf{x}_1) \cdots \delta\boldsymbol{\alpha}(\mathbf{x}_n)}\bigg|_{\boldsymbol{\alpha}=\mathbf{0}}. \tag{4.2}$$

By inserting Q (4.1) into the above functional relation (4.2) of the transformed domain

$$\begin{aligned}(-i)^n \frac{\delta^n \tilde{\Phi}[\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}); \tilde{t}]}{\delta\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}_1) \cdots \delta\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}_n)}\bigg|_{\tilde{\boldsymbol{\alpha}}=\mathbf{0}} &= (-i)^n \frac{\delta^n \left[e^q \Phi[\boldsymbol{\alpha}(\mathbf{x}); t] + (1 - e^q)\right]}{\delta\boldsymbol{\alpha}(\mathbf{x}_1) \cdots \delta\boldsymbol{\alpha}(\mathbf{x}_n)}\bigg|_{\boldsymbol{\alpha}=\mathbf{0}} \\ &= e^q \cdot (-i)^n \frac{\delta^n \Phi[\boldsymbol{\alpha}(\mathbf{x}); t]}{\delta\boldsymbol{\alpha}(\mathbf{x}_1) \cdots \delta\boldsymbol{\alpha}(\mathbf{x}_n)}\bigg|_{\boldsymbol{\alpha}=\mathbf{0}},\end{aligned} \tag{4.3}$$

we can see how the symmetry transformation Q (4.1) induces the following invariant transformation for the n-point velocity correlation functions (3.1)

$$\mathsf{Q}_\mathsf{E}: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{\mathbf{H}}_n = e^q \mathbf{H}_n, \quad n \geq 1, \tag{4.4}$$

which for the first time was derived in Khujadze & Oberlack (2004) as a "new statistical symmetry" of Navier-Stokes turbulence. For their derivation they however only considered the unclosed multi-point equations for the velocity moments up to order $n = 2$ in the limit of an inviscid parallel shear flow in ZPG turbulent boundary layer flow[†], while recently in Oberlack & Rosteck (2010) this result (4.4) was re-derived most generally without any flow restrictions by using the full infinite chain of Friedmann-Keller equations. In

[†]In Khujadze & Oberlack (2004) the iterative sequence of the "new statistical scaling symmetry" (4.4) begins only from $n = 2$ onwards, i.e only for all $n \geq 2$. The transformation for $n = 1$ is excluded, i.e. the mean velocity $\mathbf{H}_1 = \langle \mathbf{u} \rangle$ stays invariant.

both derivations the statistical invariance analysis was performed in the local differential framework based on the corresponding deterministic form (2.1), in which the pressure field is explicitly present. Thus in both derivations their results also include, next to the n-point velocity moments $\mathbf{H}_n$, all invariant transformations for the velocity-pressure correlations. Of course, these correlations are not part of our result given here, since we derived (4.4) from the Hopf equation (3.2) which is based on the underlying nonlocal deterministic integral form (2.2), in which the pressure field has been consistently eliminated from the dynamical equations.

Indeed, it can be easily verified that transformation Q_E (4.4) is admitted as an invariant transformation also by the nonlocal integro-differential Friedmann-Keller equations

$$\partial_t \mathbf{H}_n + \boldsymbol{\mathcal{A}}_n \cdot \mathbf{H}_n + \boldsymbol{\mathcal{B}}_n \cdot \widehat{\mathbf{H}}_{n+1} = \mathbf{0}, \quad n \geq 1, \tag{4.5}$$

which are defined and derived in Appendix B. However, as noted in Appendix B and discussed in more detail in Appendix C, the invariant transformation Q_E (4.4) does not act as a symmetry transformation, but only in the weaker form as an equivalence transformation. The reason is that the hierarchy (4.5) forms an unclosed system. The still missing transformation rule for the unclosed n-point function of $(n+1)$-th moment $\widehat{\mathbf{H}}_{n+1}$, which formally stands for $\widehat{\mathbf{H}}_{n+1} = \mathbf{H}_{n+1}|_{\mathbf{x}_{n+1}=\mathbf{x}_n}$, is then dictated by the given transformation rule (4.4) for the corresponding $(n+1)$-point function $\mathbf{H}_{n+1}$ as

$$\widetilde{\widehat{\mathbf{H}}}_{n+1} = \widetilde{\mathbf{H}}_{n+1}\big|_{\tilde{\mathbf{x}}_{n+1}=\tilde{\mathbf{x}}_n} = \big(e^q \mathbf{H}_{n+1}\big)\big|_{\mathbf{x}_{n+1}=\mathbf{x}_n} = e^q\big(\mathbf{H}_{n+1}\big|_{\mathbf{x}_{n+1}=\mathbf{x}_n}\big) = e^q \widehat{\mathbf{H}}_{n+1}. \tag{4.6}$$

It is clear that this simple transformation rule (4.6) is only due to the global and uniform nature of Q_E (4.4), in which all system variables transform uniformly by the same constant scaling exponent q and independent from the coordinates, which themselves stay invariant.

Important to note here is that in Oberlack & Rosteck (2010) the invariant transformation Q_E (4.4) is considered as a true *symmetry* transformation. However, as already discussed in Section 3 and explained in Appendix C, this claim is not correct. Transformation Q_E (4.4) can only act as an *equivalence* transformation and not as a *symmetry* transformation. Hence, the invariance analysis performed in Oberlack & Rosteck (2010) is based on equivalence and not on symmetry groups, simply because unclosed and thus arbitrary functions are permanently involved within the considered analysis.

But this insight now has consequences in the interpretation of their newly derived statistical scaling laws, because these laws as presented in Oberlack & Rosteck (2010) may not be interpreted as being privileged *solutions* of the underlying statistical set of equations as was done therein. They may only be interpreted as being functional relations or functional complexes which stay invariant under the derived equivalence group, nothing more! Therefore these new relations derived in Oberlack & Rosteck (2010) *only possibly but not necessarily* can serve as useful turbulent scaling functions. Moreover, a comparison to DNS results reveals that these new statistical scaling laws presented in Oberlack & Rosteck (2010) are unphysical as they clearly fail to fulfil the most basic predictive requirements of a scaling law. For ZPG turbulent boundary layer flow this investigation is presented and further discussed in Section 5.

The reason for this failure is twofold: Next to the reason just discussed, that the invariance analysis in Oberlack & Rosteck (2010) was performed upon an underdetermined statistical system which cannot admit true invariant *solutions* without establishing a correct link to the underlying deterministic equations, the second and more stronger reason is that the symmetry Q (4.1) itself is unphysical. This physical inconsistency of course transfers down to Q_E (4.4), as it is induced by Q. This will also explain why on the higher level of the probability density functionals a true symmetry transformation, such as Q (4.1), only induces an equivalence transformation, such as Q_E (4.4), and not a corresponding symmetry transformation on the lower level of the n-moment functions $\mathbf{H}_n$.

Before we proceed with the proof, it is worthwhile to see that when considering the chain only up to the second moment ($n \leq 2$), the *general* equivalence transformation

Q_E (4.4) will reduce in the smooth limit of zero correlation length to the equivalence transformation E_2 (2.8) discussed in Section 2:

$$\lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} \mathsf{Q}_\mathsf{E}^{(n\leq 2)} = \mathsf{E}_2 \backslash \langle \tilde{p} \rangle, \tag{4.7}$$

where, due to the eliminated pressure field in the underlying Hopf equation, the missing transformation rule $\langle \tilde{p} \rangle = e^q \langle p \rangle$ for the mean pressure field in Q_E (4.4) is consistently dictated by the mean solenoidal velocity field $\tilde{\mathbf{H}}_1 = \langle \tilde{\mathbf{u}} \rangle$ through the one-point momentum equation (2.7) as given in E_2 (2.8). Hence, the above mentioned and still to be proven physical inconsistency of Q (4.1) will thus even fully transfer down to E_2 (2.8).

4.1. *Proof of the physical inconsistency of symmetry* Q

The physical inconsistency of symmetry Q (4.1) can be readily observed when connecting the averaged (statistical) level back to the fluctuating (deterministic) level. For the Hopf equation (3.2) the transition rule in going from the fine-grained (fluctuating) to the coarse-grained (averaged) level is defined by the path integral (3.3), which in each time step t sums up all coarse-grained probabilities P for all possible realizations in the fine-grained velocity field $\mathbf{u}$. Now, in order to see the inconsistency, it is necessary to consider the inverse functional Fourier transform of (3.3) in the transformed domain of Q (4.1):

$$\tilde{P}[\tilde{\mathbf{u}}(\tilde{\mathbf{x}});\tilde{t}] = \int \tilde{\Phi}[\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}});\tilde{t}]\, e^{-i\int d^3\tilde{\mathbf{x}}\, \tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}})\cdot\tilde{\mathbf{u}}(\tilde{\mathbf{x}})} \mathcal{D}\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}). \tag{4.8}$$

By inserting the transformation Q (4.1) into the right-hand side of (4.8) we get

$$\begin{aligned} \tilde{P}[\tilde{\mathbf{u}};\tilde{t}] &= \int \Big[\, e^q \Phi[\boldsymbol{\alpha};t] + (1-e^q) \,\Big]\, e^{-i\int d^3\mathbf{x}\, \boldsymbol{\alpha}\cdot\mathbf{u}} \mathcal{D}\boldsymbol{\alpha} \\ &= e^q P[\mathbf{u};t] + (1-e^q)\cdot\delta[\mathbf{u}], \end{aligned} \tag{4.9}$$

which is the corresponding transformation rule for P, if Φ transforms as given in (4.1), where for better readability we suppressed the implicit dependence $\tilde{\mathbf{x}} = \mathbf{x}$ on both sides. The function $\delta[\cdot]$ is the functional analog of the Dirac δ-function, and is called the δ-functional (Klauder, 2010; Kleinert, 2013). In the lattice approximation, corresponding to (3.5) with (3.6), they are defined as an infinite product of ordinary one-dimensional δ-functions

$$\delta[\mathbf{u}] = \prod_{i=1}^{3} \delta[u^i] = \prod_{i=1}^{3} \prod_{\mathbf{n}=-\infty}^{\infty} \sqrt{2\pi/\epsilon^3}\cdot\delta(u^i(\epsilon\cdot\mathbf{n})), \tag{4.10}$$

and thus having the obvious property

$$\int \mathcal{D}\mathbf{u}\,\delta[\mathbf{u}] = 1, \quad \int \mathcal{D}\boldsymbol{\alpha}\,\delta[\boldsymbol{\alpha}] = 1. \tag{4.11}$$

Note that since the variables $\mathbf{x}$ and $\boldsymbol{\alpha}$ transform invariantly under (4.1), the velocity field $\mathbf{u}$ must stay invariant too, otherwise we would loose the definition (3.3) of a functional Fourier transform in the transformed domain.

However, on the other hand, if we insert Q (4.1) into the functional relation (4.2) in order to explicitly generate the transformation rule for the n-point velocity moments, as was already exercised in (4.3), we will get again

$$(-i)^n \frac{\delta^n \tilde{\Phi}[\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}});\tilde{t}]}{\delta\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}_1)\cdots\delta\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}_n)}\bigg|_{\tilde{\boldsymbol{\alpha}}=\mathbf{0}} = e^q\cdot(-i)^n \frac{\delta^n \Phi[\boldsymbol{\alpha}(\mathbf{x});t]}{\delta\boldsymbol{\alpha}(\mathbf{x}_1)\cdots\delta\boldsymbol{\alpha}(\mathbf{x}_n)}\bigg|_{\boldsymbol{\alpha}=\mathbf{0}}, \tag{4.12}$$

which, if using the representation of Φ (3.3), turns into the following relations

$$\begin{aligned} n=1:&\ \int \tilde{P}[\tilde{\mathbf{u}}(\tilde{\mathbf{x}});\tilde{t}]\cdot\tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1)\cdot\mathcal{D}\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = e^q \int P[\mathbf{u}(\mathbf{x});t]\cdot\mathbf{u}(\mathbf{x}_1)\cdot\mathcal{D}\mathbf{u}(\mathbf{x}), \\ n=2:&\ \int \tilde{P}[\tilde{\mathbf{u}}(\tilde{\mathbf{x}});\tilde{t}]\cdot\tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1)\otimes\tilde{\mathbf{u}}(\tilde{\mathbf{x}}_2)\cdot\mathcal{D}\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = e^q \int P[\mathbf{u}(\mathbf{x});t]\cdot\mathbf{u}(\mathbf{x}_1)\otimes\mathbf{u}(\mathbf{x}_2)\cdot\mathcal{D}\mathbf{u}(\mathbf{x}), \\ &\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned} \tag{4.13}$$

By recognizing again the already mentioned fact that the velocity field (along with its continuous index) is an invariant under the considered transformation Q (4.1), we can replace the variable $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ in (4.13) with $\mathbf{u}(\mathbf{x})$ and vice versa for all points. Then, by equating in its present and already irreducible form the left-hand side with the right-hand side for each order n, we obtain from (4.13) the transformation relation[†]

$$\tilde{P}[\tilde{\mathbf{u}};\tilde{t}] = e^q P[\mathbf{u};t], \tag{4.14}$$

which is in conflict with the previously found transformation rule (4.9) for P, i.e. there is no unique transformation rule for the probability density functional P. Consequently, via the fine- to coarse-grained transition rule (4.8) the symmetry transformation Q (4.1) induces an inconsistency. This conflict, however, can only be solved if $q = 0$, but which then turns the symmetry transformation Q (4.1) into a trivial identity transformation.

Worthwhile to note is that by physical intuition alone one already can recognize this conflict just by solely observing relation (4.9) more closely. Because, since the variables $\mathbf{x}$, $\mathbf{u}$ and t transform invariantly under Q (4.1) we can identically write the transformation rule (4.9) also as

$$\tilde{P}[\mathbf{u};t] = e^q P[\mathbf{u};t] + (1 - e^q) \cdot \delta[\mathbf{u}], \tag{4.15}$$

which states that although the system on the fine-grained level $\mathbf{u}$ stays unchanged, it nevertheless undergoes a *global* change $P \to \tilde{P}$ on the coarse-grained level, which is completely unphysical and not realized in nature.

We thus have a classical violation of cause and effect, as the system would experience an effect (change in averaged dynamics) without a corresponding cause (change in fluctuating dynamics). Note that the opposite conclusion is not the rule, i.e. a change on the fluctuating level can occur without inducing an effect on the averaged level. A macroscopic or coarse-grained (averaged) observation might be insensitive to many microscopic or fine-grained (fluctuating) details, a property of nature widely known as universality (see e.g. Marro (2014)). For example, a high-level complex coherent turbulent structure, though a consequence of the low-level fluctuating description, does not depend on all its details on its lowest level. The opposite again, however, is not realized in nature, i.e., stated differently, if the coherent structure experiences a *global* change, e.g. in scale or in a translational shift, it definitely must have a cause and thus *must* go along with a corresponding change on the lower fluctuating level — see also the discussions, e.g., in Frewer *et al.* (2015, 2016*a*).

Exactly this non-physical behavior can also be independently observed in the induced transformation rule Q_E (4.4) for the n-point velocity moments $\mathbf{H}_n$ (3.1). It can either be exposed directly on the fluctuating level as an unphysical equivalence transformation, or indirectly on the averaged level as a superfluous or artificial equivalence transformation. In any case, on each level we will gain different insights for this non-physical transformation behavior.

Hence, fully detached from the finding that the equivalence transformation Q_E (4.4) for the velocity moments is induced by an unphysical symmetry transformation Q (4.1) on the higher statistical level of the corresponding probability densities, we will now repeat our investigation on the lower statistical level of the velocity moments themselves, by only focussing on the link between Q_E (4.4) and the unclosed Friedmann-Keller equations (4.5).

4.2. *The unphysical behavior of equivalence* Q_E *on the fluctuating level*

In the case of the Friedmann-Keller chain, especially when used in the oversimplified form (4.5), particular care has to be taken when actually performing a systematic invariance analysis on these equations. The problem is that in contrast to the other two statistical approaches, i.e. the Lundgren-Monin-Novikov chain or the Hopf equation, the Friedmann-Keller chain does not naturally come along with additional physical constraints which are necessary in order to reveal the nonlinear and nonlocal connection between all constituents (see Appendix C).

[†]Note that a local relation can only be identified correctly from an integral relation if it's formulated irreducibly, i.e., in a form such that it cannot be reduced or simplified any further.

This circumstance can easily lead to misleading results, as it is the case for Q_E (4.4). Here it is necessary to recognize that Q_E (4.4) is an invariant scaling transformation which only linear systems can admit, since only the system's dependent variables get uniformly scaled, while the coordinates $(t, \mathbf{x})$ themselves stay invariant. Indeed, the corresponding dynamical system which admits Q_E (4.4) is the Friedmann-Keller chain of equations (4.5), which is a linear system, since $\boldsymbol{\mathcal{A}}_n$ and $\boldsymbol{\mathcal{B}}_n$ are both linear operators (see Appendix B).

However, this result, that the hierarchical system (4.5) admits Q_E (4.4) as an equivalence transformation, is misleading, since it suggests that all correlation functions $\mathbf{H}_n$ scale uniformly with the same scaling factor, which really is not the case as the underlying theory dictates a nonlinear correlation between all these quantities. The problem clearly lies in the notation: Using a formal symbol as $\mathbf{H}_n$, where only an external index n allows to distinguish between different multi-point functions, hides the actual underlying correlation information among them. In this sense the notation used in equation (4.5) is counterproductive from the perspective of an analysis on invariance, in that it oversimplifies the physical situation by representing the dynamics for the n-point velocity correlations as a linear PDE system, while it is actually based on a nonlinear theory.[†]

It is this misleading aspect which was not recognized and taken into account in Oberlack & Rosteck (2010). That is to say, by *explicitly* revealing the underlying nonlinear structure behind the formal symbol $\mathbf{H}_n$ in (4.5), namely that $\mathbf{H}_{n+1}$ contains one deterministic velocity field $\mathbf{u}$ more than $\mathbf{H}_n$, will ultimately break the equivalence scaling Q_E (4.4), as will be shown next.

Since the velocity correlation function $\mathbf{H}_n$ in Q_E (4.4) is nonlinearly built up by n multiplicative evaluations of the instantaneous (fluctuating) velocity field $\mathbf{u}$ according to (3.1), the following chain of reasoning instantly emerges: Since for $n = 1$ the averaged function $\mathbf{H}_1 = \langle \mathbf{u}_1 \rangle$ scales as e^q for all points $\mathbf{x}_1 = \mathbf{x}$ in the domain, the corresponding fluctuating quantity $\mathbf{u}_1$ has to scale in the same manner, since every averaging operator $\langle , \rangle$ is linearly commuting with any *constant* scale factor. But this implies that any product of n fluctuating fields $\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n$ has to scale as $e^{n \cdot q}$, which again implies that also the corresponding averaged quantity $\mathbf{H}_n = \langle \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_n \rangle$ then has to scale as $e^{n \cdot q}$. Symbolically the chain reads as

$$
\begin{aligned}
\tilde{\mathbf{H}}_1 = e^q \mathbf{H}_1 \;&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \rangle = e^q \langle \mathbf{u}(\mathbf{x}_1, t) \rangle, \text{ for all points } \mathbf{x}_1 = \mathbf{x} \\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle = e^q \langle \mathbf{u}(\mathbf{x}_k, t) \rangle, \text{ for all } k \geq 1 \\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle = \langle e^q \mathbf{u}(\mathbf{x}_k, t) \rangle, \text{ for all possible configurations } \mathbf{u} \\
&\Rightarrow\; \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) = e^q \mathbf{u}(\mathbf{x}_k, t) \\
&\Rightarrow\; \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) = e^{n \cdot q} \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) \rangle = \langle e^{n \cdot q} \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle \\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) \rangle = e^{n \cdot q} \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle \\
&\Rightarrow\; \tilde{\mathbf{H}}_n = e^{n \cdot q} \mathbf{H}_n. \qquad (4.16)
\end{aligned}
$$

For a detailed explanation of this proof in all its steps, please refer to Appendix D. Conclusion (4.16) clearly demonstrates that if the one-point function $\mathbf{H}_1$ globally scales as e^q then the n-point function $\mathbf{H}_n$ has to scale accordingly, namely as $e^{n \cdot q}$ and not as e^q as dictated by Q_E (4.4). Only the former scaling $e^{n \cdot q}$ will guarantee for an all-over consistent relation between the fluctuating and averaged level of the dynamical Navier-Stokes system. In other words, if a dynamical system experiences a *global* transformational change on the averaged level then there must exist *at least* one corresponding change on the fluctuating level (Frewer *et al.*, 2016*a*). But exactly this is not the case for Q_E (4.4) as both $\mathbf{H}_1$ and $\mathbf{H}_n$ scale therein with the *same* global factor, for which, thus, a corresponding fluctuating scaling cannot be derived or constructed, neither as a symmetry nor as any regular transformation, meaning that the system experiences a *global* change on the averaged level with *no* corresponding change on the fluctuating level — again the classical violation of cause

[†]Note that also the Hopf equation, and its discrete version, the Lundgren-Monin-Novikov equations are linear systems, but at the expense of operating on a higher statistical level than the moments of the Friedmann-Keller chain of equations, which, by definition, are all uniquely correlated in a nonlinear manner.

and effect as was already discussed before. Hence, on the lower statistical level of the velocity moments, too, the physical consistency can only be restored if $q = 0$, i.e. if the equivalence transformation Q_E (4.4) gets broken.

To conclude, it should be pointed out that this proof (4.16) clearly shows that the transformation (4.4) itself, i.e. detached from any transport equations, leads to contradictions as soon as one considers more than one point ($n \geq 2$). However, for $n = 1$, i.e. for the mean velocity $\mathbf{H}_1 = \langle \mathbf{u} \rangle$ itself, no contradiction exits. Only as from $n \geq 2$ onwards, the contradiction starts, which also can be clearly observed when comparing to DNS data as will be demonstrated in Section 5: The mismatch of the corresponding scaling laws which involve this contradictive scaling group (4.4) gets more strong as the order of the moments n increases.

Also finally note again that in order to perform the proof (4.16) we basically used the *consistency* of $n = 1$ (the first four lines of (4.16)) to show the *inconsistency* for all $n \geq 2$ (the remaining lines of (4.16)). Hence, irrelevant of whether Q_E (4.4) represents an invariance or not, the transformation itself leads for $n \geq 2$ to contradictions.

