FLEX-ON: 1st Report
November 25th, 2015

POLITECNICO DI TORINO

Time-Domain Hybrid Pulse Amplitude Modulation

Power-Ratio and Ideal B2B Performance

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OPTCOM

Outline

- BER Estimation for Square and Rectangular QAM;
  - Mathematical Formulation;
  - Numerical Results.

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- Hybrid PAM;
  - Mathematical Formulation;
  - BER Estimation;
  - Power-Ratio Management;
  - Numerical Results.

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  - Numerical Results.
- Hybrid PAM;
  - Mathematical Formulation;
  - BER Estimation;
  - Power-Ratio Management;
  - Numerical Results.
- Appendices
  - A - Mean Constellation Power;
  - B - Analytical Attempt to Determine the Optimum SNR for Minimum BER Operation;

BER Estimation for Square and Rectangular QAM

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BER Estimation from SNR: Square QAM

- Most commonly, the BER of square QAM can be estimated from the SNR, by applying the following **closed-form expression**:

$$\Psi\left[\mathrm{SNR}, M\right] \simeq \frac{2}{\log_2(M)}\left(1-\frac{1}{\sqrt{M}}\right)\mathrm{erfc}\left(\sqrt{\frac{3\,\mathrm{SNR}}{2(M-1)}}\right), \quad (1)$$

  where

  - M - number of symbols in the constellation;
  - SNR - SNR per symbol;
  - $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2} dt$ is the complementary error function.

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- However, equation 1 neglects the bit errors originated at inter-symbol distances larger than the minimum Euclidean distance;
  - It therefore tends to underestimate BER in the low SNR (high BER) regime;

BER Estimation from SNR: Square QAM

- An **exact formulation** for BER estimation in an AWGN channel has been derived in (1), as a **weighting average of all possible bit error probabilities** between all symbols in the QAM constellation:

$$\Psi\left[\text{SNR}, M\right] = \frac{2}{\log_2(M)} \sum_{k=1}^{\log_2(\sqrt{M})} \Phi(k), \quad (2)$$

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$$\Phi(k) = \frac{1}{\sqrt{M}} \sum_{i=0}^{(1-2^{-k})\sqrt{M}-1} \left\{ (-1)^{\left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} \right\rfloor} \left(2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} + \frac{1}{2} \right\rfloor \right) \operatorname{erfc}\left((2i+1)\sqrt{\frac{3\,\mathrm{SNR}}{2(M-1)}} \right) \right\}, \tag{3}$$

where $\lfloor \cdot \rfloor$ denotes the nearest integer towards minus infinity.

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where $\lfloor \cdot \rfloor$ denotes the nearest integer towards minus infinity.

- For high SNR, the **first term ($i = 0$) is dominant** in equation (3). By keeping only the first term of each $\Phi(k)$ component, we arrive at the approximated expression of equation (1).

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- Note that, following the formulation of (2), the exact BER of QPSK can be calculated by a single term (the approximated formulation of (2) is exact for QPSK);

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- Note that, following the formulation of (2), the exact BER of QPSK can be calculated by a single term (the approximated formulation of (2) is exact for QPSK);
- Then, the number of BER components increases with the QAM constellation size.
  - $k \in [1, 2]$ for 16QAM;
  - $k \in [1, 2, 3]$ for 64QAM;
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BER Estimation from SNR: Rectangular QAM

- The exact BER expression for square QAM can be **generalized for any kind of rectangular QAM** (1):

$$\Psi\left[\text{SNR}, M_{\text{I}}, M_{\text{Q}}\right] = \frac{1}{\log_2(M_{\text{I}} \cdot M_{\text{Q}})} \left(\sum_{k=1}^{\log_2(M_{\text{I}})} \Phi_{\text{I}}(k) + \sum_{n=1}^{\log_2(M_{\text{Q}})} \Phi_{\text{Q}}(n) \right), \tag{4}$$

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where M_I and M_Q are the number of in-phase and quadrature levels, respectively. $\mathrm{BER}_\mathrm{I}(k)$ and $\mathrm{BER}_\mathrm{Q}(n)$ are the correspondent k-th BER components, given by:

$$\Phi_\mathrm{I}(k) = \frac{1}{M_\mathrm{I}}\sum_{i=0}^{(1-2^{-k})M_\mathrm{I}-1}\left\{(-1)^{\left\lfloor\frac{i\cdot 2^{k-1}}{M_\mathrm{I}}\right\rfloor}\left(2^{k-1}-\left\lfloor\frac{i\cdot 2^{k-1}}{M_\mathrm{I}}+\frac{1}{2}\right\rfloor\right)\mathrm{erfc}\left((2i+1)\sqrt{\frac{3\,\mathrm{SNR}}{M_\mathrm{I}^2+M_\mathrm{Q}^2-2}}\right)\right\}, \quad (5)$$

$$\Phi_\mathrm{Q}(n) = \frac{1}{M_\mathrm{Q}}\sum_{j=0}^{(1-2^{-n})M_\mathrm{Q}-1}\left\{(-1)^{\left\lfloor\frac{j\cdot 2^{n-1}}{M_\mathrm{Q}}\right\rfloor}\left(2^{n-1}-\left\lfloor\frac{j\cdot 2^{n-1}}{M_\mathrm{Q}}+\frac{1}{2}\right\rfloor\right)\mathrm{erfc}\left((2j+1)\sqrt{\frac{3\,\mathrm{SNR}}{M_\mathrm{I}^2+M_\mathrm{Q}^2-2}}\right)\right\}. \quad (6)$$

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- Similarly to the case of square QAM, an **approximated formula for high SNR** can be obtained by keeping only the $i=0$ and $j=0$ terms:

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- Also note that, if $M_{\mathrm{I}} = M_{\mathrm{Q}} = \sqrt{M}$ (square QAM), then expression (7) reduces to equation (1);
- These exact and approximated formulations allow for a **generalized treatment of any QAM** (square or rectangular) **or even PAM** ($M_{\mathrm{I}} \times 1$ or $1 \times M_{\mathrm{Q}}$) constellation.

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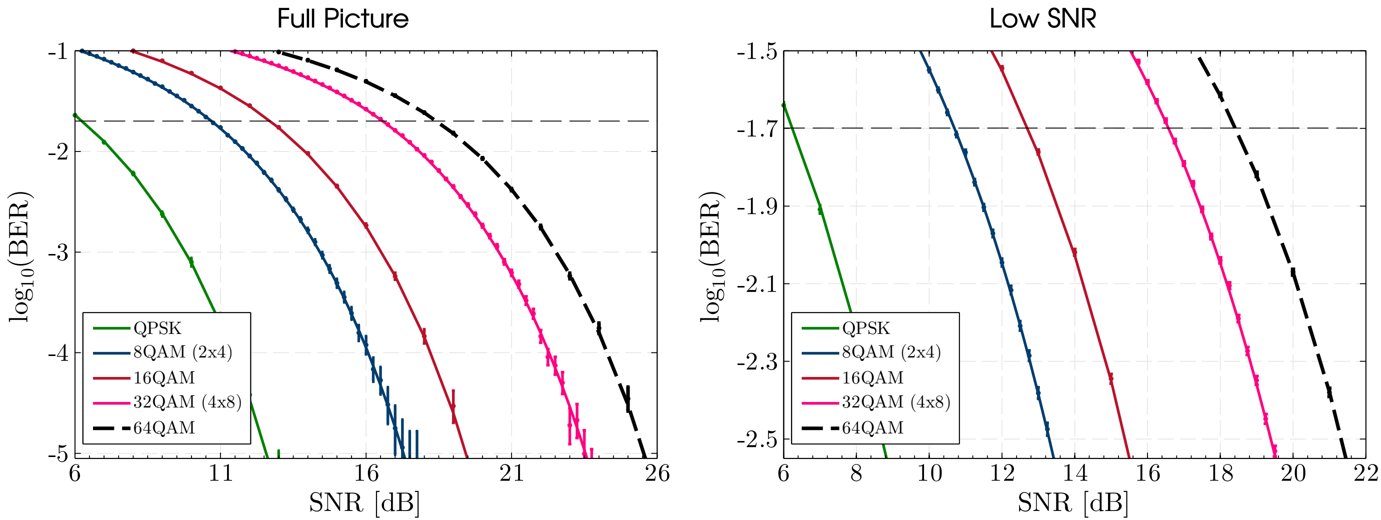
BER Estimation from SNR: Numerical Results

- We have ran **simulations in an AWGN channel**, and compared the counted BER with the exact predictions given by expression (4):



- There is a **very good agreement** both in high and low SNR regions;
- It is important to note, however, that the rectangular 8QAM and 32QAM constellations are sub-optimal when compared with their cross-QAM implementation.

OPTCOM

Hybrid PAM - Mathematical Framework

Hybrid PAM: Mathematical Formulation

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$$\mathbf{M} = \begin{bmatrix} M_{\text{I}}(1) & M_{\text{I}}(2) & \dots & M_{\text{I}}(N_{\text{PAM}}) \\ M_{\text{Q}}(1) & M_{\text{Q}}(2) & \dots & M_{\text{Q}}(N_{\text{PAM}}) \end{bmatrix}. \tag{8}$$

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  - a $[2 \times N_{\rm PAM}]$ matrix defining the **power ratio** between formats (and between I and Q):

$$\mathbf{P} = \begin{bmatrix} P_{\rm I}(1) & P_{\rm I}(2) & \dots & P_{\rm I}(N_{\rm PAM}) \\ P_{\rm Q}(1) & P_{\rm Q}(2) & \dots & P_{\rm Q}(N_{\rm PAM}) \end{bmatrix}, \quad (9)$$

  where $P_{\rm I}(n)/P_{\rm Q}(n)$ defines the power ratio between I and Q in the n-th format and $P_{\rm I/Q}(n)/P_{\rm I/Q}(k)$ defines the power ratio between the I/Q components of the n-th and k-th format in the frame;

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  - a $[1 \times N_{\mathrm{PAM}}]$ vector defining the **time ratio** of each format in the frame:

$$\mathbf{F} = \begin{bmatrix} F(1) & F(2) & \ldots & F(N_{\mathrm{PAM}}) \end{bmatrix}, \tag{10}$$

    where $F(n)$ defines the time ratio occupied by the n-th format in the frame. Note that $\sum_{n=1}^{N_{\mathrm{PAM}}} F(n) = 1$.