4.3. *The superfluous behavior of equivalence* Q_E *on the averaged level*

The immediate consequence on the averaged level in using an oversimplified statistical representation is that Q_E (4.4) will show a superfluous or artificial transformation behavior as soon as one changes to a more detailed representation which reveals more information about the underlying theory. From the perspective of an invariance analysis, it is intuitively clear that changing the statistical description for example to the Reynolds decomposed representation will be superior to the oversimplified notation used in (4.5), as it explicitly will reveal the nonlinearity within the system on the averaged lower level of the moments. Performing a Reynolds decomposition, for example, of the instantaneous 2-point velocity field $\mathbf{H}_2$ into its mean and fluctuating part, will thus lead to

$$\begin{aligned} \mathbf{H}_2 &= \langle \mathbf{u}_1 \otimes \mathbf{u}_2 \rangle = \langle (\mathbf{U}_1 + \mathbf{u}'_1) \otimes (\mathbf{U}_2 + \mathbf{u}'_2) \rangle \\ &= \mathbf{R}_{12} + \mathbf{U}_1 \otimes \mathbf{U}_2. \end{aligned} \tag{4.17}$$

This relation explicitly unfolds its nonlinear connection to the one-point velocity fields, where $\mathbf{R}_{12} = \langle \mathbf{u}'_1 \otimes \mathbf{u}'_2 \rangle$ is the corresponding 2-point correlation function for the (zero-mean) fluctuating field $\mathbf{u}'$ evaluated in the points $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{x}_2$ respectively, while $\mathbf{U}_1$ and $\mathbf{U}_2$ is the mean velocity evaluated in the same points. Next, the decomposition for the instantaneous 3-point velocity field $\mathbf{H}_3$

$$\begin{aligned} \mathbf{H}_3 &= \langle \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 \rangle = \langle (\mathbf{U}_1 + \mathbf{u}'_1) \otimes (\mathbf{U}_2 + \mathbf{u}'_2) \otimes (\mathbf{U}_3 + \mathbf{u}'_3) \rangle \\ &= \mathbf{R}_{123} + \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3 \\ &\qquad + \mathbf{R}_{12} \otimes \mathbf{U}_3 + \mathbf{R}_{13} \otimes \mathbf{U}_2 + \mathbf{R}_{23} \otimes \mathbf{U}_1, \end{aligned} \tag{4.18}$$

will not only nonlinearly connect to 1-point, but also to 2-point functions. This nonlinear connection will then iteratively continue for all higher multi-point functions. Hence, in a bijective, one-to-one manner the equivalence transformation Q_E in the oversimplified (linear) representation (4.4) then changes to the following more detailed (nonlinear) representation

$$\begin{aligned} \mathsf{Q}_\mathsf{E}: \quad & \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{\mathbf{U}}_n = e^q \, \mathbf{U}_n, \quad \cdots \\ & \tilde{\mathbf{R}}_{nm} = e^q \, \mathbf{R}_{nm} + (e^q - e^{2q}) \cdot \mathbf{U}_n \otimes \mathbf{U}_m, \quad \cdots \\ & \tilde{\mathbf{R}}_{nml} = e^q \, \mathbf{R}_{nml} \\ & \qquad + (e^q - e^{2q}) \cdot (\mathbf{R}_{nm} \otimes \mathbf{U}_l + \mathbf{R}_{nl} \otimes \mathbf{U}_m + \mathbf{R}_{ml} \otimes \mathbf{U}_n) \\ & \qquad + (e^q - 3e^{2q} + 2e^{3q}) \cdot \mathbf{U}_n \otimes \mathbf{U}_m \otimes \mathbf{U}_l, \quad \cdots \end{aligned} \tag{4.19}$$

where we explicitly expressed the transformation only up to third order in the velocity field for all point-indices $n, m, l \geq 1$ in all possible combinations.

In contrast to representation (4.4), the above representation (4.19) of $\mathsf{Q_E}$ immediately reveals its superfluous or artificial behavior as an equivalence transformation. In (4.19) one can see that the aim of all terms containing the prefactor e^q is to enforce a linear system scaling invariance which attempts to scale all field variables uniformly and independently from its coordinates. But since the underlying Navier-Stokes theory is inherently nonlinear, typical error terms proportional to e^{2q} and e^{3q} then emerge in (4.19) which need to be subtracted accordingly in order to allow for a misleading linear invariance property within a true nonlinear system of moments. In other words, although transformation (4.19) acts as a true equivalence transformation in the correspondingly Reynolds decomposed representation of the instantaneously averaged system (4.5), it acts artificially in that it interprets the nonlinear terms $\mathbf{U}_n \otimes \mathbf{U}_m$, $\mathbf{U}_n \otimes \mathbf{U}_m \otimes \mathbf{U}_l$, etc. as error terms which all are corrected for in order to achieve the desired linear system scaling invariance, but which, as was demonstrated before in (4.16), ultimately cannot exist physically as it induces inconsistencies already on the fluctuating level.

In order to avoid a misconception on this subtle issue, we will repeat the above argumentation again by using different words and by viewing it from a different perspective.

Our claim here is that for the moments the **H** notation as used by Oberlack et al. should not be used when performing an analysis on invariance, because, due to its high notational simplification, it can easily lead to misleading results, in particular when ignoring its connection to the underlying deterministic theory. Careful, our statement is only to be interpreted as a precautionary measure to avoid possible misguidance from the outset. We do not say that the **H** notation is wrong, we just say that it is counterproductive to use, because when working with this oversimplified notation without making a direct connection to the underling deterministic theory, one clearly has a higher risk to produce non-physical results than when working in the classical **R** notation. The self-evident reason is that the oversimplified **H** notation hides essential information of the underlying deterministic Navier-Stokes equations, while the **R** notation, in contrast, reveals it. In other words, when not connecting the notation to the underlying deterministic theory, the **R** notation is *physically* more transparent and helpful than the mathematically equivalent **H** notation.

Of course, as both notations are just linked by a bijective (one-to-one) mapping, the classical **R** notation is not free of the risk to induce a non-physical result, too. But such a non-physical result will be more easy to track in the detailed **R** notation than in the oversimplified **H** notation, where it's even not noticeable without properly connecting the notation to the underlying theory. In the **R** notation, however, unphysical results immediately reveal themselves by showing an artificial functional behavior, as in the case of the new unphysical scaling invariance (4.19).

It is clear that since this scaling invariance is unphysical in the **H** notation (4.4) it is also unphysical in the **R** notation (4.19). But in contrast to relation (4.4), the corresponding relation (4.19) *immediately* indicates that it's unphysical. To be explicit, let's consider the new scaling invariance in the **R** notation (4.19) for the one-point correlations up to second order

$$\tilde{t} = t, \qquad \tilde{\mathbf{x}} = \mathbf{x}, \qquad \tilde{\mathbf{U}} = e^q \mathbf{U}, \qquad \tilde{\boldsymbol{\tau}} = e^q \boldsymbol{\tau} + (e^q - e^{2q}) \cdot \mathbf{U} \otimes \mathbf{U}, \quad \cdots \tag{4.20}$$

where $\boldsymbol{\tau} = \langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ is the Reynolds-stress tensor, and compare it to the single and only scaling symmetry of the deterministic Navier-Stokes equations S (2.3), which, when transcribed into the statistical form of the **R** notation, will read

$$\tilde{t} = e^{2\varepsilon} t, \qquad \tilde{\mathbf{x}} = e^{\varepsilon} \mathbf{x}, \qquad \tilde{\mathbf{U}} = e^{-\varepsilon} \mathbf{U}, \qquad \tilde{\boldsymbol{\tau}} = e^{-2\varepsilon} \boldsymbol{\tau}, \quad \cdots \tag{4.21}$$

Although both (4.20) and (4.21) are mathematically admitted as invariant transformations of the underlying Reynolds-stress transport equations up to second order (Pope, 2000; Davidson, 2004), it is only transformation (4.20) which on this level of description immediately shows an artificial and thus a physically non-useful transformation behavior.

Thus, without making a connection to the underlying fluctuating dynamics we already can observe that (4.20) is actually a physically non-useful transformation just by solely inspecting expression (4.20). This is definitely not possible in the oversimplified $\mathbf{H}$ notation (4.4), and hence, therefore, one has the higher risk of being misguided when using this notation.

The reason why on this level of description (4.20) acts artificially and (4.21) not, is that in order to explicitly scale the values of the Reynolds-stress tensor $\boldsymbol{\tau} \to \tilde{\boldsymbol{\tau}}$ one has to involve the mean velocity field itself (in the quadratic form $\mathbf{U} \otimes \mathbf{U}$). But such a transformation (4.20) is not in accord with the idea of a Reynolds decomposition which has the intention to study turbulence statistics *relative* to the mean velocity field $\mathbf{U}$. The problem is that since the *untransformed* Reynolds-stress $\boldsymbol{\tau}$ is quadratically built up by a (zero-mean) fluctuating field $\mathbf{u}'$ which measures the mean stress *relative* to the mean velocity $\mathbf{U}$, the *transformed* quantity $\tilde{\boldsymbol{\tau}}$ in (4.20) doesn't have this 'relative measure'-property anymore because the values are now mixed with mean-velocity values. In other words, *within the transformed system* the quantity $\tilde{\boldsymbol{\tau}}$ *cannot* be identified as a Reynolds-stress anymore, which actually should measure the stress *relative* to the transformed mean velocity $\tilde{\mathbf{U}}$.

In this sense, transformation (4.20) is not physically useful, which we directly can also observe when fitting the resulting scaling laws to DNS data (see Section 5). The observed result will be a clear mismatch between theory and experiment, but, as soon as the unphysical structure of transformation (4.20) is excluded or removed from the scaling laws, the matching will improve again by several orders of magnitude which, ultimately, is a clear indication that the scaling (4.20) is unphysical.

Moreover, when returning back to the previously mentioned perspective where the additional scaling terms in (4.20) are only required to restore a misleading linear scaling within a nonlinear theory of moments, the artificial transformation behavior of (4.20) can also be immediately seen when generating invariant functions. Consider the following invariant one-point function of transformation (4.20)

$$\mathbf{F}(\mathbf{x}) = \frac{\boldsymbol{\tau} + \mathbf{U} \otimes \mathbf{U}}{\sqrt{\mathbf{U} \cdot \mathbf{U}}}. \tag{4.22}$$

Now, when explicitly performing this invariant transformation

$$\begin{aligned} \tilde{\mathbf{F}}(\tilde{\mathbf{x}}) &= \frac{\tilde{\boldsymbol{\tau}} + \tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}}{\sqrt{\tilde{\mathbf{U}} \cdot \tilde{\mathbf{U}}}} \\ &= \frac{e^q \boldsymbol{\tau} + (e^q - e^{2q}) \cdot \mathbf{U} \otimes \mathbf{U} + (e^q \mathbf{U}) \otimes (e^q \mathbf{U})}{\sqrt{(e^q \mathbf{U}) \cdot (e^q \mathbf{U})}} = \frac{\boldsymbol{\tau} + \mathbf{U} \otimes \mathbf{U}}{\sqrt{\mathbf{U} \cdot \mathbf{U}}} = \mathbf{F}(\mathbf{x}), \end{aligned} \tag{4.23}$$

we see how the transformation rule for the Reynolds-stress $\boldsymbol{\tau}$ acts artificially, in that one of its direct aims is to only cancel the disturbing nonlinear terms. Hence, it's highly questionable whether, and in which sense, the invariant function (4.22) is actually physically relevant, since its corresponding invariant transformation (4.20) is not incorporating the nonlinear terms into the transformation process itself but instead only treats them as 'error terms' which must be cancelled accordingly.

Finally, the reader should note that such a superfluous construction is not specific to the Navier-Stokes theory, i.e. the construction principle itself to yield the misleading type of invariance (4.19) is not unique or particular to the Navier-Stokes equations but can be established basically in any statistical system of any nonlinear theory. In other words, the superfluous type of linear scaling invariance (4.19) inherently also exists for example in any unclosed statistical model of the nonlinear Maxwell or the nonlinear Schrödinger equations (see Appendix E), by just reformulating the corresponding expressions accordingly. Hence, if one is not careful enough wrong conclusions will be the general consequence.

In a more general sense we can thus conclude that systematically ignoring any information about the functional structure of an either closed or an unclosed model equation, which is directly linked to an underlying theory in using an oversimplified representation (instead of an appropriate representation which explicitly reveals this information),

unawarely allows for generating unphysical and thus useless results when performing an analysis on invariance. This conclusion can be stated as the following general principle:

$\mathcal{P}$: *For every invariance analysis to be performed on an equation-based model which is linked to an underlying physical theory, it is crucial how the model equations are represented. It is necessary to reveal all information available for the system. If the notation tends to be oversimplified by not revealing all essential information, the analysis runs the risk to generate non-physical results without knowing.*

In other words, caution has to be exercised in knowing that mathematical notation, even if formally correct, always has the unfortunate ability to simplify or even oversimplify the actual physical situation and thus causing misguidance, or suggesting an intuition which by closer inspection is not supported.

4.4. *An example of a physically consistent statistical scaling symmetry*

We want to close Section 4 with a contrasting (positive) example of a statistical scaling symmetry admitted by the Hopf equation (3.2), which not only is compatible with all three constraints (3.4), but which also acts fully consistent on the coarse-grained (averaged) as well as on the fine-grained (fluctuating) level. The symmetry is

$$\mathsf{S}^{\mathsf{HEq}}: \quad \tilde{t} = e^{2\varepsilon} t, \quad \tilde{\mathbf{x}} = e^{\varepsilon} \mathbf{x}, \quad \tilde{\boldsymbol{\alpha}} = e^{-2\varepsilon} \boldsymbol{\alpha}, \quad \tilde{\Phi} = \Phi, \tag{4.24}$$

which is the only admitted physical (global) scaling symmetry S (2.3) of the Navier-Stokes equations just reformulated here for the Navier-Stokes-Hopf equation (3.2).[†] Note that (4.24) then induces the transformation rule for the

- *velocity field*: since the exponent inside the kernel of (3.3) should stay invariant, i.e. $d^3\tilde{\mathbf{x}} \cdot \tilde{\boldsymbol{\alpha}} \cdot \tilde{\mathbf{u}} = d^3\mathbf{x} \cdot \boldsymbol{\alpha} \cdot \mathbf{u}$, in order to consistently define a functional Fourier transform also in the transformed (scaled) domain, the velocity field must scale as

$$\tilde{\mathbf{u}} = e^{-\varepsilon} \mathbf{u}, \tag{4.25}$$

- *functional derivative*: since the functional derivative carries the physical dimension of the considered field variable per volume, it must scale as

$$\frac{\delta}{\delta \tilde{\boldsymbol{\alpha}}} = \frac{\partial}{\partial \tilde{\boldsymbol{\alpha}} d^3\tilde{\mathbf{x}}} = e^{-\varepsilon} \frac{\partial}{\partial \boldsymbol{\alpha} d^3\mathbf{x}} = e^{-\varepsilon} \frac{\delta}{\delta \boldsymbol{\alpha}}, \tag{4.26}$$

- *functional volume element*: since for path integrals the measure is of infinite size, it will scale accordingly for each field as

$$\begin{aligned} \mathcal{D}\tilde{\boldsymbol{\alpha}} &= \prod_{\tilde{\mathbf{x}}} \left(\sqrt{d^3\tilde{\mathbf{x}}/2\pi}\right)^3 \cdot d^3\tilde{\boldsymbol{\alpha}}(\tilde{\mathbf{x}}) \\ &= \prod_{\mathbf{x}} e^{-3\varepsilon/2} \left(\sqrt{d^3\mathbf{x}/2\pi}\right)^3 \cdot d^3\boldsymbol{\alpha}(\mathbf{x}) = \Big(\prod_{\mathbf{x}} e^{-3\varepsilon/2}\Big) \mathcal{D}\boldsymbol{\alpha}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \mathcal{D}\tilde{\mathbf{u}} &= \prod_{\tilde{\mathbf{x}}} \left(\sqrt{d^3\tilde{\mathbf{x}}/2\pi}\right)^3 \cdot d^3\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) \\ &= \prod_{\mathbf{x}} e^{+3\varepsilon/2} \left(\sqrt{d^3\mathbf{x}/2\pi}\right)^3 \cdot d^3\mathbf{u}(\mathbf{x}) = \Big(\prod_{\mathbf{x}} e^{+3\varepsilon/2}\Big) \mathcal{D}\mathbf{u}, \end{aligned} \tag{4.28}$$

[†]The 'official' theoretical development of extending classical point symmetry analysis from partial to functional differential equations is provided in Oberlack & Wacławczyk (2006), and recently also in Wacławczyk & Oberlack (2013*b*) adjusting it to Fourier space. However, it still lacks completeness, since the extension is based on an incomplete set of variables, in that the continuous index point $\mathbf{x}$ (in coordinate space) or $\mathbf{k}$ (in wavenumber space) are considered as being unchangeable quantities, which is *not* the case, simply because both variables carry a physical dimension which always, at least, must allow for a (re-)scaling in the units. A clear counter-example is given by (4.24). But also from a pure mathematical perspective, the independent variables $\mathbf{x}$ or $\mathbf{k}$ have to be included into the transformation process, even if they at most only act as integration (dummy) variables, nevertheless, their transformational change is always ruled by the Jacobian. Hence, by making sole use of the extended Lie algorithm developed in Oberlack & Wacławczyk (2006) and Wacławczyk & Oberlack (2013*b*), the fundamental scaling symmetry (4.24) can *not* be generated and essentially an important symmetry as (4.24) will thus be missed. For more details, we refer to our comments Frewer & Khujadze (2016*a*); Frewer *et al.* (2016*b*) and to our reactions Frewer & Khujadze (2016*b*); Frewer *et al.* (2016*c*), respectively.

where it should be noted that the continuous counting index (not the variable itself) stays invariant under transformation (4.24), i.e. $\prod_{\tilde{\mathbf{x}}} = \prod_{\mathbf{x}}$, since by $\tilde{\mathbf{x}} = e^{\varepsilon}\mathbf{x}$ any set of real numbers is mapped again in a one-to-one manner to a set of real numbers of the same measure,

- *probability density functional*: since the physical constraint $\int P[\mathbf{u};t]\mathcal{D}\mathbf{u} = 1$ must stay invariant, it has to scale as

$$\tilde{P}[\tilde{\mathbf{u}};\tilde{t}] = \Big(\prod_{\mathbf{x}} e^{-3\varepsilon/2}\Big) P[\mathbf{u};t], \tag{4.29}$$

which, in contrast to (4.15), makes an intuitive physical statement, in that if the system experiences a *global* change in scale on the fine-grained level $\mathbf{u} \to \tilde{\mathbf{u}}$ of type (4.25), then the system will experience this change in scale also on the coarse-grained level accordingly $P \to \tilde{P}$ (4.29).

- *n-point velocity correlation functions*: since the construction of all $\mathbf{H}_n$ in the Hopf framework are given according to rule (4.2), they scale as

$$\tilde{\mathbf{H}}_n = e^{-n\cdot\varepsilon}\,\mathbf{H}_n, \tag{4.30}$$

which is the only correct possible scaling behavior for the incompressible Navier-Stokes n-point velocity correlation functions. To date, no other statistical scaling symmetry exists!

5. Comparing to DNS results

This section will investigate if all statistical scaling laws which are based on the "new statistical scaling symmetry" Q_E (4.4) qualify as useful scaling laws. For geometrically simple wall-bounded flows the general construction principle to generate these laws as "first-principle results" in the inertial region is given in Oberlack & Rosteck (2010, 2011), which recently in Avsarkisov *et al.* (2014) got extended to include more sophisticated wall-bounded flows.[†] Of particular interest are those laws, which according to Oberlack & Rosteck (2010, 2011) should scale all higher order velocity moments beyond the log-law of the mean-velocity profile. The corresponding derivation procedure is revisited in Appendix F. Up to third moment, the explicit functional structure for all one-point velocity moments is derived in (F.10) and given as

$$\left.\begin{aligned}
U(y) &= \alpha_U \cdot \ln(y+c) + \beta_U,\\
\tau^{ij}(y) &= \alpha_H^{ij} + \beta_H^{ij}\cdot(y+c)^{\gamma} - \delta^{1i}\delta^{1j}U(y)^2,\\
T^{111} &= \alpha_H^{111} + \beta_H^{111}\cdot(y+c)^{2\gamma} - 3U(y)\cdot\tau^{11}(y) - U(y)^3,\\
T^{112} &= \alpha_H^{112} + \beta_H^{112}\cdot(y+c)^{2\gamma} - 2U(y)\cdot\tau^{12}(y),\\
T^{ij1} &= \alpha_H^{ij1} + \beta_H^{ij1}\cdot(y+c)^{2\gamma} - U(y)\cdot\tau^{ij}(y),\ \text{for } (i,j) = (2,2),(3,3),\\
T^{ij2} &= \alpha_H^{ij2} + \beta_H^{ij2}\cdot(y+c)^{2\gamma},\ \text{for } (i,j) = (2,2),(3,3).
\end{aligned}\right\} \tag{5.1}$$

At the example of ZPG turbulent boundary layer flow these "new statistical scaling laws", which by construction all apply in the inertial region of the flow, will be matched to DNS data. The investigation itself will be based on the method of least square fits (chi-square methods) for DNS data with a Reynolds number as high as $Re_\theta = 6000$. An open source software package which runs in Mathematica was used for all fitting needs, designed and programmed by Zielesny (2011). The DNS data was made available to us on the courtesy of Jiménez et al. (Simens *et al.*, 2009; Borrell *et al.*, 2013).

[†]In Oberlack & Rosteck (2010, 2011) as well as in Avsarkisov *et al.* (2014) all scaling laws for wall-bounded flows are actually based on two "new statistical symmetries". Next to the "new scaling symmetry" (4.4) also a "new translation symmetry" is involved, but which, as the scaling symmetry too, turns out to be completely unphysical. This can be easily demonstrated by using the same procedure as shown and developed in this article.