Hybrid PAM: Mathematical Formulation

- Associated with the frame ratio matrix, $\mathbf{F}$, and the number of symbols per frame, N_{sym}, we can define the following a **set of indices**:

$$[J(k)] = \left\{ j : N_{\text{sym}} \sum_{i=1}^{k-1} F(i) < j \leq N_{\text{sym}} \sum_{i=1}^{k} F(i), j \in \mathbb{N} \right\}, \quad (11)$$

where k is the modulation format index in the frame and $[J(k)]$ **are the indices of the corresponding symbols** in the hybrid PAM frame.

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- In addition, note that the sets of possible discrete values for the **electrical field in the I and Q components** (normalized to unit power) are given by:

$$[I(M_{\text{I}})] = \left\{ \frac{2k - \text{sgn}(k)}{\sqrt{\frac{2M_{\text{I}}^2 - 2}{3}}} : -\frac{M_{\text{I}}}{2} \leq k \leq \frac{M_{\text{I}}}{2}, k \in \mathbb{Z} \setminus \{0\} \right\}, \tag{13}$$

$$[Q(M_{\text{Q}})] = \left\{ \frac{2k - \text{sgn}(k)}{\sqrt{\frac{2M_{\text{Q}}^2 - 2}{3}}} : -\frac{M_{\text{Q}}}{2} \leq k \leq \frac{M_{\text{Q}}}{2}, k \in \mathbb{Z} \setminus \{0\} \right\}. \tag{14}$$

Hybrid PAM: Mathematical Formulation

- Utilizing the previously definitions, we can **mathematically describe the n-th sample of the electrical field, $E(n)$**, of an hybrid PAM signal:

$$E(n) = \mathbf{P}^{(1,[J(k)])} \cdot [I(M_{\mathrm{I}}(k))]^{(m_{\mathrm{I}}(k))} + i\mathbf{P}^{(2,[J(k)])} \cdot \left[Q(M_{\mathrm{Q}}(k))\right]^{(m_{\mathrm{Q}}(k))}, \quad (15)$$

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  - $m_{\mathrm{I}}(k)$ **and** $m_{\mathrm{Q}}(k)$ **are the in-phase and quadrature electrical field mapping indices**, where the information is encoded by mapping a given I/Q symbol to the correspondent electrical field components, $[I(M_{\mathrm{I}}(k))]$ and $[Q(M_{\mathrm{Q}}(k))]$.

Hybrid PAM - BER Estimation

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- Assuming the hybrid PAM definition enclosed by expressions (8)-(15), we may **generalize the mathematical formulation of BER estimation** as:

$$\Psi_{\mathrm{TDHMF}} = \frac{\sum_{n=1}^{N_{\mathrm{PAM}}} \log_2 \left(M_{\mathrm{I}}(n) \cdot M_{\mathrm{Q}}(n)\right) F(n) \cdot \Psi \left[\mathrm{SNR}(n), M_{\mathrm{I}}(n), M_{\mathrm{Q}}(n)\right]}{\sum_{n=1}^{N_{\mathrm{PAM}}} \log_2 \left(M_{\mathrm{I}}(n) \cdot M_{\mathrm{Q}}(n)\right) F(n)} \tag{17}$$

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$$\overline{P}(n) = \frac{P_{\mathrm{I}}(n) + P_{\mathrm{Q}}(n)}{2} \tag{19}$$

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  - $\Psi[\text{SNR}(n), M_{\text{I}}(n), M_{\text{Q}}(n)]$ **is the BER estimation function** for the n-th PAM format in the frame, either using the exact or approximate formulation of expressions (4) and (7).

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$$\frac{P_{\mathrm{I}}(n)}{P_{\mathrm{Q}}(n)} = \frac{M_{\mathrm{I}}(n)^2 - 1}{M_{\mathrm{Q}}(n)^2 - 1}, \qquad 1 \leq n \leq N_{\mathrm{sym}}, \qquad n \in \mathbb{N}. \tag{20}$$

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  - **same BER performance** for all formats at a given target BER;
  - **minimum overall BER** for a given average SNR.

Hybrid PAM: Power-Ratio - Constant Power

- In order to guarantee **constant power between modulation formats** in the hybrid PAM frame (and still guarantee the same Euclidean distance between I and Q components in each format), the power ratio matrix, $\mathbf{P}$, should be defined as follows:

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- Note that the **ratio between the numerators** of each $\mathbf{P}$ row guarantees the same Euclidean distance between I and Q;
- In turn, the **ratio between the denominators** of each $\mathbf{P}$ column guarantees **constant power** between the hybrid PAM symbols.

Hybrid PAM: Power-Ratio - Same Euclidean Distance

- Since in the $\mathbf{P}$ definition of expression (21) all quadratures are normalized to unit power, then it is a sufficient condition to guarantee the **same Euclidean distance** between all modulation formats in the frame.

- The power-ratio matrix, $\mathbf{P}$, is therefore rewritten here:

$$\mathbf{P} = \begin{bmatrix} M_{\mathrm{I}}^2(1) - 1 & M_{\mathrm{I}}^2(2) - 1 & \ldots & M_{\mathrm{I}}^2(N_{\mathrm{PAM}}) - 1 \\ M_{\mathrm{Q}}^2(1) - 1 & M_{\mathrm{Q}}^2(2) - 1 & \ldots & M_{\mathrm{Q}}^2(N_{\mathrm{PAM}}) - 1 \end{bmatrix}. \quad (23)$$

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Hybrid PAM: Power-Ratio - Same BER

- In order to guarantee the **same BER performance** at a given target BER, one must adjust the power-ratio between modulation formats so that the SNR perceived by each one of them is given by:

$$\mathrm{SNR}(n) = \Psi^{-1}\left[\mathrm{BER}, M_{\mathrm{I}}(n), M_{\mathrm{Q}}(n)\right] \quad (24)$$

where $\Psi^{-1}[\cdot]$ is the inverse of approximate BER estimation expression (7). Note that obtaining an inverse of the exact BER expression (4) would require to invert a series of $\mathrm{erfc}(\cdot)$ functions, which may be unfeasible in practice.

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- Note that, relatively to the constant-power condition, the same BER operation only requires to **multiply each P by the respective required SNR**.

Hybrid PAM: Power-Ratio - Minimum BER

- To achieve the **theoretical minimum BER** requires to minimize the BER estimation expression given by (17), which is composed by $N_{\mathrm{PAM}}\ \mathrm{erfc}(\cdot)$ terms;

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- Once the optimum SNR is found for each modulation format, the power ratio matrix is defined similarly to (25):

$$\mathbf{P} = \begin{bmatrix} \mathrm{SNR}_{\mathrm{opt}}(1)\frac{M_{\mathrm{I}}^2(1)-1}{M_{\mathrm{I}}^2(1)+M_{\mathrm{Q}}^2(1)-2} & \mathrm{SNR}_{\mathrm{opt}}(2)\frac{M_{\mathrm{I}}^2(2)-1}{M_{\mathrm{I}}^2(2)+M_{\mathrm{Q}}^2(2)-2} & \cdots & \mathrm{SNR}_{\mathrm{opt}}(N_{\mathrm{PAM}})\frac{M_{\mathrm{I}}^2(N_{\mathrm{PAM}})-1}{M_{\mathrm{I}}^2(N_{\mathrm{PAM}})+M_{\mathrm{Q}}^2(N_{\mathrm{PAM}})-2} \\ \mathrm{SNR}_{\mathrm{opt}}(1)\frac{M_{\mathrm{Q}}^2(1)-1}{M_{\mathrm{I}}^2(1)+M_{\mathrm{Q}}^2(1)-2} & \mathrm{SNR}_{\mathrm{opt}}(2)\frac{M_{\mathrm{Q}}^2(2)-1}{M_{\mathrm{I}}^2(2)+M_{\mathrm{Q}}^2(2)-2} & \cdots & \mathrm{SNR}_{\mathrm{opt}}(N_{\mathrm{PAM}})\frac{M_{\mathrm{Q}}^2(N_{\mathrm{PAM}})-1}{M_{\mathrm{I}}^2(N_{\mathrm{PAM}})+M_{\mathrm{Q}}^2(N_{\mathrm{PAM}})-2} \end{bmatrix}. \quad (26)$$

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- The essential difference between the ``*same BER*'' and ``*minimum BER*'' strategies lies on the **determination of the SNR** associated to each modulation format:
  - **same BER**: the required SNR per format is **independently determined**;
  - **minimum BER**: the optimum SNR per format requires a **joint optimization process**.

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Hybrid PAM: Power-Ratio - Minimum BER

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- Once the optimum SNR is found for each modulation format, the power ratio matrix is defined similarly to (25):

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  - **same BER**: the required SNR per format is **independently determined**;
  - **minimum BER**: the optimum SNR per format requires a **joint optimization process**.
- In general, to minimize the BER of a frame composed of N_{PAM} format requires a $N_{\text{PAM}}-1$-dimensional optimization process;
- This means that, in practice, if the frame is composed of more than two formats, then the optimization process becomes quite complex.

Hybrid PAM: Power-Ratio - Minimum BER

- In the following, we specify a possible strategy for the **BER minimization with $N_{\text{PAM}} = 2$**;

Hybrid PAM: Power-Ratio - Minimum BER

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- By defining a **range of trial values for** $\mathrm{SNR}(1)$, the corresponding vector of trial $\mathrm{SNR}(2)$ values is obtained as:

$$\mathrm{SNR}(2) = \frac{\overline{\mathrm{SNR}} - F(1)\mathrm{SNR}(1)}{F(2)} \qquad (27)$$

where $\overline{\mathrm{SNR}}$ is the given average SNR of the entire frame.

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- The estimated BERs are then calculated via the exact expression (4);

- The **optimum SNRs** are then obtained as:

$$\mathrm{SNR}_{\mathrm{opt}}(1) = \left\{ \mathrm{SNR}(1) : \Psi_{\mathrm{TDHMF}}\left[\mathrm{SNR}(1), \mathrm{SNR}(2)\right] = \min\left\{ \Psi_{\mathrm{TDHMF}}\left[\mathrm{SNR}(1), \mathrm{SNR}(2)\right] \right\} \right\} \tag{28}$$

$$\mathrm{SNR}_{\mathrm{opt}}(2) = \frac{\overline{\mathrm{SNR}} - F(1)\mathrm{SNR}_{\mathrm{opt}}(1)}{F(2)} \tag{29}$$