The curve fitting strategy is organized as follows: The complete fitting procedure will be based on an experimentally obtained fit for the mean streamwise velocity profile in the form of the modified (y-shifted) log-law according to (5.1):

$$U^+_{\text{exp}} = \alpha_{U,\text{exp}} \cdot \ln(y^+ + c_{\text{exp}}) + \beta_{U,\text{exp}}, \tag{5.2}$$

where all variables and parameters were normalized to wall units.[†] This y-shifted (non-classical) log-law (5.2) was first derived and proposed in Oberlack (2001), and then later re-derived in Oberlack & Rosteck (2010) as shown in Appendix F. The corresponding three parameters $\kappa_{\text{exp}} = 1/\alpha_{U,\text{exp}} = 0.38$, $\beta_{U,\text{exp}} = 4.1$ and $c_{\text{exp}} = 5.0$ are taken from Lindgren *et al.* (2004), which were determined as an average-fit over an ensemble of experimental data sets ranging in Re_θ from 2500 up to 27000. Hence, this particular specification in the log-law parameters should also apply to our chosen DNS data set, as its Reynolds number value of $Re_\theta = 6000$ lies within that ensemble range. The idea behind this strategy is to make use of the universal scaling behavior of the log-law, under the critical assumption, of course, that it's valid. In other words, if we assume the universality of the log-law to be correct, it is more than reasonable to take a well-established average-fit over an ensemble of experiments within a wide range of different Reynolds numbers and to imbed it into a numerical data set for a Reynolds number which lies inside that range. In this way, as will be done herein, one obtains a robust reference for all upcoming fits to be generated in this section.

Figure 1, left plot (L), convinces with a good comparison for our chosen DNS data set of $Re_\theta = 6000$ to other sets with different Reynolds numbers. Also the above universal modified log-law (5.2) is included to get an impression for its range of validity.

As the general aim of this section is to perform proper fits by using basic methods from statistical data analysis, the error of the considered data needs to be known. Fortunately the numerical error for some simulated quantities in the case of ZPG turbulent boundary layer flow are easily determined. Due to statistical symmetries in the flow the mean spanwise velocity component as well as all moments involving an uneven number of fluctuating spanwise velocity fields should be exactly zero by theory. However, every numerical simulation is and always will be unable to resolve these zero-fields exactly to zero. Hence the difference can be interpreted as the error of the simulation for that particular field, i.e. the mean spanwise velocity component W then serves as the error for U, and e.g. the Reynolds stress τ^{23} can serve as the error for τ^{22} and τ^{33}, etc.

5.1. *Curve fitting for $Re_\theta = 6000$ in the inertial region*

The course of action will follow the underlying statistical hierarchy, by first fitting the lowest order moments to be then used as input information to fit the next higher order ones. The process starts with Figure 1, right plot (R), for the mean streamwise velocity profile. It shows the result of comparison between the above in (5.2) specified and discussed log-law

$$U^+_{\text{exp}} = 1/0.38 \cdot \ln(y^+ + 5) + 4.1, \tag{5.3}$$

and the corresponding DNS data at $Re_\theta = 6000$. According to Lindgren *et al.* (2004) the fit should be valid down to about $y^+ \sim 100$. The upper limit is then fixed symmetrically by taking the same residual as found at the lower limit. This then gives a convincing fit ranging from $y^+ \sim 100$ to $y^+ \sim 330$, shown in Figure 1R as the shaded region, with a rather good reduced chi-square (χ^2_{red}) value of 75. Here the mean spanwise velocity W locally served in each point of the considered range as the underlying error field to determine the necessary quality of that fit.

[†]The normalization into wall units is based on the kinematic viscosity ν and the mean streamwise friction velocity $U_\tau = \sqrt{\nu \cdot \partial_y U|_{y=0}}$, which needs to be extracted from DNS data. In order to avoid an overloading of notation, the '+'-index in the parameters will be suppressed.

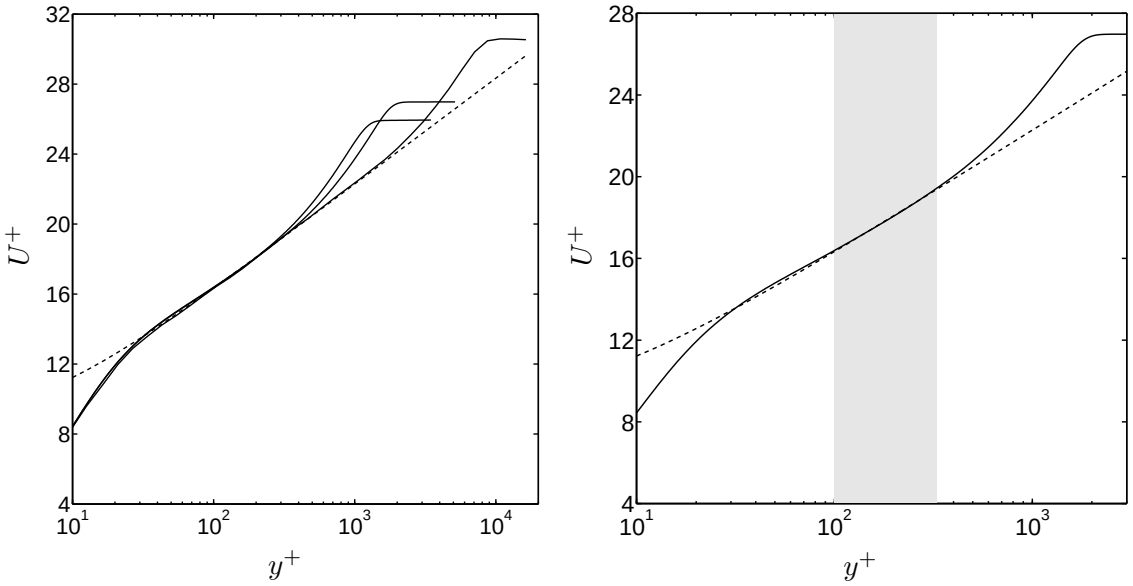

Figure 1: *Left plot:* Comparison between DNS data from Borrell *et al.* (2013) of the mean streamwise velocity profile for $Re_\theta = 6000$ (middle solid line from top end), and from Schlatter & Örlü (2010) for $Re_\theta = 4500$ (lower solid line from top end), and the experimental data set from Österlund *et al.* (2000) for $Re_\theta = 27000$ (upper solid line from top end). The dashed line shows the log-law (5.3), which was experimentally gained as an average-fit over an ensemble of Re_θ ranging from 2500 up to 27000 (Lindgren *et al.*, 2004).
Right plot: The solid line displays the DNS data of the mean streamwise velocity profile for $Re_\theta = 6000$ (Borrell *et al.*, 2013), while the dashed line shows again the experimentally fitted log-law (5.3). Inside the grey-shaded region, ranging from $y^+ = 100$ to 330, the statistical fitting error is $\delta_{\text{fit}} = 1.3 \cdot 10^{-3}$ and $\chi^2_{\text{red}} = 7.5 \cdot 10^1$ when referred to the underlying DNS error. Hereby the shaded region visualizes the inertial range, which from hereon will now consistently define the fitting domain in all plots.

As a result the log-law in Figure 1R thus slightly underfits the DNS data in this range, as its χ^2_{red}-value is above the optimal value of one (Zielesny, 2011). In particular, the rule is that *relative to the underlying numerical error of the data*, any χ^2_{red}-value smaller than one indicates a so-called *overfit* of the data meaning that the considered fitting range of the domain is possibly too small, while a value larger than one indicates an *underfit* of the data showing that either the considered fitting range of the domain is too large or that the considered function itself is inappropriate. However, despite this small underfit residing in Figure 1R we preferably gained an universal (Reynolds number independent) reference for all remaining scaling laws still to be fitted in this section — however only under the critical assumption of course that the universality principle holds.

Note that we excluded, on purpose, the lower end range from $y^+ \sim 30$ to $y^+ \sim 100$ as part of the log-region. Our aim here is to operate and to continue the investigation with a convincing fit for the log-law (5.3). Including the range below $y^+ \sim 100$ would only deteriorate the quality of the fit (by nearly one order of magnitude in χ^2_{red}), since this range clearly shows a small overshoot which noticeably deviates from a typical log-behavior in both the experimental and the DNS data as seen in Figures 1L and 1R.

Finally to note is that in *all* plots to be discussed from hereon, the solid line represents the DNS data for $Re_\theta = 6000$ from Borrell *et al.* (2013), while the dashed line represents the corresponding *best-fit* according to the functional form as given in (5.1). To enforce consistency, all fits will only be performed within the shaded region covering here the inertial range between $y^+ \sim 100$ and $y^+ \sim 330$. Additionally, all fits will be compared relative to the underlying DNS error in a consistent manner by employing either the corresponding zero-fields or the budget residuals. The explicit values of the fitted parameters will be stated up to three digits precise in the corresponding caption of each figure.

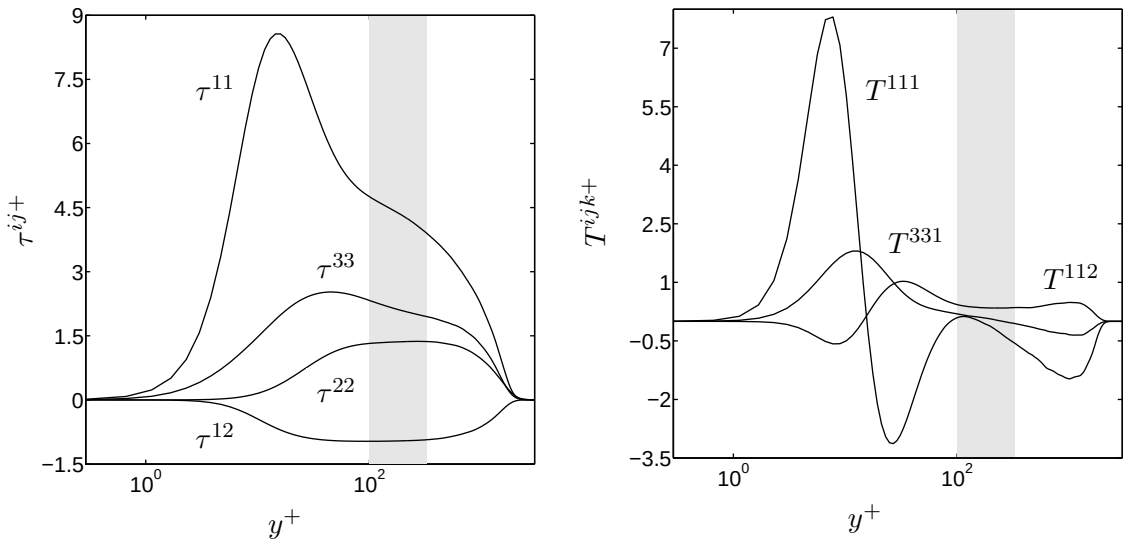

Figure 2: *Left plot:* All Reynolds stresses for $Re_\theta = 6000$ (Borrell *et al.*, 2013) plotted over the complete wall-normal range.
Right plot: A selection of third order velocity moments for $Re_\theta = 6000$ (Borrell *et al.*, 2013) plotted in the same wall-normal range as left. In both plots the grey-shaded area shows the inertial region.

Before we start investigating the higher order moments inside the shaded inertial range, it is helpful to first visualize the second and third order velocity moments in the complete wall-normal domain of the flow. While Figure 2L presents all second moments, Figure 2R only shows those third moments which were selected later on for a detailed investigation. In both plots the shaded inertial range was made visible in order to get a strong comparative impression of the complete functional structure inside and outside that domain.

Figure 3L shows the *best-fit* for the Reynolds-stress component (5.1)

$$\tau^{11+} = \alpha_H^{11} + \beta_H^{11} \cdot (y^+ + c_{\exp})^\gamma - U_{\exp}^{+2}, \tag{5.4}$$

the new mixed scaling law first derived in Oberlack & Rosteck (2010) as the zero-correlation-length limit of the corresponding 2-point function, which next to a power-law also includes the universal log-law $U_{\exp}^+$ (5.3). Figure 3R however shows the *best-fit* for

$$\hat{\tau}^{11+} = \hat{\alpha}_H^{11} + \hat{\beta}_H^{11} \cdot (y^+ + c_{\exp})^{\hat{\gamma}}, \tag{5.5}$$

an alternative three-parametric scaling law, which, just for the sake of interest, excludes the unphysical log-part $U_{\exp}^{+2}$ from the newly proposed scaling law (5.4). Our motivation to use (5.5) as comparison to (5.4) stems from the idea to observe the difference in functional behavior when a *physical* subgroup of $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2 \circ \mathsf{Q}_1 \circ \mathsf{Q}_2$ (F.1)-(F.5) is being considered, that is, to compare function (5.4) with an alternative function which is not linked to the unphysical "new statistical symmetries" Q_1 (F.4) and Q_2 (F.5).

When excluding these two from the group $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2 \circ \mathsf{Q}_1 \circ \mathsf{Q}_2$, the physical subgroup $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2$ is obtained, from which then, if the scaling in the mean velocity profile U (F.7) is not forced to be broken, only pure power laws as given by (5.5) can be induced as invariant functions, thereby explicitly demonstrating that the functional shift $U_{\exp}^{+2}$ in (5.4) has its origin solely in the unphysical "new statistical symmetries" Q_1 and Q_2. However, the reader should note that the power-law as specifically defined in (5.5) is not a genuine invariant function of the considered physical subgroup $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2$, due to the constant offset $\hat{\alpha}_H^{11}$ we explicitly included in order to ensure a same level of competitiveness as for function (5.4) which features three open parameters. In other words, although (5.5) *cannot* be identified as an exact invariant function of $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2$, it is nevertheless motivated through this physical transformation group to take the structural form of an invariant power-law, providing for (5.5) thus a far more stronger physical background than $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2 \circ \mathsf{Q}_1 \circ \mathsf{Q}_2$

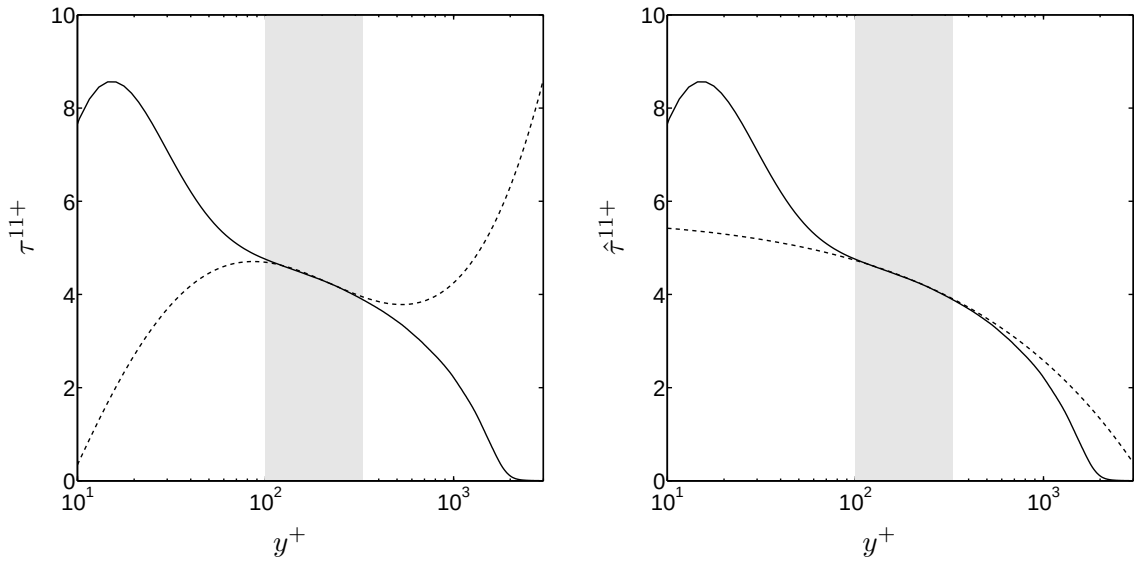

Figure 3: *Left plot:* Best-fit of τ^{11+} according to the new statistical scaling law (5.4). The corresponding parameters are $\alpha_H^{11} = -321.3$, $\beta_H^{11} = 302.6$, $\gamma = 0.145$, with the quality measures $\delta_{\text{fit}} = 5.5 \cdot 10^{-3}$, $\chi^2_{\text{red}} = 3.3 \cdot 10^2$.
Right plot: Best-fit of $\hat{\tau}^{11+}$ according to the pure power-law (5.5), without the unphysical log-squared-term U^{+2}_{exp} appearing in the new scaling law (5.4). The corresponding parameters are $\hat{\alpha}_H^{11} = 5.893$, $\hat{\beta}_H^{11} = -0.134$, $\hat{\gamma} = 0.464$, with the quality measures $\delta_{\text{fit}} = 1.9 \cdot 10^{-3}$, $\chi^2_{\text{red}} = 3.1 \cdot 10^1$.

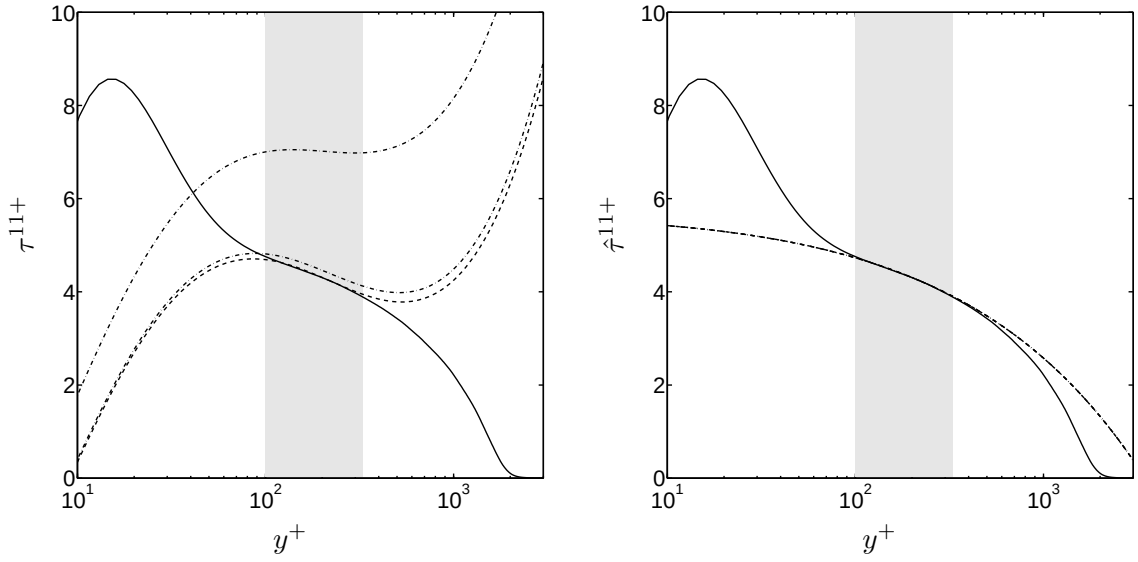

Figure 4: *Left plot:* Best-fit of τ^{11+} according to the new statistical scaling law (5.4) using different fitting precisions. It shows the high sensitivity against small perturbations in the parameters values, when going from six digits (lower and best matching fit), to four digits (middle fit) down to three digits (upper fit). The corresponding parameters are given in the left part of the table below.
Right plot: Best-fit of $\hat{\tau}^{11+}$ according to the power-law (5.5), without the unphysical log-squared-term U^{+2}_{exp} appearing in the new scaling law (5.4). In contrast to the fit in the left plot it shows complete insensitivity against small perturbations in the parameters values when going from six down to three digits. The corresponding parameters are given in the right part of the table below.

Digits	$\alpha_H^{11} \cdot 10^{-3}$	$\beta_H^{11} \cdot 10^{-3}$	γ	$\hat{\alpha}_H^{11} \cdot 10^{-1}$	$\hat{\beta}_H^{11}$	$\hat{\gamma}$
6	−0.321261	0.302558	0.144571	0.589339	−0.134415	0.463731
4	−0.3213	0.3026	0.1446	0.5893	−0.1344	0.4637
3	−0.321	0.303	0.145	0.589	−0.134	0.464

Parameters for Figure 4.

can provide for scaling law (5.4) — a difference which becomes easily visible now when determining and comparing their best fits.

Although the quality of both fits for (5.4) and (5.5) is comparable relative to the statistical measure δ_{fit}[†], we clearly see however that the fit in Figure 3L for (5.4) definitely has a far more *unnatural* functional behavior than the fit in Figure 3R for (5.5). In other words, although the fit on the left is of nearly equal quality as that on the right regarding its residual δ_{fit}, the scaling law (5.5), which *excludes* the unphysical log-term U^{+2}_{exp}, follows the DNS data more naturally on the right than the scaling law (5.4) on the left of Figure 3, which *includes* the log-term U^{+2}_{exp}. This is also quantitatively expressed by the fact that the two fitting parameters α^{11}_H and β^{11}_H in the left plot resulted in far higher values than $\hat{\alpha}^{11}_H$ and $\hat{\beta}^{11}_H$ in the plot on the right, showing that in order to optimally fit an unnatural behavior is at the expense of having to use large (unnatural) values in the parameters.

Additionally the fit in Figure 3L is by far more sensitive to small changes in the parameters than the fit in Figure 3R, which, in contrast to the left plot, shows a very robust behavior against small perturbations in the parameter values. In order to reproduce Figure 3L at least six digits of precision are necessary in the values for the fitting parameters: $\alpha^{11}_H = -0.321261 \cdot 10^3$, $\beta^{11}_H = 0.302558 \cdot 10^3$, $\gamma = 0.144571$. To reproduce however Figure 3R only three digits of precision turn out to be already sufficient. This very strong dependence upon small changes in the parameters demonstrates the artificial behavior of the new scaling law (5.4), as its corresponding fit in Figure 3L can only be generated under great effort in maintaining a high (unnatural) degree of precision. Figure 4 explicitly shows and compares this behavior, thus explicitly displaying that (5.4) essentially cannot be regarded as a physically useful scaling law.

Another measure is χ^2_{red}, which in Figure 3L is greater than one and by one magnitude larger than in Figure 3R, indicating that the mixed scaling law (5.4) of Oberlack et al. underfits the data more strongly than an alternative scaling law (5.5) with a pure power-law behavior would do, hence ultimately indicating again that relative to the underlying DNS error the pure power-law in Figure 3R fits the DNS data more naturally. For both fits the local error field was taken to be the zero-field τ^{13+}.

This fitting procedure is then repeated for the streamwise triple velocity moment T^{111+} shown in Figure 5L and Figure 5R. Without even referring to any statistical measures one clearly sees that the corresponding mixed scaling law of Oberlack et al. (5.1)

$$T^{111+} = \alpha^{111}_H + \beta^{111}_H \cdot (y^+ + c_{\text{exp}})^{2\gamma} - 3U^+_{\text{exp}} \cdot \tau^{11+} - U^{+3}_{\text{exp}}, \tag{5.6}$$

shown in Figure 5L, is completely incapable to predict the corresponding DNS data in the shaded inertial region correctly. To guarantee for a consistent fit of (5.6) the parameters γ and those of τ^{11+} were taken as determined in the previous fit for Figure 3L, thus essentially dealing only with a 2-parametric fitting function for α^{111}_H and β^{111}_H. In stark contrast, Figure 5R shows a huge improvement in the fitting results as soon as the two unphysical log-terms appearing in the above scaling law are removed, and, instead of (5.6), the following pure power-law is used

$$\hat{T}^{111+} = \hat{\alpha}^{111}_H + \hat{\beta}^{111}_H \cdot (y^+ + c_{\text{exp}})^{3\hat{\gamma}/2}. \tag{5.7}$$

As before for the Reynolds stress $\hat{\tau}^{11+}$ (5.5), this is again motivated by us to compare the newly proposed function (5.6) to an alternative scaling law which is not linked to the unphysical "new statistical symmetries" Q_1 (F.4) and Q_2 (F.5), as they form the origin of the peculiar functional shift $(-3U^+_{\text{exp}} \cdot \tau^{11+} - U^{+3}_{\text{exp}})$ in (5.6). Carefully note here again, that although the power-law for the triple moment (5.7) with exponent $3\hat{\gamma}/2$ has been consistently generated from the physical subgroup $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2 \subset \mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2 \circ \mathsf{Q}_1 \circ \mathsf{Q}_2$ (F.1)-(F.5) in accord with the power-law for the double moment (5.5) with exponent $\hat{\gamma}$,

[†]The dimensionless quality measure δ_{fit} is the normalized root mean squared error relative to the maximum value inside the considered domain to be fitted.

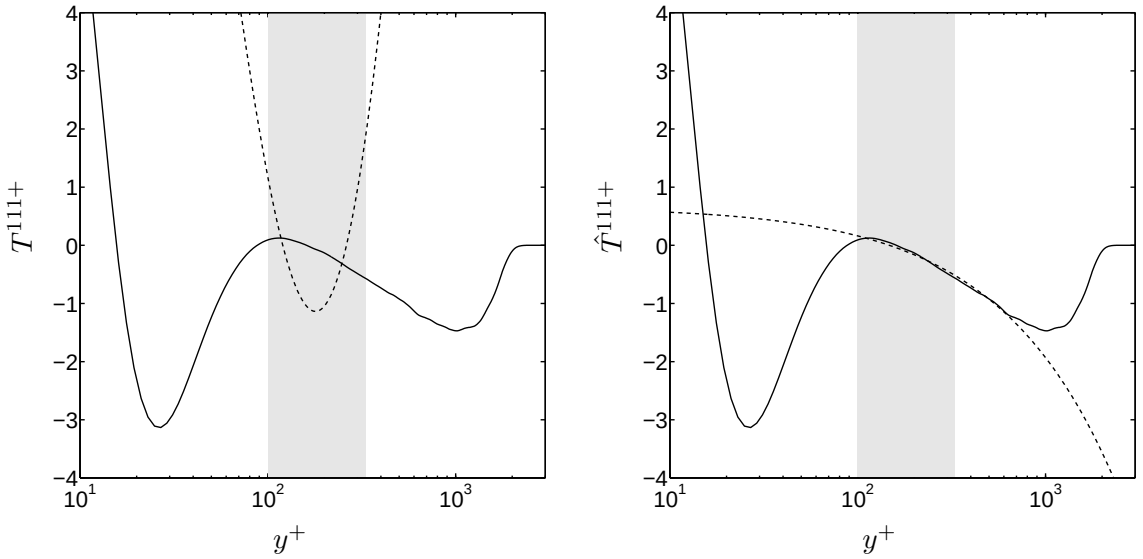

Figure 5: *Left plot:* Best-fit of T^{111+} according to the new statistical scaling law (5.6). Based on the result $\gamma = 0.145$ from Figure 3L, the best fitted parameters are $\alpha_H^{111} = -2755$, $\beta_H^{111} = 1913$, with the quality measures $\delta_{\text{fit}} = 1.8$, $\chi^2_{\text{red}} = 7.8 \cdot 10^6$.
Right plot: Best-fit of $\hat{T}^{111+}$ according to the pure power-law (5.7), excluding both physically inconsistent log-terms appearing in the new scaling law (5.6). Based on the result $\hat{\gamma} = 0.464$ from Figure 3R, the corresponding parameters are $\hat{\alpha}_H^{111} = 0.711$, $\hat{\beta}_H^{111} = -0.021$, with the quality measures $\delta_{\text{fit}} = 6.6 \cdot 10^{-2}$, $\chi^2_{\text{red}} = 1.7 \cdot 10^4$.

the complete scaling function as specifically defined in (5.7) is not a genuine invariant function of the considered physical subgroup $\mathsf{T} \circ \mathsf{S}_1 \circ \mathsf{S}_2$ itself. This is due to the constant offset $\hat{\alpha}_H^{111}$ which we included again to ensure for both functions (5.6) and (5.7) the same size of parameter space, as only this will allow for an equal and fair comparison.

Referring to χ^2_{red} in Figure 5, the improvement in choosing the pure power-law (5.7) instead of the mixed law (5.6) spans nearly three orders of magnitude. Of course, to guarantee for a consistent fit in this case, too, the parameter $\hat{\gamma}$ was taken from the previous result of Figure 3R. For both fits in Figure 5L and Figure 5R the local error field was approximated by the second moment zero-field $(\tau^{13+})^{3/2}$, as unfortunately no zero-fields for third order velocity moments were generated during the DNS.

Figure 6L shows the *best-fit* for the off-diagonal stress component (5.1)

$$\tau^{12+} = \alpha_H^{12} + \beta_H^{12} \cdot (y^+ + c_{\text{exp}})^{\gamma}, \tag{5.8}$$

where the local error field was taken to be the zero-field τ^{23+}. This result was then absorbed into the next fit shown in Figure 6R, which again clearly demonstrates the poor prediction ability of an Oberlack et al. proposed scaling law of mixed type. In particular, Figure 6R shows the *best-fit* of the triple velocity moment (5.1)

$$T^{112+} = \alpha_H^{112} + \beta_H^{112} \cdot (y^+ + c_{\text{exp}})^{2\gamma} - 2U^+_{\text{exp}} \cdot \tau^{12+}, \tag{5.9}$$

with the fixed parameter $\gamma = 0.145$, which was already consistently determined for τ^{11+} in Figure 3L. Here the local error field was approximated by the zero-field $(\tau^{23+})^{3/2}$.

Finally, Figure 7L shows the *best-fit* for the Reynolds-stress component (5.1)

$$\tau^{33+} = \alpha_H^{33} + \beta_H^{33} \cdot (y^+ + c_{\text{exp}})^{\gamma}, \tag{5.10}$$

and Figure 7R the *best-fit* for the third order velocity moment (5.1)

$$T^{331+} = \alpha_H^{331} + \beta_H^{331} \cdot (y^+ + c_{\text{exp}})^{2\gamma} - U^+_{\text{exp}} \cdot \tau^{33+}, \tag{5.11}$$

facing again the same poor quality issues. The zero-fields τ^{23+} and $(\tau^{23+})^{3/2}$ were respectively chosen again as the underlying DNS error.

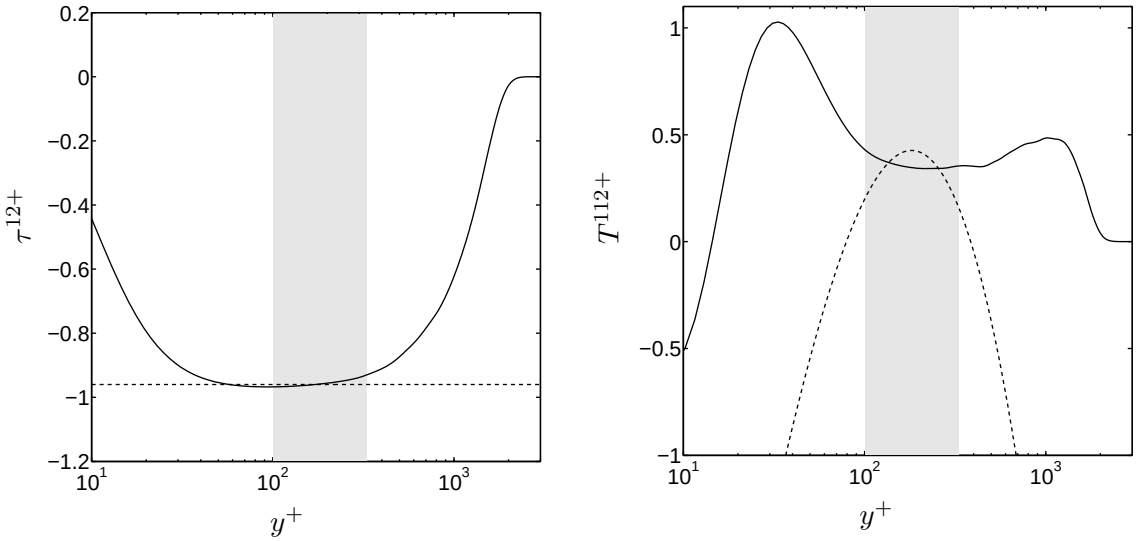

Figure 6: *Left plot:* Best-fit of τ^{12+} (5.8). The fitted parameters are $\alpha_H^{12} = -0.961$ and $\beta_H^{12} = 0$, with the quality measures $\delta_{\text{fit}} = 1.4 \cdot 10^{-2}$, $\chi^2_{\text{red}} = 1.5 \cdot 10^2$.
Right plot: Best-fit of T^{112+} according to the new statistical scaling law (5.9). Based on the result $\tau^{12+} = -0.961$ from the left plot in this figure, the best fitted parameters are $\alpha_H^{112} = -16.38$, $\beta_H^{112} = -3.851$, with the quality measures $\delta_{\text{fit}} = 2.2 \cdot 10^{-1}$, $\chi^2_{\text{red}} = 4.9 \cdot 10^6$.

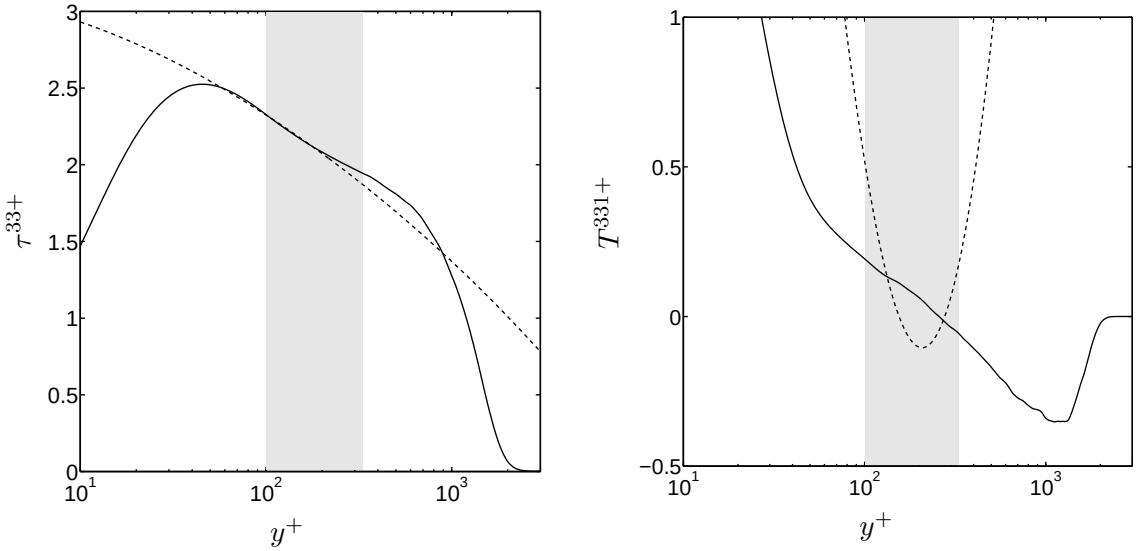

Figure 7: *Left plot:* Best-fit of τ^{33+} according to (5.10). Based on the result $\gamma = 0.145$ from Figure 3L, the parameters are $\alpha_H^{33} = 4.798$, $\beta_H^{33} = -1.262$, with the quality measures $\delta_{\text{fit}} = 1.4 \cdot 10^{-2}$, $\chi^2_{\text{red}} = 1.7 \cdot 10^2$.
Right plot: Best-fit of T^{331+} according to the new statistical scaling law (5.11). Based on the result τ^{33+} from the left plot in this figure, the best fitted parameters are $\alpha_H^{331} = 43.56$, $\beta_H^{331} = -1.315$, with the quality measures $\delta_{\text{fit}} = 7.3 \cdot 10^{-1}$, $\chi^2_{\text{red}} = 6.2 \cdot 10^5$.

6. Concluding remarks

The investigation done in the previous section could independently demonstrate that the "new statistical scaling laws", which in particular are based on the "new scaling symmetry" Q_E (4.4) recently proposed in Oberlack & Rosteck (2010), are by no means useful and, in our opinion, should be discarded in future work as they clearly fail to fulfil the most basic predictive requirements of a scaling law. This is addressed in particular to those scalings which are of mixed type, containing next to a power-law also a log-law, where the latter, due to using for the statistical moments an overall inconsistent invariance analysis in not properly incorporating the underlying deterministic theory, arise as unphysical terms proportional to the mean streamwise velocity profile.

Section 4 explicitly revealed this failure. When taking the perspective of the equivalence transformation Q_E (4.4) for the n-point velocity correlations, the reason is twofold, depending on the particular level of the statistical description:

i) The lower level *equivalence* Q_E (4.4) is *induced* by the higher level *symmetry* Q (4.1), which itself is physically inconsistent. By considering the fine- to coarse-grained transition rule (4.8), this inconsistency in Q (4.1) is exposed as a violation of cause and effect, in that the system on the statistical higher level would experience an effect without a corresponding cause.

ii) The invariance Q_E (4.4) is *admitted* by a specific system of equations (4.5) whose statistical representation hides essential information about the underlying deterministic system. Since nonlinear aspects of turbulence theory are not revealed, Q_E (4.4) misleadingly represents itself as an invariance which only linear systems can admit.

Both reasons now turn the invariance Q_E (4.4) into an unphysical transformation. Reason i), because the physical inconsistency on the higher level is directly transferred onto the lower level. Reason ii), because when revealing the hidden nonlinear information, the invariance gets broken as shown in (4.16), since the linear scaling, in which all system variables scale uniformly and independently from its coordinates, is fully incompatible with any nonlinear structure.

This very same conclusion also applies to the second "new statistical symmetry" (F.5), first proposed in Oberlack & Rosteck (2010) and recently again in Wacławczyk *et al.* (2014). Admitted by the unclosed system (4.5) as a translation equivalence, in which again the coordinates stay invariant and only the system variables get transformed due to the equation's oversimplified and thus misleading representation as a linear system of gradient-type, it straightforwardly can be exposed as a further unphysical invariance just by using the same procedure developed in this study.

Thus the claim made in Oberlack & Rosteck (2010) and Wacławczyk *et al.* (2014), in dealing with a first-principle construction method to generate scaling laws for wall-bounded turbulent flows does not comply, all the more so as the previous section clearly demonstrated a mismatch between theory and numerical experiment. For the third order moments this was more strongly pronounced than for the second order moments, and would most probably continue to decline in quality if the order of the moments is increased further. In fact, the quality of these fits relative to the measure χ^2_{red} declined drastically by several orders of magnitude as the moments increased to the next higher order. This just reflects the physical inconsistency of the invariance Q_E (4.4), which, as proven in (4.16), manifestly intensifies as the order of the transformed moments increase. The same is true for second unphysical invariance (F.5).

Important to note in this respect is that although the comparison was explicitly based only for the case of a ZPG turbulent boundary layer flow over a flat plate, it is obvious that all conclusions and results generalize and transfer in a one-to-one manner when comparing this "new scaling theory" to any other wall-bounded flow configuration, for example as to a channel-, pipe- or to a Couette-type of flow. For example, the result recently formulated in Avsarkisov *et al.* (2014) for a more sophisticated wall-bounded flow heavily relies

on those unphysical "new statistical symmetries". Thus, generating a useful statistical scaling law within Avsarkisov *et al.* (2014) for any higher order velocity moment, which goes beyond the mean velocity profile, is predetermined to fail when considering all facts discussed herein. Indeed, this failure is confirmed in Khujadze & Frewer (2016).

To conclude, we finally want to stress again in a brief historical outline that using the terminology 'invariant solution' in the theory of turbulence is more than misleading.

For example, the classical von Kármán log-law was definitely not derived as a solution of the (unclosed) statistical wall-bounded Navier-Stokes equations, but only as a self-similar candidate function upon pure dimensional arguments (Monin & Yaglom, 1971). In other words, although the classical von Kármán log-law is based on a dimensional scaling symmetry, it is definitely not a first principle solution; it only performs as an invariant function which *only possibly but not necessarily* can serve as a useful turbulent scaling function inside the inertial region. In this region also other different functions exist which all scale equally well, sometimes even better than the classical log-law (Barenblatt, 1993; Barenblatt & Prostokishin, 1993; Barenblatt *et al.*, 2014). The reason for this non-uniqueness is clearly to be seen as the consequence of the statistical closure problem of turbulence, in that no true solutions or true invariant solutions can be analytically constructed by just employing the method of Lie symmetry groups alone (Frewer *et al.*, 2014*b*; Frewer & Khujadze, 2016*c*). Hence, also a so-called 'advanced' or 'modern' invariance theory still faces the same problems as von Kármán had at his time.

Another prominent historical example which shows the complexity in the statistical description of turbulence is that of anomalous scaling and the breaking of global self-similarity, which both interdependently can be attributed to the complex property of intermittency (Frisch, 1985, 1995). For example in the inertial range of homogenous isotropic turbulence the results clearly show that the flow cannot be globally invariant under scaling, neither in a deterministic nor in a statistical sense. Inertial range intermittency when measured with the longitudinal multi-point structure functions show a clear lack of global statistical self-similarity (Frisch, 1995; Biferale *et al.*, 2003). The point we want to stress here is that even if we would only consider a highly idealized turbulent flow as that of a homogeneous isotropic turbulent flow, the statistical solutions, in particular the higher order correlations, are still by far more complicated than we currently can imagine and that it's actually unrealistic to believe that this complicated behavior can be captured by some *global* scaling symmetries. In particular, when realizing the fact that intermittency is essentially a property which rather breaks a symmetry than statistically restoring it.

Appendix A. Invariant solutions for underdetermined systems

Since the strong property of a Lie *symmetry* transformation only applies for fully determined equations (or overdetermined systems of equations) it can be exploited to construct invariant *solutions* of the considered equations. The reader should note that we emphasize the word *'solution'*. Because, for unclosed and thus underdetermined equations, for which only Lie *equivalence* transformations can be generated, this situation is different, as we want to explicitly demonstrate in the following.

For fully- or overdetermined systems of equations the word 'solution' is clearly defined. However, for *under*determined equations the word 'solution' is defined in a broader context. It is defined as a mathematical expression which directly solves the *under*determined equation without having to solve it again if more information is added to this equation, and, of course, that the solution must be *in principle* constructible without having to initially model the equation.

A.1. *Basic illustrative examples*

First consider the following algebraic but underdetermined (unclosed) equation for x

$$x^2 - y = 0. \tag{A.1}$$

Formally this equation can be solved for the desired variable x, to give the general solution

$$x = \pm\sqrt{y}, \tag{A.2}$$

as a set of infinitely many (non-unique) possible solutions, where the solution set itself has the geometrical structure of the parabola $y = x^2$. Note that at this stage all infinite solutions along this parabola are equally privileged, i.e. no preferred solution exists. But this situation of course will change as soon as we will get additional information about equation (A.1). Imagine to get the following additional information, either

i) that the unclosed value y can be uniquely constructed from some existing but still unknown equation $z(y) = 0$, i.e. that y is determined by a specific but analytically non-accessible process z which as a hidden process uniquely acts in the background,

or

ii) that the unclosed variable y is showing some uniquely existing but still unknown substructure $y = y(x)$, i.e. that y is an arbitrary but fixed function of x.

It's clear that although we give additional information as i) or ii), in both cases equation (A.1) remains underdetermined and thus unclosed. But, for its solution manifold the situation completely changed:

a) For information i) the general (infinitely non-unique) solution (A.2) turns into the unique (up to the option of choosing either plus or minus) but still unknown relation

$$x = \pm\sqrt{y_0}, \quad \text{with} \quad z(y_0) \equiv 0, \tag{A.3}$$

which, from all possible solutions (A.2) which lie on the parabola $y = x^2$, is now a privileged solution on this parabola. That means, once the underlying process z in (A.3) is known, all other (infinitely many) solutions lying on this parabola with $y \neq y_0$ must be discarded then as they no longer satisfy equation (A.3) anymore.

b) For information ii) the general solution (A.2) turns into an own unclosed equation for x

$$x = \pm\sqrt{y(x)}, \tag{A.4}$$

which needs to solved again for x, depending of course on the specification of the function $y(x)$.

In both cases we therefore cannot determine or construct a solution for x without modelling the process z or without modelling the substructure $y(x)$. Using the word 'solution' for the unmodelled expressions (A.3) or (A.4) would thus be completely misleading.

Hence, for underdetermined equations as given in (A.1) we can either construct non-unique and equally privileged (mostly infinitely many) solutions (A.2), or no solution at all if the unclosed term emerges from an underlying but unmodelled process (A.3), or if it shows an existing but unmodelled substructure (A.4).

The same reasoning also holds for differential equations. For example, consider the following simple but underdetermined (unclosed) first-order ODE for f

$$\frac{d}{dx} f(x) - g(x) = 0, \tag{A.5}$$

which, if the arbitrary function g is integrable, can be generally solved as

$$f(x) = f(x_0) + \int_{x_0}^{x} g(x')\, dx', \tag{A.6}$$

to give a solution set with infinitely many possible and equally privileged solutions for $f = f_g$ (infinite dimensional solution manifold), depending in each case on the particular specification of the arbitrary function g.