Hybrid PAM: Power-Ratio - Minimum BER

- Alternatively, it is also possible to **analytically find the minimum BER** by evaluating the derivative of the approximate BER estimation formula of expression (7):

$$\frac{\partial \Psi_{\mathrm{TDHMF}}\left[\mathrm{SNR}(1), \frac{\overline{\mathrm{SNR}} - F(1)\mathrm{SNR}(1)}{F(2)}\right]}{\partial \mathrm{SNR1}} = 0, \tag{30}$$

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which, after some algebra (see Appendix B), yields the following transcendental equation:

$$\left[\frac{\alpha}{x} - \beta\right] e^{\kappa x} = \gamma, \qquad (31)$$

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where

- $x = \mathrm{SNR}(1)$;
- $\alpha = e^{2B_2} B_2$;
- $\beta = e^{2B_2} C_2$;
- $\kappa = -2(C_1 + C_2)$;
- $\gamma = \dfrac{A_2^2 C_2^2}{A_1^2 C_1}$;
- $A_1 = F(1)\left(\dfrac{M_{\mathrm{I}}(1) - 1}{M_{\mathrm{I}}(1)} + \dfrac{M_{\mathrm{Q}}(1) - 1}{M_{\mathrm{Q}}(1)}\right)$;
- $A_2 = F(2)\left(\dfrac{M_{\mathrm{I}}(2) - 1}{M_{\mathrm{I}}(2)} + \dfrac{M_{\mathrm{Q}}(2) - 1}{M_{\mathrm{Q}}(2)}\right)$;
- $B_2 = 3\dfrac{\overline{\mathrm{SNR}}}{F(2)\left(M_{\mathrm{I}}(2)^2 + M_{\mathrm{Q}}(2)^2 - 2\right)}$;
- $C_1 = \dfrac{3}{\left(M_{\mathrm{I}}(1)^2 + M_{\mathrm{Q}}(1)^2 - 2\right)}$;
- $C_2 = \dfrac{3F(1)}{F(2)\left(M_{\mathrm{I}}(2)^2 + M_{\mathrm{Q}}(2)^2 - 2\right)}$.

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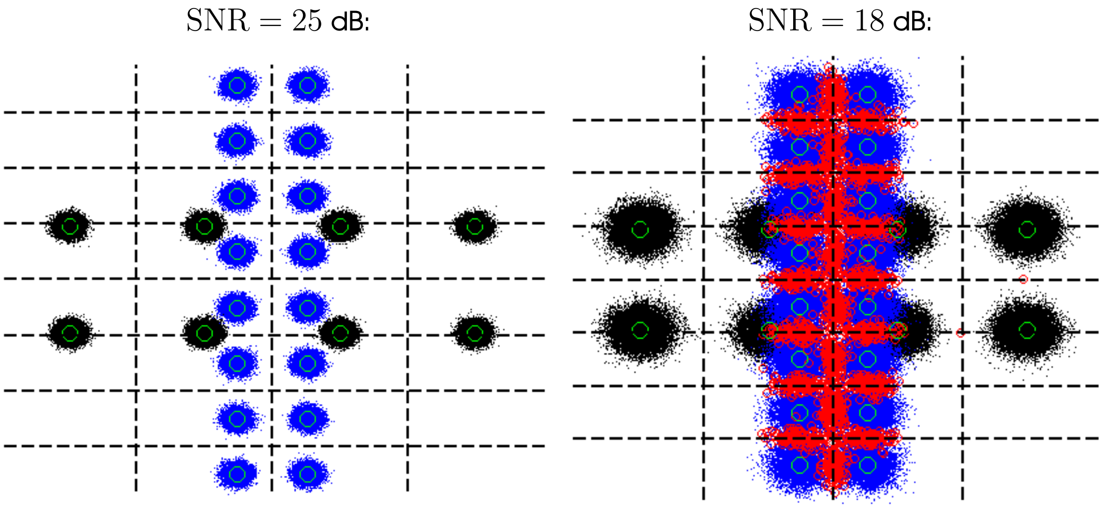
- Unfortunately, my math skills do not allow me to find a closed-form solution for equation (31)... A numerical optimization process is still needed to find x.

Some Examples: Same Power

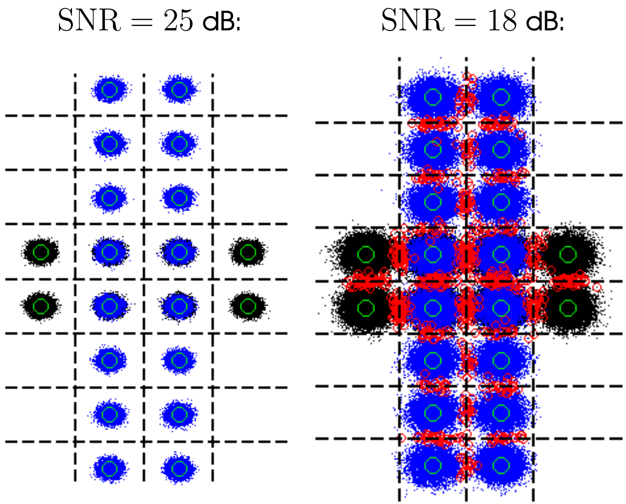
- Example of an exotic hybrid PAM constellation composed of **4×2QAM** and **2×8QAM**;
- The power ratio between formats has been defined based on the **same power** criterion.