Now, important to note in the context we are considering in this study is that this infinite set of formal solutions can also include those functions which will stay invariant under a given invariance (equivalence) group. For example, consider the following Lie equivalence (scaling) transformation

$$\tilde{x} = e^{-\alpha}x, \qquad \tilde{f} = e^{\alpha}f, \qquad \tilde{g} = e^{2\alpha}g, \tag{A.7}$$

which leaves the unclosed equation (A.5) form-invariant

$$\frac{d}{d\tilde{x}}\tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) = 0. \tag{A.8}$$

Then, from (A.7), the following two invariant functions can be constructed

$$f(x) = \frac{c_f}{x}, \qquad g(x) = \frac{c_g}{x^2}, \tag{A.9}$$

which, when choosing the integration constants as $c_g = -c_f$, will satisfy (A.6) and thus will form a solution of (A.5). In other words, among the infinite possible and equally privileged solutions of (A.5), we have picked out one specific solution (A.9) which has the additional property of staying invariant under the arbitrarily chosen transformation group (A.7). But careful, there is no reason at all that solution (A.9) should be identified as a privileged or preferred solution among the infinite set of all other possible solutions (A.6). Relative to their corresponding equation (A.5) *all* solutions including (A.9) are still *equally* privileged — solution (A.9) only has some special additional transformational property, that's all!

In this sense we can define, in contrast to the strong form of a Lie-symmetry-based invariant solution, the opposite weak form of an invariant solution when based on a Lie *equivalence* transformation, namely as only being a particular and non-privileged solution within an infinite set of other possible and equally privileged solutions.

But, if we now would get the additional information that the arbitrary function $g(x)$ in (A.5) either i) can be uniquely determined from some existing but still unknown process h, e.g. from a functional relation of the form $h[g(x)] = 0$, or, ii) that it shows some existing but unknown substructure, e.g. in the functional form $g(x) = g[f(x), x]$, then the general solution (A.6) looses its status of being an explicit and constructible solution. Because, for i) the general non-unique solution (A.6) will turn into the unique and privileged relation

$$f(x) = f(x_0) + \int_{x_0}^{x} g_0(x')\,dx', \quad \text{with} \quad h[g_0(x)] \equiv 0, \tag{A.10}$$

while for ii) it will turn itself into an underdetermined (unclosed) integral equation

$$f(x) = f(x_0) + \int_{x_0}^{x} g[f(x'), x']\,dx', \tag{A.11}$$

for which in both cases no solution and thus also no invariant solution for $f(x)$ can be determined or constructed; of course, only as long as the functional h in (A.10) or the kernel g in (A.11) stays unspecified. In other words, no solution and thus also no invariant solution can be determined or constructed without invoking a modelling procedure for the underlying process h or the substructure g.

Hence, for an underdetermined (unclosed) differential equation as (A.5) we thus have the same situation as before for the unclosed algebraic equation (A.1), in that either infinitely many and equally privileged solutions (including all possible invariant solutions) can be constructed, or in that, depending on whether the unclosed term underlies an

unique but analytically non-accessible process (A.10) or on whether it shows an existing but analytically unknown substructure (A.11), no solutions and thus also no invariant solutions can be determined without a prior modelling assumption in both cases. It's clear that all arguments given in this example can be easily extended also to underdetermined (unclosed) PDEs.

A.2. *Examples from turbulence theory*

First consider the underdetermined (unclosed) differential system (2.7) of Example 2 in Section 2. We obviously face both issues i) and ii) as defined in the beginning of the previous Section A.1. First of all, the unclosed second moment $\mathbf{T}$ definitely shows a substructure since it can be uniquely determined from the underlying instantaneous (fluctuating) velocity field $\mathbf{u}$. Furthermore, since the mean velocity field $\langle\mathbf{u}\rangle$ is the most basic element which can be constructed from $\mathbf{u}$, the second moment $\mathbf{T}$ is mostly assumed to be in first approximation a functional of the following form (Pope, 2000; Davidson, 2004)

$$\mathbf{T} = \mathbf{T}\big[\langle\mathbf{u}\rangle, \nabla\otimes\langle\mathbf{u}\rangle\big]. \tag{A.12}$$

Hence, if this arbitrary functional $\mathbf{T}$ stays unspecified, then no solution and thus also no invariant solution can be determined, which thus corresponds to situation ii). Even if we would suppress the substructure and would only demand a dependence on the coordinates

$$\mathbf{T} = \mathbf{T}\big[\mathbf{x}, t\big], \tag{A.13}$$

which then, according to system (2.7), would allow (similar to (A.6)) for a formal construction of infinitely many and equally privileged solutions for the mean velocity field $\langle\mathbf{u}\rangle$, the usage of the word 'solution' would nonetheless be misleading in this case because we basically reside in situation i). The reason for it is twofold: Not only because the unclosed second moment (A.13) suffices an own unclosed one-point transport equation $\boldsymbol{\mathcal{E}}[\mathbf{T}] = \mathbf{0}$ (Pope, 2000; Davidson, 2004), which is structurally different to system (2.7), but also because since the second moment (A.13) is uniquely determined by the underlying instantaneous (fluctuating) velocity field $\mathbf{u}$, there can be only one physical (privileged) realization for $\mathbf{T}$. That means that all other solutions within this infinite dimensional solution manifold have to be discarded as unphysical, once this physical solution is determinable. But, the probability to find this particular specification (A.13) which belongs to this one physical solution (also within only a locally pre-specified spatiotemporal range) is practically zero. Even more unlikely is the case if this one particular specification and its corresponding physical solution would additionally stay invariant e.g. under the global scaling group E_2 (2.8). Hence, for this reason we claim that without a prior modelling assumption for the unclosed system (2.7) the determination of its solutions and thus also of its invariant solutions is misleading and essentially ill-defined.

A further, more general example is the infinite differential chain of n-point moment equations based on the full instantaneous velocity and pressure fields of the incompressible Navier-Stokes equations as presented in Oberlack & Rosteck (2010), and also recently in Wacławczyk *et al.* (2014)

$$\frac{\partial H_{i_{\{n\}}}}{\partial t} + \sum_{l=1}^{n}\left[\frac{\partial H_{i_{\{n+1\}}[i_{(n+1)}\mapsto k_{(l)}]}[\mathbf{x}_{(n+1)}\mapsto\mathbf{x}_{(l)}]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n-1\}}[l]}}{\partial x_{i_{(l)}}} - \nu\frac{\partial^2 H_{i_{\{n\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right] = 0, \quad n = 1,\ldots,\infty, \tag{A.14}$$

$$\left.\begin{aligned}&\frac{\partial H_{i_{\{n\}}[i_{(l)}\mapsto k_{(l)}]}}{\partial x_{k_{(l)}}} = 0, \quad \text{for } l = 1,\ldots,n\\ &\frac{\partial I_{i_{\{n-1\}}[k][i_{(l)}\mapsto m_{(l)}]}}{\partial x_{m_{(l)}}} = 0, \quad \text{for } k,l = 1,\ldots,n \text{ and } k\neq l,\end{aligned}\right\} \tag{A.15}$$

where (A.14) are the transport equations for the (equal-time) multi-point velocity correlation functions (3.1) of tensor order $n \geq 1$

$$H_{i_{\{n\}}} := H_{i_{(1)}i_{(2)}\ldots i_{(n)}} := \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)}) \rangle, \tag{A.16}$$

and the n-point pressure-velocity correlation functions of tensor order $(n-1)$

$$I_{i_{\{n-1\}}[l]} := \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \cdot \ldots \cdot u_{i_{(l-1)}}(\mathbf{x}_{(l-1)}) \cdot p(\mathbf{x}_{(l)}) \cdot u_{i_{(l+1)}}(\mathbf{x}_{(l+1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)}) \rangle, \tag{A.17}$$

along with (A.15) as the two continuity constraints.

In the following it will be helpful to briefly introduce the notation of Oberlack et al. The first index of $i_{\{n\}}$ in (A.16) defines the tensor character of the quantity $\mathbf{H}$, while the second index in braces denotes its tensor order. The curly brackets point out that not an index of a tensor but an enumeration is meant. On the other hand, the spatial component index $i_{(n)}$ runs in general from 1 to 3 for all points $n \geq 1$. For $n = 1$ one has the connection to the mean velocity field according to

$$H_{i_{\{1\}}} = H_{i_{(1)}} = \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \rangle =: U_{i_{(1)}}(\mathbf{x}_{(1)}), \tag{A.18}$$

and to the mean scalar pressure

$$I_{i_{\{0\}}[1]} =: I_{[1]} =: P(\mathbf{x}_{(1)}). \tag{A.19}$$

To note is that for convenience and a better readability we denote, in contrast to Oberlack & Rosteck (2010) and Wacławczyk *et al.* (2014), the instantaneous fields by small and all averaged fields by capital (latin) letters and, as well, we let all indices run from 1 and not from 0 upwards. Also note that in all definitions to follow the explicit time dependence in all functions will be suppressed. Next, if the list of indices gets interrupted by one or more other indices it is pointed out by attaching the replaced value in brackets to the index

$$H_{i_{\{n\}}[i_{(l)} \mapsto k_{(l)}]} := \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \cdot \ldots \cdot u_{i_{(l-1)}}(\mathbf{x}_{(l-1)}) \cdot u_{k_{(l)}}(\mathbf{x}_{(l)}) \cdot u_{i_{(l+1)}}(\mathbf{x}_{(l+1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)}) \rangle, \tag{A.20}$$

which is further extended by

$$H_{i_{\{n+1\}}[i_{(n+1)} \mapsto k_{(l)}]}[\mathbf{x}_{(n+1)} \mapsto \mathbf{x}_{(l)}] := \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)}) \cdot u_{k_{(l)}}(\mathbf{x}_{(l)}) \rangle \tag{A.21}$$

$$=: \widehat{H}_{i_{\{n+1\}}[l]}, \tag{A.22}$$

where not only the index $i_{(n+1)}$ is replaced by $k_{(l)}$, but also the independent variable $\mathbf{x}_{(n+1)}$ is replaced by $\mathbf{x}_{(l)}$. If it is clear from the context, quantity (A.21) will be constantly abbreviated as (A.22).

Now, although infinite in dimension, the hierarchal system (A.14)-(A.15) is by construction unclosed and thus underdetermined, where, due to that $\widehat{H}_{i_{\{n+1\}}[l]} \neq H_{i_{\{n+1\}}}$,[†] the lower dimensional moments $\widehat{H}_{i_{\{n+1\}}[l]}$ (A.22) are to be identified as the unclosed terms since they do not *directly* enter the system's next higher order correlation equation (for a more detailed explanation we refer to Appendix C).

Exactly as in the example before, system (A.14)-(A.15) again represents a specific underdetermined system of equations for which no solutions can be determined endogenously. The obvious reason is that we again face both issues i) and ii) as defined in the beginning

[†]Note that although *all* components of $\widehat{H}_{i_{\{n+1\}}[l]}$ (A.22) can be uniquely constructed from the higher dimensional moments $H_{i_{\{n+1\}}}$ once they are known, which can be formally written as the process $\widehat{H}_{i_{\{n+1\}}[l]} = \lim_{\mathbf{x}_{(n+1)} \to \mathbf{x}_{(l)}} H_{i_{\{n+1\}}}$, the necessary inverse construction, however, fails.

of Section A.1. First of all, every unclosed term $\widehat{H}_{i_{\{n+1\}}[l]}$ (A.22) definitely shows a substructure as they all can be uniquely determined from a single instantaneous (fluctuating) velocity field $\mathbf{u}$, and therefore will show at each order n a high correlation at least among its neighboring orders, i.e. in first approximation they will at least be functionals of the form (for all $k = 1, \ldots, n$):

$$\widehat{H}_{i_{\{n+1\}}[l]} = \widehat{H}_{i_{\{n+1\}}[l]}\Big[H_{i_{\{n\}}};\ \widehat{H}_{i_{\{n+1\}}[k\neq l]};\ \lim_{(\mathbf{x}_{(n+2)},\mathbf{x}_{(n+1)})\to\mathbf{x}_{(k)}} H_{i_{\{n+2\}}}\Big]. \tag{A.23}$$

Hence, if these arbitrary functionals $\widehat{H}_{i_{\{n+1\}}[l]}$ stay unspecified, then no solutions and thus also no invariant solutions can be determined. Secondly, even if we would suppress the substructure and would only demand a dependence on the coordinates

$$\widehat{H}_{i_{\{n+1\}}[l]} = \widehat{H}_{i_{\{n+1\}}[l]}\big[\mathbf{x}_{(k)}, t\big], \tag{A.24}$$

which then, according to system (A.14)-(A.15), would allow (similar again to (A.6)) for a formal construction of infinitely many and equally privileged solutions $H_{i_{\{n\}}}$ for each order n, the usage of the word 'solution' would be again still misleading.

As before, the reason is again twofold: Not only because the unclosed terms (A.24), defined by (A.21), suffice own unclosed transport equations $\mathcal{E}_n[\widehat{H}_{i_{\{n+1\}}[l]}] = 0$ (Pope, 2000; Davidson, 2004), which are structurally different to system (A.14)-(A.15) due to the non-commuting property of the zero-correlation-length limit (see Appendix C), but also because since all arbitrary functions (A.24) are uniquely determined by the underlying instantaneous (fluctuating) velocity field $\mathbf{u}$, there can be only one physical (privileged) realization for each order n. That means that all other solutions within this infinite dimensional solution manifold have to be discarded as unphysical, once this physical solution is determinable. But, the probability again to find this particular specification (A.24) which belongs to this one physical solution (also within only a locally pre-specified spatiotemporal range) is practically zero. Even more unlikely is the case if this one particular specification and its corresponding physical solution would additionally stay invariant e.g. under the global scaling group Q_E (4.4). Hence, for this reason we again claim that without any prior modelling assumption for the unclosed system (A.14)-(A.15), the determination of its solutions and thus also of its invariant solutions is misleading and essentially ill-defined.

Appendix B. Formal derivation of the Friedmann-Keller hierarchy

Following the procedure from Fursikov (1999), we briefly revisit the formal derivation of the integro-differential Friedmann-Keller chain of equations for incompressible and spatially unbounded Navier-Stokes turbulence.

The starting point is the deterministic Navier-Stokes equation in differential form (2.1). The pressure gradient can be eliminated by acting to both sides of (2.1) with the operator $\boldsymbol{\mathcal{P}}$ of orthoprojection to solenoidal vector fields. As a result we then get the following equations being absent of the gradient pressure field

$$\partial_t \mathbf{u} - \boldsymbol{\mathcal{P}}\left[\nu\Delta\mathbf{u}\right] + \boldsymbol{\mathcal{P}}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \mathbf{0}, \quad \text{with } \nabla\cdot\mathbf{u} = 0, \tag{B.1}$$

which in a more compact form can also be written as

$$\partial_t \mathbf{u} + \mathbf{A}\cdot\mathbf{u} + \mathbf{B}(\mathbf{u}) = \mathbf{0}, \tag{B.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ is the corresponding linear and nonlinear (quadratic) operator respectively. The explicit expression for $\boldsymbol{\mathcal{P}}$ acting on any arbitrary vector-field $\mathbf{f}$ has the form

$$\boldsymbol{\mathcal{P}}[\mathbf{f}] = \int d^3\mathbf{x}'\, \boldsymbol{\rho}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{f}(\mathbf{x}'), \tag{B.3}$$

where the kernel $\boldsymbol{\rho}$ is given by

$$\begin{aligned}\boldsymbol{\rho}(\mathbf{x}-\mathbf{x}') &= \delta^3(\mathbf{x}-\mathbf{x}')\cdot\mathbf{1} + \nabla\otimes\nabla\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\right)\\ &= \frac{2}{3}\delta^3(\mathbf{x}-\mathbf{x}')\cdot\mathbf{1} - \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\left[\mathbf{1} - \frac{3(\mathbf{x}-\mathbf{x}')\otimes(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right].\end{aligned} \tag{B.4}$$

By construction, the projection properties of $\boldsymbol{\mathcal{P}}$ are

$$\left.\begin{aligned}\boldsymbol{\mathcal{P}}[\mathbf{f}] &= \mathbf{f}, \quad \text{if } \nabla\cdot\mathbf{f} = 0,\\ \boldsymbol{\mathcal{P}}[\mathbf{f}] &= \mathbf{0}, \quad \text{if } \nabla\times\mathbf{f} = \mathbf{0}, \text{ e.g. if } \mathbf{f} = \nabla\phi.\end{aligned}\right\} \tag{B.5}$$

With these definitions at hand, it is now straightforward to show that the corresponding dynamical equations for the n-point velocity moments $\mathbf{H}_n$ (3.1) can be iteratively organized into the following infinite hierarchy of *linear* equations, known as the (integro-differential) Friedmann-Keller equations:

$$\partial_t\mathbf{H}_n + \boldsymbol{\mathcal{A}}_n\cdot\mathbf{H}_n + \boldsymbol{\mathcal{B}}_n\cdot\widehat{\mathbf{H}}_{n+1} = \mathbf{0}, \quad n\geq 1, \tag{B.6}$$

where the two spatial linear operators are defined as

$$\boldsymbol{\mathcal{A}}_n\cdot\mathbf{H}_n = \sum_{i=1}^{n}\boldsymbol{\mathcal{P}}_{\mathbf{x}_i}\left[-\nu\Delta_{\mathbf{x}_i}\mathbf{H}_n\right], \tag{B.7}$$

$$\boldsymbol{\mathcal{B}}_n\cdot\widehat{\mathbf{H}}_{n+1} = \sum_{i=1}^{n}\boldsymbol{\mathcal{P}}_{\mathbf{x}_i}\left[\nabla_{\mathbf{x}_i}\cdot\widehat{\mathbf{H}}_{i,n+1}\right], \quad \text{with } \widehat{\mathbf{H}}_{i,n+1} = \mathbf{H}_{n+1}\big|_{\mathbf{x}_{n+1}=\mathbf{x}_i}, \tag{B.8}$$

in which the projection operator $\boldsymbol{\mathcal{P}}_{\mathbf{x}_i}$, as well as the differential operators $\Delta_{\mathbf{x}_i}$ and $\nabla_{\mathbf{x}_i}$ are to be evaluated for each summand at the specific point $\mathbf{x}=\mathbf{x}_i$, for all $i=1,\ldots,n$.

Two important things should be noted here. Firstly, since $\widehat{\mathbf{H}}_{n+1}\neq\mathbf{H}_{n+1}$, i.e. since $\widehat{\mathbf{H}}_{n+1}$ is not a $(n+1)$-point function but only a lower level n-point function of $(n+1)$-th moment, the hierarchy of equations (B.6), although infinite in dimension, is not formally closed. For more details we refer to Appendix C. Hence, the hierarchy of equations (B.6) is always a permanently underdetermined system, for which any invariance analysis only will generate the weaker class of equivalence transformations.

Secondly, great care has to be taken when performing a systematic invariance analysis upon the linear system (B.6). Because, as no natural or additional physical constraints come along with this system, it can lead to misleading and ultimately to unphysical invariance results when not revealing the actual nonlinear and nonlocal structure behind the oversimplified notation for the symbols $\mathbf{H}_n$ and $\widehat{\mathbf{H}}_{n+1}$. In other words, system (B.6) represents itself as a formal linear system only because an oversimplified notation is being used, which hides essential information about the underlying deterministic system. In this sense it admits formal equivalences which also every underdetermined linear system of gradient-type will admit, but as soon as one unfolds the oversimplified notation, as thoroughly discussed in Section 4 at the example of a scaling equivalence, these equivalences lead to physical inconsistencies.

Appendix C. Formally closed and unclosed infinite systems

For this discussion let us consider all three statistical and infinite dimensional approaches to turbulence:

Approach 1: The functional Hopf equation (HEq)

$$\frac{\partial\Phi}{\partial t} = \int d^3\mathbf{x}\,\alpha_k\left(i\frac{\partial}{\partial x_l}\frac{\delta^2}{\delta\alpha_k\delta\alpha_l} + \nu\Delta\frac{\delta}{\delta\alpha_k}\right)\Phi, \tag{C.1}$$

for the probability density functional

$$P[\mathbf{u}(\mathbf{x});t] = \int \Phi[\boldsymbol{\alpha}(\mathbf{x});t]\, e^{-i\int d^3\mathbf{x}\,\boldsymbol{\alpha}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})}\mathcal{D}\boldsymbol{\alpha}(\mathbf{x}) \geq 0, \tag{C.2}$$

of the velocity field sampled at infinitely many, non-denumerable (continuum) number of points (Hopf, 1952; McComb, 1990; Shen & Wray, 1991). The evolution equation (C.1) goes along with the three physical constraints

$$\Phi^*[\boldsymbol{\alpha}(\mathbf{x});t] = \Phi[-\boldsymbol{\alpha}(\mathbf{x});t], \quad \Phi[0;t] = 1, \quad \big|\Phi[\boldsymbol{\alpha}(\mathbf{x});t]\big| \leq 1, \tag{C.3}$$

in order to guarantee for a physical solution for all times t.

Approach 2: The integro-differential Lundgren-Monin-Novikov (LMN) hierarchy

$$\left[\partial_t + \sum_{i=1}^{n}\mathbf{v}_i\cdot\boldsymbol{\nabla}_i\right] f_n = -\sum_{i=1}^{n}\frac{\partial}{\partial\mathbf{v}_i}\Big[\lim_{\mathbf{x}_{n+1}\to\mathbf{x}_i}\nu\Delta_{n+1}\int d^3\mathbf{v}_{n+1}\mathbf{v}_{n+1}f_{n+1} - \int d^3\mathbf{x}_{n+1}d^3\mathbf{v}_{n+1}\left(\boldsymbol{\nabla}_i\frac{1}{4\pi|\mathbf{x}_i-\mathbf{x}_{n+1}|}\right)(\mathbf{v}_{n+1}\cdot\boldsymbol{\nabla}_{n+1})^2 f_{n+1}\Big], \tag{C.4}$$

for the n-point probability density function (PDF)

$$f_n = f_n(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_n,\mathbf{v}_n;t) \geq 0, \tag{C.5}$$

of the velocity field sampled at a finite number of points, being thus the discrete version of the above Hopf equation (Lundgren, 1967; Monin, 1967; Friedrich *et al.*, 2012). Apart from the usual continuity constraints, the evolution equation (C.4) goes along with infinitely many physical constraints in order to guarantee for a physical solution for all times t. The most cited ones are i) the reduction or normalization constraint

$$\int d\mathbf{v}_1\, f_1(\mathbf{x}_1,\mathbf{v}_1;t) = 1,$$
$$\int d\mathbf{v}_{n+1}\, f_{n+1}(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_{n+1},\mathbf{v}_{n+1};t) = f_n(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_n,\mathbf{v}_n;t), \quad n\geq 1, \tag{C.6}$$

ii) the coincidence constraint for $n \geq 2$, and $1 \leq (i,j) \leq n$, $i \neq j$

$$\int d\mathbf{x}_i\,\delta(\mathbf{x}_i-\mathbf{x}_j)f_n(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_n,\mathbf{v}_n;t) \sim \lim_{|\mathbf{x}_i-\mathbf{x}_j|\to 0} f_n(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_n,\mathbf{v}_n;t)$$
$$= f_{n-1}(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_{i-1},\mathbf{v}_{i-1};\mathbf{x}_{i+1},\mathbf{v}_{i+1};\ldots;\mathbf{x}_n,\mathbf{v}_n;t)\cdot\delta(\mathbf{v}_i-\mathbf{v}_j), \tag{C.7}$$

and iii) the separation property for $n \geq 2$, and $1 \leq (i,j) \leq n$, $i \neq j$, with $|\mathbf{x}_j| < \infty$

$$\lim_{|\mathbf{x}_i-\mathbf{x}_j|\to\infty} f_n(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_n,\mathbf{v}_n;t)$$
$$= f_{n-1}(\mathbf{x}_1,\mathbf{v}_1;\ldots;\mathbf{x}_{i-1},\mathbf{v}_{i-1};\mathbf{x}_{i+1},\mathbf{v}_{i+1};\ldots;\mathbf{x}_n,\mathbf{v}_n;t)\cdot f_1(\mathbf{x}_i,\mathbf{v}_i;t). \tag{C.8}$$

Yet the reader should note that besides these three usually mentioned LMN constraints as 'normalization', 'coincidence' and 'separation', there exists a fourth and even more strong constraint which unfortunately is not mentioned anymore in the recent literature, as e.g. in Friedrich *et al.* (2012). We are talking about the additional constraint first derived in Ievlev (1970) (listed therein as constraint (2.6)), and also presented in Monin & Yaglom (1975) as constraint (19.139).

Approach 3,1: The differential Friedmann-Keller or multi-point correlation (MPC) hierarchy

$$\frac{\partial H_{i_{\{n\}}}}{\partial t} + \sum_{l=1}^{n}\left[\frac{\partial H_{i_{\{n+1\}}[i_{(n+1)}\mapsto k_{(l)}]}[\mathbf{x}_{(n+1)}\mapsto\mathbf{x}_{(l)}]}{\partial x_{k_{(l)}}} + \frac{\partial I_{i_{\{n-1\}}[l]}}{\partial x_{i_{(l)}}} - \nu\frac{\partial^2 H_{i_{\{n\}}}}{\partial x_{k_{(l)}}\partial x_{k_{(l)}}}\right] = 0, \quad n\geq 1, \tag{C.9}$$

for the n-point velocity moments (A.16)

$$H_{i_{\{n\}}} := H_{i_{(1)}i_{(2)}\ldots i_{(n)}} := \langle u_{i_{(1)}}(\mathbf{x}_{(1)}) \cdot \ldots \cdot u_{i_{(n)}}(\mathbf{x}_{(n)}) \rangle, \tag{C.10}$$

and the n-point pressure-velocity moments as defined in (A.17), where both sets of moments are based on the full instantaneous fields of the incompressible Navier-Stokes equations (Oberlack & Rosteck, 2010; Wacławczyk *et al.*, 2014). Note that

$$\widehat{H}_{i_{\{n+1\}}[l]} := H_{i_{\{n+1\}}[i_{(n+1)} \mapsto k_{(l)}]}[\mathbf{x}_{(n+1)} \mapsto \mathbf{x}_{(l)}] = \lim_{\mathbf{x}_{(n+1)} \to \mathbf{x}_{(l)}} H_{i_{\{n+1\}}}, \tag{C.11}$$

is not a $(n+1)$-point moment, but only a n-point moment of $(n+1)$th order, i.e. $\widehat{H}_{i_{\{n+1\}}[l]} \neq H_{i_{\{n+1\}}}$.

Approach 3,2: The integro-differential Friedmann-Keller or multi-point correlation (MPC) hierarchy

$$\partial_t \mathbf{H}_n + \boldsymbol{\mathcal{A}}_n \cdot \mathbf{H}_n + \boldsymbol{\mathcal{B}}_n \cdot \widehat{\mathbf{H}}_{n+1} = \mathbf{0}, \quad n \geq 1, \tag{C.12}$$

for the n-point velocity moments

$$\mathbf{H}_n = \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle, \quad n \geq 1, \tag{C.13}$$

based on the instantaneous velocity field $\mathbf{u}$ (Fursikov, 1999). The integral operators $\boldsymbol{\mathcal{A}}_n$ and $\boldsymbol{\mathcal{B}}_n$ are defined in (B.7) and (B.8) respectively. Note that

$$\widehat{\mathbf{H}}_{n+1} = \lim_{\mathbf{x}_{n+1} \to \mathbf{x}_n} \mathbf{H}_{n+1}, \tag{C.14}$$

is not a $(n+1)$-point moment, but only a n-point moment of $(n+1)$th order, i.e. $\widehat{\mathbf{H}}_{n+1} \neq \mathbf{H}_{n+1}$.

In the discussion to follow, it is essential to recognize that in contrast to the Hopf equation and the LMN hierarchy, the MPC equations do *not* go along with additional physical constraints (besides the usual continuity constraints), neither in the differential nor in the integro-differential form.

Now, before we investigate these three infinite-dimensional approaches on formal closure, it is necessary to distinguish the terminology of *'closed'* from *'formally closed'*. The latter is obviously a much more weaker concept than the former.

First of all, all three statistical approaches to turbulence are *unclosed*, because in a *practical* sense one either has to discretize the continuous functional formulation or to truncate the infinite discrete hierarchy of equations. Discretizing the Hopf equation leads to the LMN equations and truncating these at a certain order then leads to unclosed terms (arbitrary functions) which need to be modelled in order to close the equations in each case. The same situation we face for the MPC equations in either form.

But, if we would *not* discretize the Hopf equation or truncate either system of LMN and MPC, i.e. if we would *formally* consider for each approach the full continuous and infinite formulation, then from a formal point of view, for example from the viewpoint of an invariance analysis, the Hopf and LMN equations act completely different than the MPC equations: In contrast to the MPC equations, the Hopf and LMN equations act in a *formally closed* manner. In other words, on the *formal (non-truncated)* level the Hopf equation and the LMN equations can be regarded as formally closed systems, while the MPC equations not.

The key point here is that since the LMN system is just the discrete version of the functional Hopf equation, which for itself undoubtedly acts as a formally closed system, the LMN system will thus induce in its *non-truncated* form an infinite dimensional but functionally *unique* solution manifold. This is not the case for the MPC system, which in its *non-truncated* form would induce infinitely many functionally different and thus *non-unique* solutions manifolds, each being itself of course infinite dimensional and equally privileged.

This different *formal* behavior can only be understood when directly comparing the LMN system (which in its well-known form only holds for spatially *unbounded* flow configurations) with the MPC system: Next to the usual continuity constraints, the LMN system goes along with several additional and independent physical constraints, which all will naturally restrict the general solution space down to a physical (unique) solution space (like boundary conditions restricting the general solution space for usual PDEs), while the MPC system is completely free of such constraints (up to the usual continuity constraints). In other words, the infinite many *unphysical* solution manifolds of the MPC system cannot be separated from the *physical* solution manifold as it automatically happens in the LMN system.

This restriction in the solution manifold of the LMN system can already be observed when performing any invariance analysis upon them. Because, for such an analysis the LMN constraints are obstructive, as they all favor the mechanism of symmetry breaking. In strong contrast of course to a performed invariance analysis of the MPC system, where, due to the absence of physical constraints, no invariance breaking mechanism exists; *all* admitted invariant transformations of the MPC system, physical or unphysical, thus agglomerate to the invariant solution manifold.

In order to demonstrate the difference between 'formally unclosed' and 'formally closed' in an explicit manner, let's consider e.g. the following infinite chain of second order PDEs for the different n-dimensional scalar functions $f_n := f_n(x_1, x_2, \ldots, x_n)$

$$\mathcal{Z}[f_n] := \frac{\partial^2 f_n}{\partial x_n^2} + \frac{\partial}{\partial x_n}\left[\lim_{x_{n+1}\to x_n} f_{n+1}\right] = 0, \quad \text{for} \quad n = 1, \ldots, \infty, \tag{C.15}$$

which *in principle* should mimic the basic behavior of the MPC equations (C.9) in a very primitive form: The first term in (C.15) stands for the dissipative term and the second one for the convective term.

First of all, the infinite hierarchy (C.15) is unclosed, because if we would truncate this system at an arbitrary but fixed order $n = n_0$, we would explicitly gain the unclosed and thus arbitrary function f_{n_0+1}, which then needs to be modelled in order to close this system at order n_0.

But now, since (C.15) is free of any constraints, it is also *formally unclosed*, i.e. even if we would *not* truncate system (C.15) and therefore would formally consider *all* (infinite) equations, system (C.15) still has to be regarded as unclosed. The simple reason is that infinitely many disjoint and thus different solution manifolds can be generated, i.e. system (C.15) does not induce a *unique* (infinite dimensional) solution manifold. In other words, also on the formal (non-truncated) level system (C.15) is still underdetermined.

That (C.15) really induces infinitely many different and independent solution manifolds can be easily seen. For example consider the following *special* solution set of (C.15)

$$f_n = 2^{-\frac{1}{2}n^2+\frac{5}{2}n-2+c} \cdot e^{-\frac{x_n}{2^{n-2}}-\sum_{i=1}^{n-1}\frac{x_i}{2^{i-1}}}, \quad c \in \mathbb{R}, \quad \text{and for all } n \geq 1, \tag{C.16}$$

which will be part of a more *general* solution manifold, say, of S_1, where *all* functions f_n will be non-zero:

$$S_1 = \left\{ \mathcal{Z}[f_n] = 0 \;\middle|\; f_n \neq 0, \quad \text{for all } n \geq 1 \right\}. \tag{C.17}$$

To construct for (C.15) a from S_1 disjoint and thus independent general solution manifold can now be easily achieved. By choosing at an arbitrary but fixed order $n = n_0$ a specific functional relation for the next higher order variable $f_{n_0+1} = f^0_{n_0+1}$, one can iteratively determine all other (infinitely many) possible functions f_n for $n \leq n_0$ as well as for $n > n_0$. Now, since the choice of n_0 and the choice for $f^0_{n_0+1}$ are arbitrary one can consequently construct infinitely many independent general solution manifolds S_n. For example, if we would choose $n_0 = 1$ and $f^0_2 = 0$, we obtain the following to (C.16) different *special* solution

$$f_1 = c_1 \cdot x_1 + c_2, \qquad f_n = 0, \quad \text{for} \quad n \geq 2, \tag{C.18}$$

which will be part of a more *general* solution manifold S_2, where the function $f_{n_0+1} = f_{n=2}$ is permanently zero and all functions f_n below $n = 2$ are strictly non-zero, while all functions f_n beyond $n = 2$ remain unrestricted in this regard:[†]

$$S_2 = \Big\{ \mathcal{Z}[f_n] = 0 \;\Big|\; f_1 \neq 0,\ f_2 = 0 \ \text{ and } \ (f_n \neq 0 \ \text{ or } \ f_n = 0),\ \text{for } n \geq 3 \Big\}. \tag{C.19}$$

It's clear that this (infinite) solution manifold S_2 is independent and disjoint to the previous (infinite) solution manifold S_1 (C.17). If we would choose $n_0 = 2$ and $f_3^0 = 0$, then we will obtain the following to (C.18) different *special* solution

$$\left.\begin{aligned} f_1 &= -\int_{x_1} dz_1 \int_{z_1} f_2(z_2, z_2)\, dz_2 + c_1 \cdot x_1 + c_2, \\ f_2 &= F_1(x_1) \cdot x_2 + F_2(x_1), \qquad f_n = 0, \ \text{ for } \ n \geq 3, \end{aligned}\right\} \tag{C.20}$$

which now will be part of a more *general* solution manifold S_3, where now the function $f_{n_0+1} = f_{n=3}$ is permanently zero and all functions f_n below $n = 3$ are strictly non-zero, while all functions f_n beyond $n = 3$ remain unrestricted again:

$$S_3 = \Big\{ \mathcal{Z}[f_n] = 0 \;\Big|\; f_1 \neq 0,\ f_2 \neq 0,\ f_3 = 0 \ \text{ and } \ (f_n \neq 0 \ \text{ or } \ f_n = 0),\ \text{for } n \geq 4 \Big\}. \tag{C.21}$$

The solution manifold S_3 is then independent and disjoint to S_2 (C.19) and S_1 (C.17). This process can then be continued to infinity to gain infinitely many *independent* general solution manifolds S_n.

But what is actually the deeper reason that (C.15) is formally underdetermined, although each equation is linked to the next higher order one. The main reason is of course that the system (C.15) is infinite dimensional. But another reason is also that the limit appearing in (C.15) is artificial. The limit pretends a formal closure although there is none (Frewer, 2015*a*). The term in brackets is a n-dimensional function for which no *direct* equation corresponds to. It thus presents for each order the unclosed term. For example, consider (for simplicity) the case $n = 1$. The limit in (C.15) then takes the explicit form

$$\widehat{f}_2(x_1) := \lim_{x_2 \to x_1} f_2(x_1, x_2) = f_2(x_1, x_1). \tag{C.22}$$

The problem is that from a given 2-dimensional function $f_2(x_1, x_2)$, i.e. from 'above', one can uniquely construct the corresponding lower 1-dimensional function $f_2(x_1, x_1)$, but not vice versa, i.e. from 'below', that is, from a given 1-dimensional function $\widehat{f}_2(x_1)$ one *cannot* uniquely construct the corresponding higher 2-dimensional function $f_2(x_1, x_2)$. But this latter process is exactly what happens when writing the limit in (C.15), namely that on the lower n-dimensional level, i.e. from 'below', a higher $(n + 1)$-dimensional function is identified. In other words, the limit in (C.15) artificially forces a lower n-dimensional function $\widehat{f}_{n+1}$ into a higher $(n + 1)$-dimensional function f_{n+1}.

Hence, for a system as (C.15) the formal level of unclosedness (degree of underdeterminedness) even is *higher* than, for example, if we would consider instead of (C.15) the following, also formally unclosed system (Frewer, 2015*a*,*b*)

$$\frac{\partial^2 f_n}{\partial x_n^2} + \frac{\partial f_{n+1}}{\partial x_n} = 0, \ \text{ for } \ n = 1, \ldots, \infty, \tag{C.23}$$

which, in contrast to (C.15), *directly* and thus *uniquely* links each equation to the next higher order one. Because, when choosing an arbitrary but fixed order $n = n_0$ for (C.23), one only has two variable options to generate a solution: either to specify f_{n_0} or to specify f_{n_0+1}, from which, in each case, *all* (infinite) remaining functions can be formally determined then. For system (C.15), however, the two corresponding specifications, either

[†]The explicit form is: $S_2 = \big\{ f_1 = c_1 \cdot x_1 + c_2 \neq 0,\ f_2 = 0,\ f_3 = f_3(x_1, x_3 - x_2),\ f_4(x_1, x_2, x_3, x_4),\ \ldots \big\}$.

f_{n_0} or $\widehat{f}_{n_0+1} := \lim_{x_{n_0+1}\to x_{n_0}} f_{n_0+1}$, are not sufficient, because all functions for $n > n_0$ cannot be *uniquely* determined anymore without exogenously also specifying the functions f_{n_0+1}, f_{n_0+2}, etc. Thus, explicitly writing the limit in (C.15) does not provide a positive contribution to formally close this system.

Now, regarding the original MPC system (C.9)-(C.14), we see that within these equations in either form, too, the lower dimensional unclosed terms $\widehat{H}_{i_{\{n+1\}}[l]}$ (C.11) and $\widehat{\mathbf{H}}_{n+1}$ (C.14) can be explicitly written as a limit of higher dimensional functions in order to apparently establish a connection between a lower order equation and the next higher one. But, as was just explained above, this connection is artificially enforced, since the formal closure problem is not eliminated by explicitly writing this limit; the closure problem still exists independently of whether this limit is being written or not. In other words, this 'lim' notation is misleading as it suggests a formal closure for the MPC system although in reality there is none (Frewer, 2015*a*).

Note that the LMN equations (C.4) also involve such an artificial connection between the lower and higher order functions, but, in contrast to the MPC equations, the LMN equations give something in return in that they come along with additional physical constraints in order to constitute themselves as a *formally* closed system. In fact, these additional constraints will restrict the infinitely many possible solution manifolds of the LMN evolution equations down to a physical solution manifold.

To explicitly demonstrate such a restriction, let us mimic for example the normalization property of the LMN equations (C.6), by demanding next to our infinite chain of emulated equations (C.15) the following e.g. half-sided 'normalization' constraint

$$\int_0^\infty f_1 dx_1 = 1, \quad \text{and} \quad \int_0^\infty f_{n+1} dx_{n+1} = f_n, \quad \text{for all} \quad n \geq 1, \tag{C.24}$$

which is a well-defined constraint for a differential system of type (C.15). If we regard (C.24) as a physical constraint, then the *special* solutions (C.18) and (C.20) must be regarded as unphysical solutions and thus have to be discarded, due to the non-convergence of the integral if $c_1 \neq 0$ or $c_2 \neq 0$. But, not only the special solutions, even the *general* solution manifolds S_2 and S_3 themselves have to be discarded as unphysical, because the functional break $f_{n_0} \neq f_{n_0+1} = 0$ at the chosen order $n = n_0$ is not compatible with (C.24). Also the *special* closed form solution (C.16) must be discarded in this sense as unphysical, since also its functional form is not compatible with the constraint (C.24). However, the general solution manifold S_1 itself must not be discarded, because there may still exist different special closed form solutions which in contrast to (C.16) are compatible with (C.24).

Thus we see that the more physical constraints go along with an infinite hierarchy of equations, the more the general non-unique solution manifold gets restricted down to a unique (physical) solution manifold. In other words, through a sufficient number of constraints a formally underdetermined (formally unclosed) system can turn into a formally fully determined (formally closed) system. And exactly this is the case for the LMN equations as they are just the discrete version of the functional Hopf equation, which itself, after all, represents a formally closed equation.

Only due to the fact that the LMN equations go along with additional physical constraints, they constitute in contrast to the MPC equations a formally (non-truncated) closed system. Hence, the LMN system in its *non-truncated* form thus constitutes a more physical system than the corresponding *non-truncated* MPC equations, as already said by Ievlev (1970): "However, the equations for the probability distributions (the LMN equations) yield a more complete and compact statistical description of turbulence than do the usual moment equations (the Friedman-Keller equations) and apparently permit an easier formulation of the approximate conditions closing the equations."

Now, in the case of the MPC equations, what would be the appropriate procedure to turn them into a *formally* closed system? The only answer is to extend the MPC equations at each order with the lower order moment equations of the corresponding unclosed

terms. But this is a non-manageable task, as the lower-order moment equations cannot be condensed anymore into a single hierarchy as it is the case for the MPC equations, neither in the differential form (C.9) nor in the integral form (C.12). The reason for this jump in complexity is that the above artificial limit does not commute with any differential operator, e.g. as in the relevant expression of (C.9)

$$\frac{\partial}{\partial \mathbf{x}_{(l)}}\left[\lim_{\mathbf{x}_{(n+1)}\to\mathbf{x}_{(l)}} H_{i_{\{n+1\}}}\right] \neq \lim_{\mathbf{x}_{(n+1)}\to\mathbf{x}_{(l)}} \frac{\partial}{\partial \mathbf{x}_{(l)}} H_{i_{\{n+1\}}}, \tag{C.25}$$

nor with any integral operator as in (C.12)

$$\boldsymbol{\mathcal{B}}_n \cdot \left[\lim_{\mathbf{x}_{n+1}\to\mathbf{x}_n} \mathbf{H}_{n+1}\right] \neq \lim_{\mathbf{x}_{n+1}\to\mathbf{x}_n} \boldsymbol{\mathcal{B}}_n \cdot \mathbf{H}_{n+1}, \tag{C.26}$$

which everyone working in fluid mechanics may have already experienced, when writing the transport equations for the lower n-point order moments as a limit from the higher $(n+1)$-point equations. Due to this non-commuting property, the number of unclosed terms increases, while at the same time the number constraints decreases. Here a small explicit example when considering the limit of the two-point continuity constraint (A.15), which just reduces to a non-useful zero-identity due to the continuity constraint of the instantaneous velocity field:

$$\begin{aligned}
0 \;&=\; \lim_{\mathbf{x}_{(1)}\to\mathbf{x}_{(0)}} \frac{\partial H_{k_{(0)}i_{(1)}}}{\partial x_{k_{(0)}}} \;\neq\; \frac{\partial}{\partial x_{k_{(0)}}}\left[\lim_{\mathbf{x}_{(1)}\to\mathbf{x}_{(0)}} H_{k_{(0)}i_{(1)}}\right] \\
&= \lim_{\mathbf{x}_{(1)}\to\mathbf{x}_{(0)}} \frac{\partial}{\partial x_{k_{(0)}}}\Big\langle u_{k_{(0)}}(\mathbf{x}_{(0)})u_{i_{(1)}}(\mathbf{x}_{(1)})\Big\rangle \;\Big|\; = \frac{\partial}{\partial x_{k_{(0)}}} H_{k_{(0)}i_{(0)}} \\
&= \lim_{\mathbf{x}_{(1)}\to\mathbf{x}_{(0)}} \Big\langle \Big(\frac{\partial}{\partial x_{k_{(0)}}} u_{k_{(0)}}(\mathbf{x}_{(0)})\Big) u_{i_{(1)}}(\mathbf{x}_{(1)})\Big\rangle \;\Big|\; = \frac{\partial}{\partial x_{k_{(0)}}}\Big\langle u_{k_{(0)}}(\mathbf{x}_{(0)})u_{i_{(0)}}(\mathbf{x}_{(0)})\Big\rangle \\
&= \Big\langle \Big(\frac{\partial}{\partial x_{k_{(0)}}} u_{k_{(0)}}(\mathbf{x}_{(0)})\Big) u_{i_{(0)}}(\mathbf{x}_{(0)})\Big\rangle \;\Big|\; = \frac{\partial}{\partial x_k}\langle u_k u_i\rangle \;\neq\; 0 \\
&= \frac{\partial}{\partial x_k}\langle u_k u_i\rangle - \Big\langle \Big(u_k \frac{\partial}{\partial x_k}\Big) u_i\Big\rangle \;\equiv\; 0 \;\Big|
\end{aligned} \tag{C.27}$$

Hence, the overall degree of unterdeterminism of the MPC equations even increases when trying to formally close them; and exactly in this sense we can say that the MPC equations (as they are standardly used by Oberlack et al.) are underdetermined in that they involve more unknowns than determining equations. Thus, as a consequence, any invariance analysis performed upon them will only result into weak *equivalence* transformations and *not* into a strong invariance relation as that of a true *symmetry* transformation.