- Clearly, one of the formats is being strongly penalized relatively to the other.

Some Examples: Same Euclidean Distance

- Example of an exotic hybrid PAM constellation composed of **4×2QAM** and **2×8QAM**;
- The power ratio between formats has been defined based on the **same Euclidean distance** criterion.



- **All symbols** have the same probability of error. However, this may be sub-optimal, since the two formats carry different number of bits per symbol.

Some Examples: Same BER

- Example of an exotic hybrid PAM constellation composed of **4×2QAM** and **2×8QAM**;
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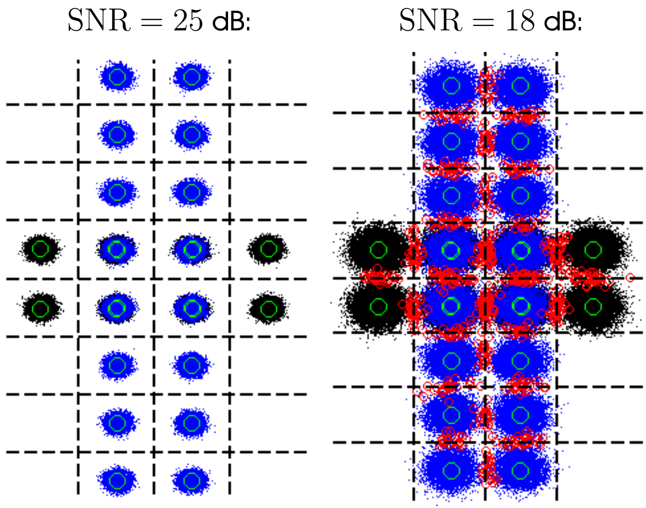
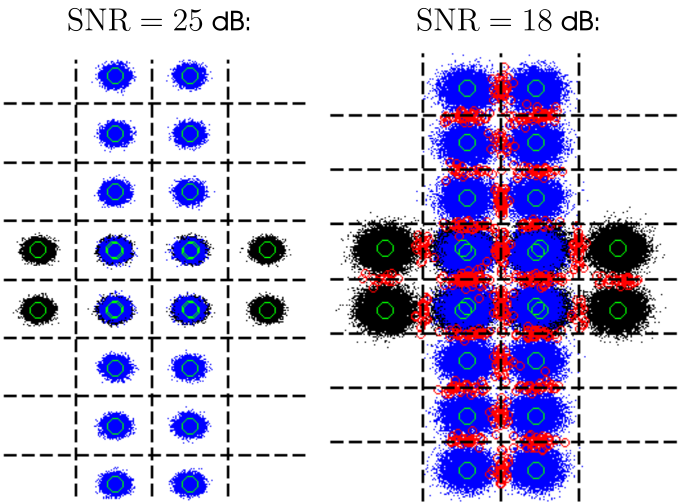


- The probability of error is now the same for both formats. It is still sub-optimum, since the two formats carry different number of bits per symbol.

Some Examples: Minimum BER

- Example of an exotic hybrid PAM constellation composed of **4×2QAM** and **2×8QAM**;
- The power ratio between formats has been defined based on the **minimum BER** criterion.

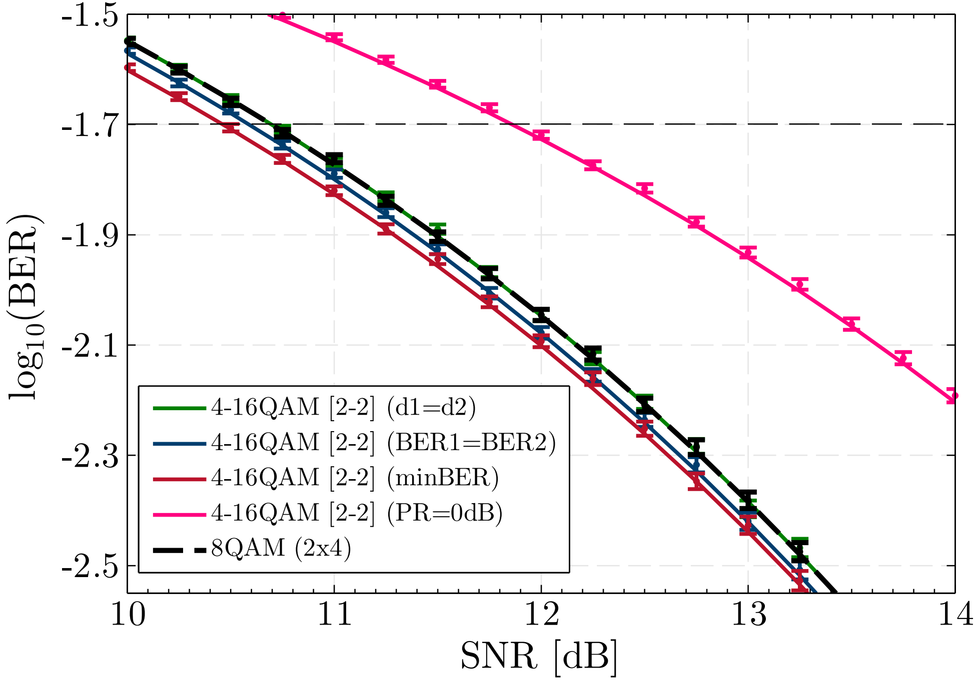


- The BER of the ensemble hybrid PAM frame is minimized - **optimum solution**.
- Note that the power-ratio between formats actually depends on the SNR.

Hybrid PAM - Numerical Results

Hybrid 4/16QAM: Numerical Results

- Performance of Hybrid 4-16QAM with **50%-50%** power ratio (**3 bits per symbol**):



- Good agreement between numerical results and analytical BER estimation;
- Good agreement with the OFC 2014 results (except for the 8QAM, which is rectangular instead of star QAM).

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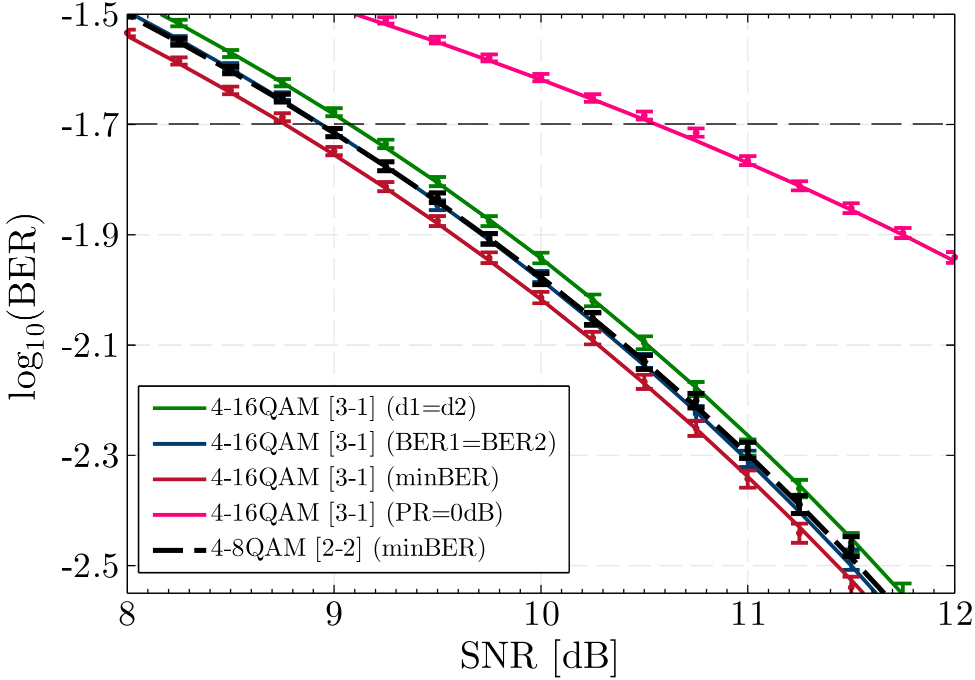
Hybrid 4/16QAM: Numerical Results

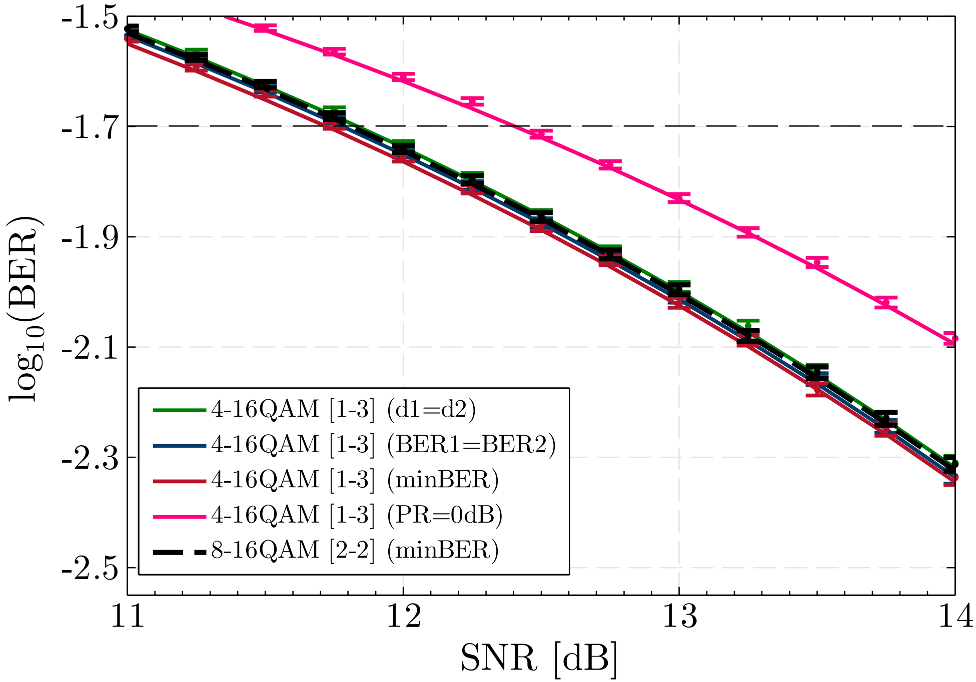
- Performance of Hybrid 4-16QAM with **75%-25%** power ratio (**2.5 bits per symbol**):



- Hybrid 4-16QAM (75%-25%) is a slightly better solution than Hybrid 4-8QAM (50%-50%);
- Most probably due to the sub-optimum performance of rectangular 8QAM.