C.1. *The non-equivalent relation between LMN and MPC*

Regarding the above discussion, the reader should finally note that the following statement made in Friedrich *et al.* (2012) *"... It can be shown that the LMN approach is completely equivalent to the statistical description of turbulence by moment equations. ..."* can be misleading if not carefully drawn from the context.

This cited *"equivalence"* refers to the fact that not only *if* a n-point PDF is known *then* the n-point moments can be determined from its expectation value, but also that *if all* n-point moments are known *then* the n-point PDF can be reconstructed. The latter 'inverse' construction is achieved by making use of Taylor series of the PDFs characteristic functions (see e.g. Monin (1967)).

The problem now is that in order to perform such a construction either the n-point PDF or the n-point moments must be known before the other one can be determined. For that one needs to solve the underlying evolution or transport equations, either for the PDFs or

for the moments. While the transport equations for the moments (MPC) can be uniquely determined from the evolution equations of the PDFs (LMN) (see e.g. Monin (1967)), the reverse cannot be established. In other words, the LMN equations cannot be *uniquely* determined from the MPC equations. Ievlev (1970) clearly has shown, which is also mentioned in Monin & Yaglom (1975) (the statements following the constraint (19.139)), that at least two *different* evolution equations for the PDFs can be constructed which all precisely result into the *same* MPC equations. Careful, although the two evolution equations for the PDFs differ by the fact that one is an approximation of the other one, this approximation is *not* transferred down to the MPC equations, which is clearly shown in Ievlev's proof in section 4.5 on page 89 (Ievlev, 1970).

That the LMN equations cannot be uniquely constructed from the MPC equations can also be easily understood from a different perspective, which already has been discussed in detail before: The additional physical constraints which go along with the LMN equations have no counterpart in the MPC equations and thus, in turn, are unable to uniquely induce the full LMN equations (including *all* possible constraint equations).

Hence, the above cited *"equivalence"* only refers to the defining relations of the PDFs and its moments (in Monin (1967) given by (3.1)-(3.3)), but definitely not to their underlying evolution equations. The LMN equations imply the MPC equations, but not oppositely, which is also clear from the aspect that a PDF formulation always operates on a higher statistical level than a formulation of the moments.

Appendix D. Additional comments on the proof for inconsistency

The proof (4.16) reads:

$$\begin{aligned}
\tilde{\mathbf{H}}_1 = e^q \mathbf{H}_1 \;&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \rangle = e^q \langle \mathbf{u}(\mathbf{x}_1, t) \rangle, \text{ for all points } \mathbf{x}_1 = \mathbf{x} && \text{(D.1)}\\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle = e^q \langle \mathbf{u}(\mathbf{x}_k, t) \rangle, \text{ for all } k \geq 1 && \text{(D.2)}\\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle = \langle e^q \mathbf{u}(\mathbf{x}_k, t) \rangle, \text{ for all possible configurations } \mathbf{u} && \text{(D.3)}\\
&\Rightarrow\; \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) = e^q \mathbf{u}(\mathbf{x}_k, t) && \text{(D.4)}\\
&\Rightarrow\; \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) = e^{n \cdot q} \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) && \text{(D.5)}\\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) \rangle = \langle e^{n \cdot q} \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle && \text{(D.6)}\\
&\Rightarrow\; \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \otimes \cdots \otimes \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_n, \tilde{t}) \rangle = e^{n \cdot q} \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \cdots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle && \text{(D.7)}\\
&\Rightarrow\; \tilde{\mathbf{H}}_n = e^{n \cdot q} \mathbf{H}_n. && \text{(D.8)}
\end{aligned}$$

D.1. *Comment No.1*

In the first step (D.1) we identify $\tilde{\mathbf{H}}_1 = \widetilde{\langle \mathbf{u}_1 \rangle}$ as $\langle \widetilde{\mathbf{u}}_1 \rangle := \langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_1, \tilde{t}) \rangle$. This conclusion is based on the simple fact that the transformation of $\tilde{\mathbf{H}}_1$ is a trivial one, in which all values of $\mathbf{H}_1$ just get globally scaled by a constant factor e^q.

In the *general case*, however, a careful distinction must be made between the two transformed expressions $\widetilde{\langle \mathbf{u} \rangle}$ and $\langle \widetilde{\mathbf{u}} \rangle$, since the former directly refers to the transformed mean velocity field while the latter refers to the transformed instantaneous (fluctuating) velocity field which is then averaged, and thus, *in general*, is mathematically distinct from the former expression. But here we are not considering the case of such a *general* variable (point) transformation

$$\widetilde{t} = \widetilde{t}\,(t, \mathbf{x}_k, \mathbf{H}_l), \qquad \widetilde{\mathbf{x}}_n = \widetilde{\mathbf{x}}_n(t, \mathbf{x}_k, \mathbf{H}_l), \qquad \widetilde{\mathbf{H}}_n = \widetilde{\mathbf{H}}_n(t, \mathbf{x}_k, \mathbf{H}_l), \qquad k, l = 1, \ldots, n, \tag{D.9}$$

between the independent variables $(t, \mathbf{x}_n)$ and the dependent variables $\mathbf{H}_n$, but only, as given by (4.4), the far more simpler *specific case* of a globally uniform scaling in the dependent variables

$$\widetilde{t} = t, \qquad \widetilde{\mathbf{x}}_n = \mathbf{x}_n, \qquad \widetilde{\mathbf{H}}_n = e^q \mathbf{H}_n, \tag{D.10}$$

which, when written for example for the one-point moment at $\mathbf{x}_1 = \mathbf{x}$

$$\widetilde{t} = t, \qquad \widetilde{\mathbf{x}} = \mathbf{x}, \qquad \widetilde{\langle \mathbf{u} \rangle} = e^q \langle \mathbf{u} \rangle, \tag{D.11}$$

acts as a trivial subset of (D.9). Note that in the following we only investigate the mathematical property of the transformation (D.11) *itself*, i.e. whether it additionally represents an equational invariance or not is irrelevant. In other words, we will investigate (D.11) very generally, solely as a transformation of variables detached from any underlying transport equations.

Now, it is straightforward to recognize that particularly in this trivial case (D.11), the two above mentioned transformed one-point expressions $\widetilde{\mathbf{H}}_1 = \widetilde{\langle \mathbf{u} \rangle}$ and $\langle \widetilde{\mathbf{u}} \rangle$ are identical

$$\widetilde{\langle \mathbf{u} \rangle} \equiv \langle \widetilde{\mathbf{u}} \rangle. \tag{D.12}$$

This conclusion is based on the following argument, in that we can write

$$\widetilde{\mathbf{H}}_1 = e^q \mathbf{H}_1 \quad \Leftrightarrow \quad \widetilde{\langle \mathbf{u} \rangle} = e^q \langle \mathbf{u} \rangle \tag{D.13}$$

$$= \langle e^q \mathbf{u} \rangle \tag{D.14}$$

$$\underset{\text{def.}}{=:} \langle \mathbf{u}^* \rangle, \tag{D.15}$$

due to the fact that any constant factor as e^q commutes with every averaging operator $\langle , \rangle$. Hence one is able to define a unique transformation relation $\mathbf{u} \to \mathbf{u}^*$ on the instantaneous level having the *same* transformational structure

$$\mathbf{u}^* = e^q \mathbf{u}, \tag{D.16}$$

as its averaged value given in (D.11), namely a simple multiplication of a constant factor e^q on some field values.[†] This, then, uniquely allows us to identify

$$\mathbf{u}^* = \widetilde{\mathbf{u}}. \tag{D.17}$$

In other words, since the symbol $\mathbf{u}^*$ on the left-hand side of (D.16) is defined by the mathematical operation on the right-hand side (a simple multiplication of a constant factor e^q), and since this mathematical operation is exactly identical to the right-hand side of the initial transformation (D.11), one can therefore uniquely identify the transformed symbol on the left-hand side of (D.16) with the same transformation symbol as it's used on the left-hand side of (D.11), i.e. $* = \sim$.

Again, the reason is that (D.11) and (D.16) show exactly the same transformation structure on their right-hand sides, namely a simple multiplication of a constant factor e^q on some field values, which then define their left-hand sides. But since we are dealing here with the same transformational process in (D.16) as in (D.11), we should also explicitly display it, namely by using $\widetilde{\mathbf{u}}$ and not $\mathbf{u}^*$, which would only unnecessarily overload the notation. Exactly this fact was implicitly assumed when writing the first line of (4.16).

But, as soon as we would consider a more complicated transformation than (D.11), as for example

$$\widetilde{t} = t, \qquad \widetilde{\mathbf{x}} = \mathbf{x}, \qquad \widetilde{\langle \mathbf{u} \rangle} = e^{q(\mathbf{x})} \langle \mathbf{u} \rangle, \tag{D.18}$$

where, instead of a globally constant scaling exponent q, we now would have a *local* scaling exponent $q(\mathbf{x})$ which explicitly depends on the spatial coordinates, the identification (D.12), of course, generally no longer holds and becomes invalid. The reason simply is that

[†]Note that if (D.11) would be additionally admitted as a symmetry of some mean field transport equations, then we may *not* conclude that (D.16) is a symmetry, too, of the underlying instantaneous (fluctuating) equations. Because, on the mean field level one can have a symmetric structure which on the fluctuating level must not exist.

in contrast to (D.11) the scaling factor in (D.18) is no longer a global constant anymore which can commute with every averaging operator $\langle , \rangle$. In other words, since generally

$$\widetilde{\langle \mathbf{u} \rangle} = e^{q(\mathbf{x})} \langle \mathbf{u} \rangle \neq \langle e^{q(\mathbf{x})} \mathbf{u} \rangle, \tag{D.19}$$

we are no longer able to define a corresponding transformation relation $\mathbf{u} \to \tilde{\mathbf{u}}$ on the instantaneous level which has the *same* transformational structure as its averaged value (D.18). On the contrary, its *real* corresponding transformation rule $\mathbf{u} \to \mathbf{u}^{**}$ will rather show a far more complex functional structure than given by (D.18), which in the first instance also cannot be mathematically determined in a straightforward manner.

Hence, since the situation of proof (4.16) is not dealing with a complex situation like (D.18), but only with a trivial one as (D.11), the notation used throughout (4.16) is correct and not misleading.

Note that already from an intuitive point of view the identification (D.12) must be valid if we consider a simple transformation as (D.11). Because, since in (D.11) only the field values and not the coordinates get transformed we can perform the following thought experiment: Imagine we have an ensemble of DNS results for the instantaneous velocity field $\mathbf{u}$ (hereby it is irrelevant from which specific equations this data set was numerically generated). From the field $\mathbf{u}$ we now construct the mean field $\langle \mathbf{u} \rangle$ (either as an ensemble average over a set of different $\mathbf{u}$, or, if we have a statistically homogenous direction, over an integral of a single $\mathbf{u}$ in this direction). Thus we then obtain all functional values of $\langle \mathbf{u} \rangle$, which we now collectively multiply with a same constant factor, say by $e^q = 2$, to get the new transformed values $\widetilde{\langle \mathbf{u} \rangle}$ of (D.11).

Now, the critical question: How should the underlying DNS data for the instantaneous field $\mathbf{u}$ be transformed in order to generate with the same corresponding averaging process the just previously constructed values $\widetilde{\langle \mathbf{u} \rangle}$? The intuitive and correct answer is that all DNS data must be *coherently* multiplied with the same factor $e^q = 2$. Only then will the new transformed data $\tilde{\mathbf{u}}$, if it emerges from the operation $\tilde{\mathbf{u}} := 2 \cdot \mathbf{u}$ and if it's correspondingly averaged to $\langle \tilde{\mathbf{u}} \rangle$, give the ability to reconstruct the functional values $\widetilde{\langle \mathbf{u} \rangle}$. Since there is no other option, we hence obtain within this process the unique result $\langle \tilde{\mathbf{u}} \rangle = \widetilde{\langle \mathbf{u} \rangle}$. Of course, this reasoning is only valid for a global (coherent) transformation as given by (D.11); for a more complicated (local) transformation as (D.18) this reasoning no longer holds.

D.2. *Comment No.2*

The conclusion (D.4) is based on the relation (D.3) which goes along with the explicit comment that this relation, by definition, must hold for *all* possible configurations or functional realizations of the fluctuating velocity field $\mathbf{u}$, and not only for any certain functional specification $\mathbf{u} = \mathbf{u}_0(\mathbf{x}, t)$. In this case, of course, conclusion (D.4) would *not* be correct, because for a certain specification $\mathbf{u} = \mathbf{u}_0$, we generally have the situation that $\langle \tilde{\mathbf{u}}_0 \rangle = \langle \mathbf{u}_0 \rangle$ although $\tilde{\mathbf{u}}_0 \neq \mathbf{u}_0$.

Now, for the reason that (D.3) must hold for *all* possible configurations $\mathbf{u}$, it is important to recognize that (D.3) is not an equation to be solved for, but that it represents a definition. And exactly this is the argument when going from the third (D.3) to fourth line (D.4). The third line

$$\langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle = \langle e^q \mathbf{u}(\mathbf{x}_k, t) \rangle, \tag{D.20}$$

does not stand for an equation but for a definition (as it's the case for any variable transformation in mathematics)

$$\langle \tilde{\mathbf{u}}(\tilde{\mathbf{x}}_k, \tilde{t}) \rangle \underset{\text{def.}}{:=} \langle e^q \mathbf{u}(\mathbf{x}_k, t) \rangle, \tag{D.21}$$

since the left-hand side (transformed side) is defined by the mathematical expression and operation given on the right-hand side. Now, since the right-hand side by definition must hold *for all* possible (functional) configurations of the instantaneous velocity field $\mathbf{u}$, and

since both functions $e^q\mathbf{u}$ and $\widetilde{\mathbf{u}}$ undergo the *same* operation of averaging, we thus can only conclude that both functions themselves must be identical. In other words, definition (D.21) implies the definition

$$\widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_k,\tilde{t}) \underset{\text{def.}}{:=} e^q\mathbf{u}(\mathbf{x}_k,t), \tag{D.22}$$

in that the transformed instantaneous velocity field $\widetilde{\mathbf{u}}$ is defined by the expression $e^q\mathbf{u}$. This then gives the fourth line (D.4).

Two things should be noted here. Firstly, the above conclusion is similar to the arguments which are standardly used in fluid mechanics when deriving a differential conservation law from its corresponding integral version. The similarity is given in so far as the argument for the validity of the integral conservation law is also based on the requirement *'for all'*, however here, for *all* possible volumes or surfaces. Hence the integral operator from the integral conservation law can be dropped and the integrand itself is identified as the corresponding conservation law on the differential level.

Secondly, according to the arguments given in Comment No.1, we are not obliged to make the direct conclusion (D.4) from relation (D.3). Conclusion (D.4) can already be directly obtained from (D.1) by considering the result (D.17), i.e. we can directly conclude that

$$\widetilde{\mathbf{H}}_1 = e^q\mathbf{H}_1 \quad\Rightarrow\quad \widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_k,\tilde{t}) = e^q\mathbf{u}(\mathbf{x}_k,t), \tag{D.23}$$

which just explicitly expresses the fact again that result (D.17) was uniquely obtained (for all $\mathbf{x}_k$ within the physical space $\mathbf{x}$) as the induced transformation rule $\mathbf{u}\to\mathbf{u}^*=\widetilde{\mathbf{u}}$ (the cause on the fluctuating level) from the transformation rule of the mean velocity $\mathbf{H}_1\to\widetilde{\mathbf{H}}_1$ (the effect on the averaged level).

D.3. *Comment No.3*

In (D.8) we identify the transformed n-point correlation function $\widetilde{\mathbf{H}}_n$ as the transformed expression $\langle\widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_1,\tilde{t})\otimes\cdots\otimes\widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_n,\tilde{t})\rangle$. This is just the obvious consequence in knowing the fact that only from the transformed velocity field $\widetilde{\mathbf{u}}$, as it is defined in (D.17) and thus in (D.4), *all* transformed correlation functions $\widetilde{\mathbf{H}}_n$ can be uniquely defined and constructed without inducing contradictions and without violating the principle of causality. In other words, making the conclusion

$$\widetilde{\mathbf{H}}_1 = e^q\mathbf{H}_1 \quad\Rightarrow\quad \widetilde{\mathbf{H}}'_n = e^{n\cdot q}\mathbf{H}_n, \tag{D.24}$$

where $\widetilde{\mathbf{H}}'_n$ represents the mean product of n spatial coordinate evaluations of the transformed and for all points $\mathbf{x}_n$ unique instantaneous velocity field $\widetilde{\mathbf{u}}=\widetilde{\mathbf{u}}(\tilde{\mathbf{x}},\tilde{t})$, and in which it then gets identified as the transformed n-point correlation function $\widetilde{\mathbf{H}}_n$, as done in (D.8),

$$\widetilde{\mathbf{H}}_n := \widetilde{\mathbf{H}}'_n = \langle\widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_1,\tilde{t})\otimes\cdots\otimes\widetilde{\mathbf{u}}(\tilde{\mathbf{x}}_n,\tilde{t})\rangle, \tag{D.25}$$

is the *only* possible conclusion without running into any contractions and without violating the principle of cause and effect.

D.4. *Comment No.4*

Note that already from an intuitive point of view only the conclusion

$$\widetilde{\mathbf{H}}_1 = e^q\mathbf{H}_1 \quad\Rightarrow\quad \widetilde{\mathbf{H}}_n = e^{n\cdot q}\mathbf{H}_n, \tag{D.26}$$

given through (D.1)-(D.8), makes physically sense, while the conclusion induced by (4.4)

$$\widetilde{\mathbf{H}}_1 = e^q\mathbf{H}_1 \quad\Rightarrow\quad \widetilde{\mathbf{H}}_n = e^q\mathbf{H}_n, \tag{D.27}$$

is physically senseless. This can be seen for example by making again the following small thought experiment: Imagine we have the following arbitrary but fixed mean velocity profile $\mathbf{H}_1=\langle\mathbf{u}\rangle$ based on some instantaneous (fluctuating) velocity field $\mathbf{u}$. Now, according

to the left-hand side of (D.26) or (D.27), if we scale this mean profile $\mathbf{H}_1$ by, say, a constant factor $e^q = 2$, we will get the two times amplified mean velocity profile $\widetilde{\mathbf{H}}_1$. Hereby it should be noted that this scaling is performed globally, i.e. for *all* points in the considered physical space $\mathbf{x}$ the mean velocity values $\mathbf{H}_1$ are scaled uniformly by a constant factor two.

Intuitively it's clear that a *globally* two times higher amplitude in the mean profile can only go along with a *globally* two times higher amplitude in the instantaneous velocity. In other words, in order to account for a global scaling $\widetilde{\mathbf{H}}_1 = 2\mathbf{H}_1$ on the averaged level (the effect), the underlying instantaneous velocity must transform accordingly $\widetilde{\mathbf{u}} = 2\mathbf{u}$ on the fluctuating level (the cause), otherwise we would not manage to reproduce this coherent amplification of a factor two on the averaged level.

But now, if the instantaneous velocity $\mathbf{u}$ globally scales (i.e. for all points $\mathbf{x}_n$ in physical space $\mathbf{x}$) by a factor two, then e.g. the two-point correlation function $\mathbf{H}_2$ will globally scale with a factor $e^{2q} = 4$ as given in (D.26), and *not* as in (D.27) with the same factor $e^q = 2$ as the mean velocity $\mathbf{H}_1$ is scaling. Hence the conclusion (D.27) is obviously unphysical.

Appendix E. Statistical scaling of the nonlinear Schrödinger equation

This section will demonstrate that the unphysical statistical scaling invariance Q_E (4.4) is not specific to the incompressible Navier-Stokes equation (2.1), or (2.2), when transcribed into its statistical form for the n-point velocity correlation moments $\mathbf{H}_n$ (3.1), respectively, either into its local differential form (A.14)-(A.15) or into its nonlocal integro-differential form (B.6). Because, as a representative example, the unphysical statistical scaling invariance Q_E (4.4) is also admitted e.g. by the following cubic nonlinear Schrödinger equation (see e.g. Sulem & Sulem (1999))

$$i\partial_t\psi + \Delta\psi + \kappa|\psi|^2\psi = 0, \tag{E.1}$$

when transcribed into its statistical form (in the thermodynamical sense) for the corresponding (equal-time) multi-point correlation moments

$$\left.\begin{aligned} H_1 &= \langle\psi_1\rangle, \\ H_{2n-1} &= \langle\psi_1\cdots\psi_n\cdot\psi^*_{n+1}\cdots\psi^*_{2n-1}\rangle,\ n\geq 2,\ n\in\mathbb{N}, \end{aligned}\right\} \tag{E.2}$$

of the full instantaneous (fluctuating) scalar wave function $\psi = \psi(t,\mathbf{x})$, where, respectively, ψ_n and ψ^*_n stand for the evaluation of the field and its complex conjugate at the specific point $\mathbf{x} = \mathbf{x}_n$ within the single physical domain $\mathbf{x}\in\mathbb{R}^3$. Note that in (E.1) the parameter κ is defined as a real constant, and that $|\psi|^2 = \psi\cdot\psi^*$ stands for the square modulus of the wave function. Further note that the main difference between the deterministic Navier-Stokes equation (2.1) and the deterministic nonlinear Schrödinger equation (E.1) is that the former involves a quadratic nonlinearity while the latter shows a cubic nonlinearity, thus having the effect that the complete Schrödinger hierarchy of multi-point moments (E.2) is ordered by odd numbers only.

To derive the corresponding transport equations for the multi-point moments (E.2) we proceed as given in Oberlack & Rosteck (2010) for the Navier-Stokes equation. In the first step the necessary deterministic equations for the wave function and its complex conjugate will be abbreviated as

$$\left.\begin{aligned} N_\psi &= i\partial_t\psi + \Delta\psi + \kappa\cdot\psi\cdot\psi\cdot\psi^* = 0, \\ N_{\psi^*} &= -i\partial_t\psi^* + \Delta\psi^* + \kappa\cdot\psi\cdot\psi^*\cdot\psi^* = 0, \end{aligned}\right\} \tag{E.3}$$

in order to then allow, in the second step, the construction of the statistical transport

equations for all moments (E.2)

$$n=1: \quad \langle N_{\psi_1}\rangle = 0, \tag{E.4}$$

$$\begin{aligned} n\geq 2: \quad & \langle N_{\psi_1}\cdot\psi_2\cdots\psi_n\cdot\psi^*_{n+1}\cdots\psi^*_{2n-1}\rangle + \langle\psi_1\cdot N_{\psi_2}\cdot\psi_3\cdots\psi_n\cdot\psi^*_{n+1}\cdots\psi^*_{2n-1}\rangle \\ & + \cdots + \langle\psi_1\cdots\psi_{n-1}\cdot N_{\psi_n}\cdot\psi^*_{n+1}\cdots\psi^*_{2n-1}\rangle \\ & - \langle\psi_1\cdots\psi_n\cdot N_{\psi^*_{n+1}}\cdot\psi^*_{n+2}\cdots\psi^*_{2n-1}\rangle - \langle\psi_1\cdots\psi_n\cdot\psi^*_{n+1}\cdot N_{\psi^*_{n+2}}\cdot\psi^*_{n+3}\cdots\psi^*_{2n-1}\rangle \\ & - \cdots - \langle\psi_1\cdots\psi_n\cdot\psi^*_{n+1}\cdots\psi^*_{2n-2}\cdot N_{\psi^*_{2n-1}}\rangle = 0, \end{aligned} \tag{E.5}$$

which can be condensed and written as the following infinite hierarchy of linear equations

$$n=1: \quad i\partial_t H_1 + \Delta H_1 + \kappa\widehat{H}_3 = 0, \tag{E.6}$$

$$\begin{aligned} n\geq 2: \quad & i\partial_t H_{2n-1} + \sum_{k=1}^{n}\Delta_{\mathbf{x}_k} H_{2n-1} - \sum_{k=n+1}^{2n-1}\Delta_{\mathbf{x}_k} H_{2n-1} \\ & + \kappa\sum_{k=1}^{n}\widehat{H}_{k;\,2(n+1)-1} - \kappa\sum_{k=n+1}^{2n-1}\widehat{H}_{k;\,2(n+1)-1} = 0. \end{aligned} \tag{E.7}$$

Exactly as in the case of the Navier-Stokes hierarchy, either as shown in its local differential form (A.14)-(A.15), or as in its nonlocal integro-differential form (B.6), the corresponding lower dimensional Schrödinger moments $\widehat{H}_3$ in (E.6) and $\widehat{H}_{k;\,2(n+1)-1}$ in (E.7) can be uniquely constructed from the corresponding higher dimensional Schrödinger moments H_3 and $H_{2(n+1)-1}$, too, namely as

$$\widehat{H}_3 = \lim_{\mathbf{x}_3\to\mathbf{x}_1,\,\mathbf{x}_2\to\mathbf{x}_1} H_3, \tag{E.8}$$

and for all $n\geq 2$ as

$$\widehat{H}_{k;\,2(n+1)-1} = \lim_{\mathbf{x}_{2n+1}\to\mathbf{x}_{n+1}}\left[\lim_{\mathbf{x}_{2n}\to\mathbf{x}_k,\,\mathbf{x}_{n+1}\to\mathbf{x}_k} H_{2(n+1)-1}\right], \quad \text{for } k=1,\ldots,2n-1. \tag{E.9}$$

However, since of course the inverse construction fails, i.e. since the higher dimensional moments H_3 and $H_{2(n+1)-1}$ cannot be uniquely constructed from the lower dimensional moments $\widehat{H}_3$ and $\widehat{H}_{k;\,2(n+1)-1}$, and therefore, since these latter moments do not *directly* enter the next higher correlation equation in the hierarchy (E.7), they have to be identified as unclosed terms. In particular, since this infinite hierarchy of Schrödinger moments also does not come along with additional constraint equations in order to realize the single physical solution (E.2), which uniquely emerges from the underlying deterministic equation (E.1) as a multiple evaluation of the single instantaneous (fluctuating) wave function $\psi = \psi(t,\mathbf{x})$, the system of equations (E.6)-(E.7) is permanently underdetermined and therefore unclosed if *no* prior modelling assumption is invoked beforehand (see Appendix C for a more detailed explanation, which, in a one-to-one way, can be carried over from the Navier-Stokes to the present Schrödinger case). The unclosed Schrödinger moments $\widehat{H}_3$ (E.8) and $\widehat{H}_{k;\,2(n+1)-1}$ (E.9) thus play the same role as the unclosed Navier-Stokes moments $\widehat{H}_{i_{\{n+1\}}[l]} = \lim_{\mathbf{x}_{n+1}\to\mathbf{x}_l} H_{i_{\{n+1\}}}$ (A.22) in the local differential formulation, or equivalently $\widehat{\mathbf{H}}_{i,n+1} = \lim_{\mathbf{x}_{n+1}\to\mathbf{x}_i}\mathbf{H}_{n+1}$ (B.8) in the nonlocal integro-differential formulation. However, note that the sequence in the limit process of (E.9) is crucial as it's not interchangeable, i.e. the inner and outer limits have to be taken only in this order as given in (E.9), otherwise the limit process would give a different result.

Now, since the infinite hierarchy of linear equations (E.6)-(E.7) is unclosed it can at most only admit equivalence transformations. It is straightforward to see that this system (E.6)-(E.7) admits the same unphysical scaling equivalence Q_E (4.4) for the Schrödinger moments as for the Navier-Stokes moments. For the Schrödinger moments (E.2) the unphysical invariance Q_E thus reads

$$\mathsf{Q}_\mathsf{E}: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{H}_{2n-1} = e^q H_{2n-1}, \quad n\geq 1, \tag{E.10}$$

which, again, only arises due to the fact that for the underlying *nonlinear* deterministic equation (E.1) the corresponding statistical hierarchy of moments (E.6)-(E.7) is misleadingly formulated as a *linear* system. In other words, as was thoroughly explained and demonstrated in this study at the example of the Navier-Stokes moments, the linear statistical formulation (E.6)-(E.7) for the Schrödinger moments is misleading too when employing an invariance analysis upon them, since one directly obtains the inconsistent and unphysical result (E.10).

Note again that as the invariance (E.10) only scales the dependent variables while leaving the coordinates unchanged, the transformation rule for the lower dimensional (unclosed) moments $\widehat{H}_3$ (E.8) and $\widehat{H}_{k;\,2(n+1)-1}$ (E.9) is identical to the transformation rule (E.10) for the corresponding higher dimensional moments H_3 and $H_{k;\,2(n+1)-1}$, i.e.

$$\widetilde{\widehat{H}}_3 = e^q \widehat{H}_3, \tag{E.11}$$

$$\widetilde{\widehat{H}}_{k;\,2(n+1)-1} = e^q \widehat{H}_{k;\,2(n+1)-1}, \text{ for } k = 1, \ldots, 2n-1, \text{ and } n \geq 2, \tag{E.12}$$

as it was also respectively derived in more detail for the corresponding lower dimensional (unclosed) Navier-Stokes moments in (4.6). Finally note that when decomposing the instantaneous wave function into its mean and purely fluctuating field $\psi = \langle\psi\rangle + \psi'$, the equivalence transformation for the instantaneous moments (E.10) will then bijectively change to the corresponding more detailed representation (4.19), from which, then, the artificial transformation behavior of (E.10) is immediately recognized.

To conclude, this demonstrating example makes it clear that the scaling invariance Q_E (4.4) for the statistical moments is neither specific to the Navier-Stokes nor specific to any nonlinear Schrödinger equation. It is just (besides its unphysical nature) only a non-specific and thus a non-significant equivalence transformation, since it will be admitted by any nonlinear deterministic system which necessitates a statistical description (in the thermodynamical sense) for its solution manifold when considering a certain hierarchy of multi-point moments.

Appendix F. Generating scaling laws from invariant transformations

According to Oberlack & Rosteck (2010) the key invariant Lie-point transformations to generate useful statistical scaling laws for wall-bounded flows in the inertial region are, at first, the physical translation symmetry in the space-time coordinates

$$\mathsf{T}: \quad \tilde{t} = t + c_{0,0}, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n + \mathbf{c}_{0,n}, \quad \tilde{\mathbf{H}}_n = \mathbf{H}_n, \tag{F.1}$$

and the two physical scaling symmetries of the inviscid Euler equations, which inherently translate to the instantaneous multi-point functions in their most general form as

$$\mathsf{S}_1: \quad \tilde{t} = e^{a_1} t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{\mathbf{H}}_n = e^{-n\cdot a_1} \mathbf{H}_n, \tag{F.2}$$

$$\mathsf{S}_2: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = e^{a_2} \mathbf{x}_n, \quad \tilde{\mathbf{H}}_n = e^{n\cdot a_2} \mathbf{H}_n, \tag{F.3}$$

and then finally the two new *unphysical* invariant transformations, which are not reflected by the deterministic Euler or Navier-Stokes equations (2.1), as they only apply to the notationally oversimplified statistical transport equations (4.5)

$$\mathsf{Q}_1: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{\mathbf{H}}_n = e^q \mathbf{H}_n, \tag{F.4}$$

$$\mathsf{Q}_2: \quad \tilde{t} = t, \quad \tilde{\mathbf{x}}_n = \mathbf{x}_n, \quad \tilde{\mathbf{H}}_n = \mathbf{H}_n + \mathbf{c}_{1,n}, \tag{F.5}$$

where only the globally invariant *scaling* $\mathsf{Q}_1 = \mathsf{Q}_\mathsf{E}$ (4.4) has been explicitly exposed in this study as an unphysical transformation. However, by using the procedure developed in Section 4, in particular (4.16), it is straightforward to also expose the globally invariant *translation* Q_2 (F.5) as an unphysical transformation. Up to now, no higher level functional

Hopf-symmetry has been found which can induce this translation invariance via (4.2), as it is the case for the scaling invariance Q_1 (F.4) which is induced by the Hopf-symmetry Q (4.1). It is the non-holonomic constraint in (3.4) which mostly breaks every symmetry, as it is too restrictive to find a symmetry which is compatible to it for all times $t \geq 0$. Note that in all five transformations (F.1)-(F.5) we suppressed the transformation rule for all pressure correlations, in contrast to the listing in Oberlack & Rosteck (2010), since we only consider the solenoidal (nonlocal) statistical transport equations (4.5), based on the deterministic form (2.2) in which the pressure got eliminated by the continuity equation. Indeed, both Q_1 (F.4) and Q_2 (F.5) are admitted by the underlying (unclosed) Friedmann-Keller equations (4.5) as equivalence transformations.

Already these few invariant Lie-group transformations in their most general form (F.1)-(F.5) allow now to generate a vast range of turbulent scaling laws. Our interest here however is to focus only on the considerably smaller set of scaling laws for one-point statistics. To generate these scalings, the above transformations for multi-point correlations has to be reduced to one-point statistics, as was also done in Oberlack & Rosteck (2010) by performing the smooth and regular limit $(\mathbf{x}_n - \mathbf{x}_1) \to \mathbf{0}$, for all $n \geq 2$.

Furthermore, since we are also only interested in the scaling behavior of geometrically simple wall-bounded flows in the inertial region, we will additionally perform next to the one-point limit the following approximation: According to Oberlack (2001); Oberlack & Rosteck (2010) and Lindgren *et al.* (2004) the flow in the inertial region between the inner (near-wall) and outer region is approximated as a *stationary inviscid parallel shear flow*. Under this assumption the wall-normal velocity component can be neglected, and all fields only depend on one independent coordinate, the wall-normal coordinate y, described then by an inviscid set of balance equations.

If we now consider the specific flow configuration of a ZPG turbulent boundary layer flow, in order to compare to the DNS results in Section 5, the invariant transformations (F.1)-(F.5) will then simplify even further. Due to spanwise homogeneity and a spanwise reflection symmetry in the flow, the mean spanwise velocity as well as all moments involving an uneven number of spanwise velocity fields vanish.

In this approximation within the one-point limit for ZPG turbulent boundary layer flow, the above transformations (F.1)-(F.5) reduce to

$$\left.\begin{aligned}
\mathsf{T}: &\quad \tilde{y} = y + c_0^2, \quad \tilde{U} = U, \quad \tilde{H}^{ij} = H^{ij}, \quad \tilde{H}^{ijk} = H^{ijk}, \\
\mathsf{S}_1: &\quad \tilde{y} = y, \quad \tilde{U} = e^{-a_1} U, \quad \tilde{H}^{ij} = e^{-2a_1} H^{ij}, \quad \tilde{H}^{ijk} = e^{-3a_1} H^{ijk}, \\
\mathsf{S}_2: &\quad \tilde{y} = e^{a_2} y, \quad \tilde{U} = e^{a_2} U, \quad \tilde{H}^{ij} = e^{2a_2} H^{ij}, \quad \tilde{H}^{ijk} = e^{3a_2} H^{ijk}, \\
\mathsf{Q}_1: &\quad \tilde{y} = y, \quad \tilde{U} = e^{q} U, \quad \tilde{H}^{ij} = e^{q} H^{ij}, \quad \tilde{H}^{ijk} = e^{q} H^{ijk}, \\
\mathsf{Q}_2: &\quad \tilde{y} = y, \quad \tilde{U} = U + c_1^1, \quad \tilde{H}^{ij} = H^{ij} + c_1^{ij}, \quad \tilde{H}^{ijk} = H^{ijk} + c_1^{ijk},
\end{aligned}\right\} \tag{F.6}$$

where we considered the chain only up to third moment ($n \leq 3$), and where the number of superscript indices on each term denotes the tensor rank of the corresponding variable, i.e. $\mathbf{x} = (x^i)$, $\mathbf{H}_1 = (H^i)$, $\mathbf{H}_2 = (H^{ij})$, $\mathbf{H}_3 = (H^{ijk})$, etc., with the short-hand notation $x^2 = y$ and $H^1 = U$.

Hence the corresponding invariant surface condition (Olver, 1993; Ibragimov, 1994; Bluman & Kumei, 1996) to generate invariant functions from (F.6) reads

$$\begin{aligned}
\frac{dy}{a_2 \cdot y + c_0^2} &= \frac{dU}{(-a_1 + a_2 + q)U + c_1^1} \\
&= \frac{dH^{ij}}{(-2a_1 + 2a_2 + q)H^{ij} + c_1^{ij}} = \frac{dH^{ijk}}{(-3a_1 + 3a_2 + q)H^{ijk} + c_1^{ijk}}.
\end{aligned} \tag{F.7}$$

To note is that in the above expression all indices are open and not contracted, i.e. they are not being summed over, and, due to the symmetries in the flow, only those indexed moments give a contribution which involve an even number of spanwise velocity fields.

According to Oberlack (2001); Oberlack & Rosteck (2010) and Lindgren *et al.* (2004) equation (F.7) provokes a further critical assumption. The line of argumentation is, since the friction velocity at the wall $U_\tau \sim \sqrt{\partial_y U|_{y=0}}$ can be seen as an external parameter or boundary condition which inhibits the free scaling of the streamwise velocity U, the corresponding scaling invariance must get broken, i.e. the scaling coefficient of U must be put to zero: $-a_1 + a_2 + q = 0$. As in Oberlack & Rosteck (2010) we choose solving for $a_1 = a_2 + q$. Inserting this restriction and dividing the equation by $a_2 \neq 0$ gives instead of (F.7) the more simplified zero-surface condition

$$\frac{dy}{y+\bar{c}_0^2} = \frac{dU}{\bar{c}_1^1} = \frac{dH^{ij}}{-\bar{q}\cdot H^{ij} + \bar{c}_1^{ij}} = \frac{dH^{ijk}}{-2\bar{q}\cdot H^{ijk} + \bar{c}_1^{ijk}}, \tag{F.8}$$

where all overbared constants represent the original constants relative to a_2. Finally, solving these equations lead to the following set of invariant scaling laws in the *inertial region* on the basis of the full *instantaneous*-fields

$$\left.\begin{aligned} U(y) &= \alpha_U \cdot \ln(y+c) + \beta_U, \\ H^{ij}(y) = \alpha_H^{ij} + \beta_H^{ij}\cdot(y+c)^\gamma, \quad H^{ijk}(y) &= \alpha_H^{ijk} + \beta_H^{ijk}\cdot(y+c)^{2\gamma}, \end{aligned}\right\} \tag{F.9}$$

where all the β's are integration constants, whereas the rest of the parameters comprise the group constants relative to a_2 as: $\gamma = -\bar{q}$, $c = \bar{c}_0^2$, $\alpha_U = \bar{c}_1^1$, $\alpha_H^{ij} = \bar{c}_1^{ij}/\bar{q}$, and $\alpha_H^{ijk} = \bar{c}_1^{ijk}/2\bar{q}$. Note the strong dependence of the scaling laws (F.9) on the two *unphysical* invariance transformations Q_1 (F.4) and Q_2 (F.5). Except for the parameter $\bar{c}_0^2$, all other parameters stem from Q_1 and Q_2.

Finally, using the Reynolds decomposition as given in (4.17) and (4.18) will give the transformation rule for the corresponding moments of the *fluctuating* fields

$$\left.\begin{aligned} U(y) &= \alpha_U \cdot \ln(y+c) + \beta_U, \\ \tau^{ij}(y) &= \alpha_H^{ij} + \beta_H^{ij}\cdot(y+c)^\gamma - \delta^{1i}\delta^{1j}U(y)^2, \\ T^{111} &= \alpha_H^{111} + \beta_H^{111}\cdot(y+c)^{2\gamma} - 3U(y)\cdot\tau^{11}(y) - U(y)^3, \\ T^{112} &= \alpha_H^{112} + \beta_H^{112}\cdot(y+c)^{2\gamma} - 2U(y)\cdot\tau^{12}(y), \\ T^{ij1} &= \alpha_H^{ij1} + \beta_H^{ij1}\cdot(y+c)^{2\gamma} - U(y)\cdot\tau^{ij}(y), \text{ for } (i,j) = (2,2),(3,3), \\ T^{ij2} &= \alpha_H^{ij2} + \beta_H^{ij2}\cdot(y+c)^{2\gamma}, \text{ for } (i,j) = (2,2),(3,3). \end{aligned}\right\} \tag{F.10}$$

We see that the above set of invariant scaling functions is lead by a generalized log-law in the mean velocity profile U, which differs from the classical von Kármán log-law by the presence of a constant shift c. This non-classical log-law was first obtained in Oberlack (2001), however, on basis of a different invariance procedure as developed in Oberlack & Rosteck (2010) and as presented herein.[†] In Lindgren *et al.* (2004) it is referred to as the *modified log-law* extending the predictability in the lower end of the inertial region. By construction all parameters are completely independent of the Reynolds number, which obviously stems from the strong approximation that the inertial region was identified as an inviscid parallel shear layer.

Important to note in (F.10) are the new results for the streamwise Reynolds stress τ^{11} and for several triple moments T^{ijk}, which, due to the presence of the mean streamwise velocity profile U, all scale in the inertial region as a combination of a log-law and a power-law. It should be clear that this peculiar scaling behavior simply and only has its origins in the unphysical invariance transformations Q_1 (F.4) and Q_2 (F.5), which again are due to having performed an invariance analysis on the notational oversimplified

[†]To note is that the Lie-group based derivation of the generalized log-law as given in (F.10) is not only heavily misleading in Oberlack & Rosteck (2010), but also in Oberlack (2001). While the derivation in Oberlack & Rosteck (2010) is based on an unphysical invariance, the derivation in Oberlack (2001) is based on an incorrectly concluded invariance (Frewer *et al.*, 2014*b*).

set of equations (4.5), which misleadingly represent themselves as a linear gradient-type set of equations due to not incorporating the underlying deterministic equations into the analysis. It should also be clear that this unphysical scaling in the streamwise direction is not restricted to one-point quantities only, but essentially is incorporated in all multi-point quantities.

Finally also note again that, independent of the fact that the scaling laws (F.9) or (F.10) are based on two unphysical invariant transformations, it would be more than misleading to identify these invariant functions as a set of privileged *solutions* to the underlying unmodelled system (4.5). As it was already discussed in Section 2.1, the reason is that the unclosed system (4.5) is not arbitrarily underdetermined, but underdetermined in the sense that all its unknown terms can be physically and uniquely determined from the underlying but yet analytically not accessible instantaneous (fluctuating) velocity field **u** according to (3.1).

Hence, the result (F.9) or (F.10) can only be interpreted as a set of functional relations which stay invariant under the global group transformations (F.1)-(F.5), and which *possibly but not necessarily* could scale the region of the inertial layer within ZPG turbulent boundary layer flow. But, since there can be only *one* physical realization for *all* moments, which *all* are driven by the *same* single deterministic velocity field **u** according to (3.1), and since the performed statistical invariance analysis in Oberlack & Rosteck (2010) did not appropriately involve this underlying deterministic layer of description, the chances are extremely low that exactly this determined set (F.9) or (F.10) of invariant functions should represent the statistically correct and thus for all correlation orders consistent solution to the complex inertial scaling problem of incompressible wall-bounded turbulence (aside from the fact, of course, that these functions are additionally based on unphysical reasoning and thus being physically void). The obvious reason for this negative result is that currently no method (including the invariant Lie-group method) exists to establish a profound and at the same time an *analytically* accessible and correct connection between the deterministic and the statistical description of the wall-bounded Navier-Stokes theory.

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