Hybrid 4/16QAM: Numerical Results

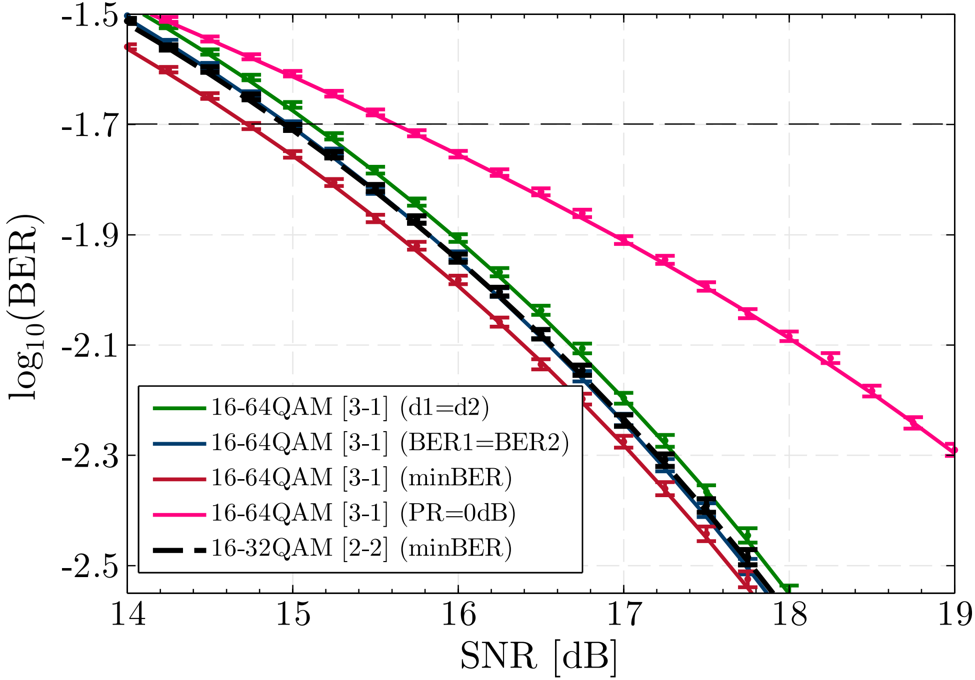
- Performance of Hybrid 4-16QAM with **25%-75%** power ratio (**3.5 bits per symbol**):



- Hybrid 4-16QAM (25%-75%) is a slightly better solution than Hybrid 8-16QAM (50%-50%);
- Most probably due to the sub-optimum performance of rectangular 8QAM.

Hybrid 16/64QAM: Numerical Results

- Performance of Hybrid 16-64QAM with **75%-25%** power ratio (**4.5 bits per symbol**):



- Hybrid 16-64QAM (75%-25%) is a slightly better solution than Hybrid 16-32QAM (50%-50%);
- Most probably due to the sub-optimum performance of rectangular 32QAM.

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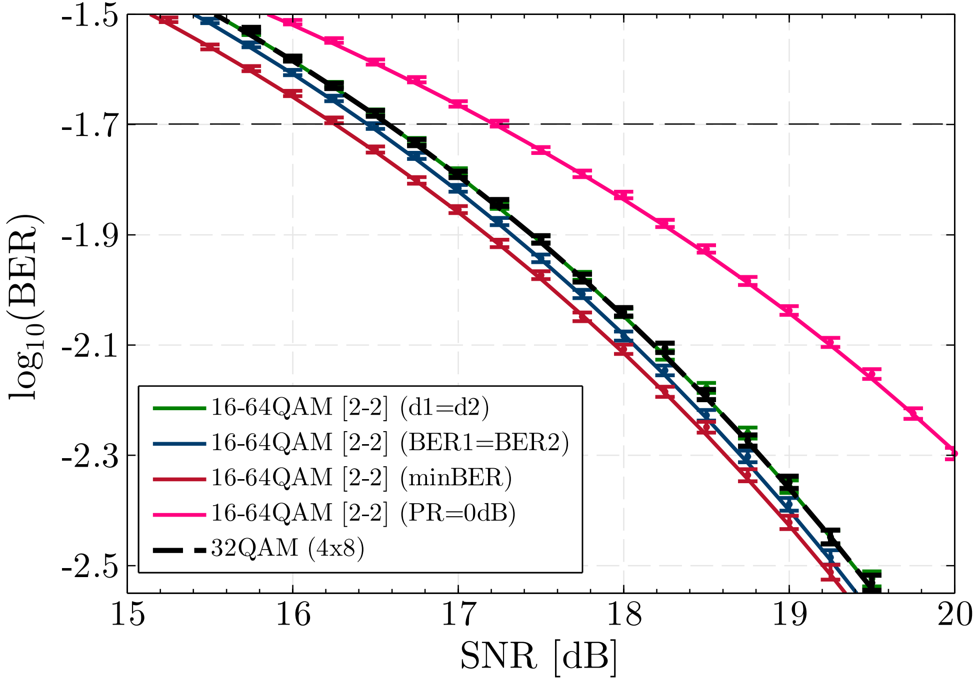
Hybrid 16/64QAM: Numerical Results

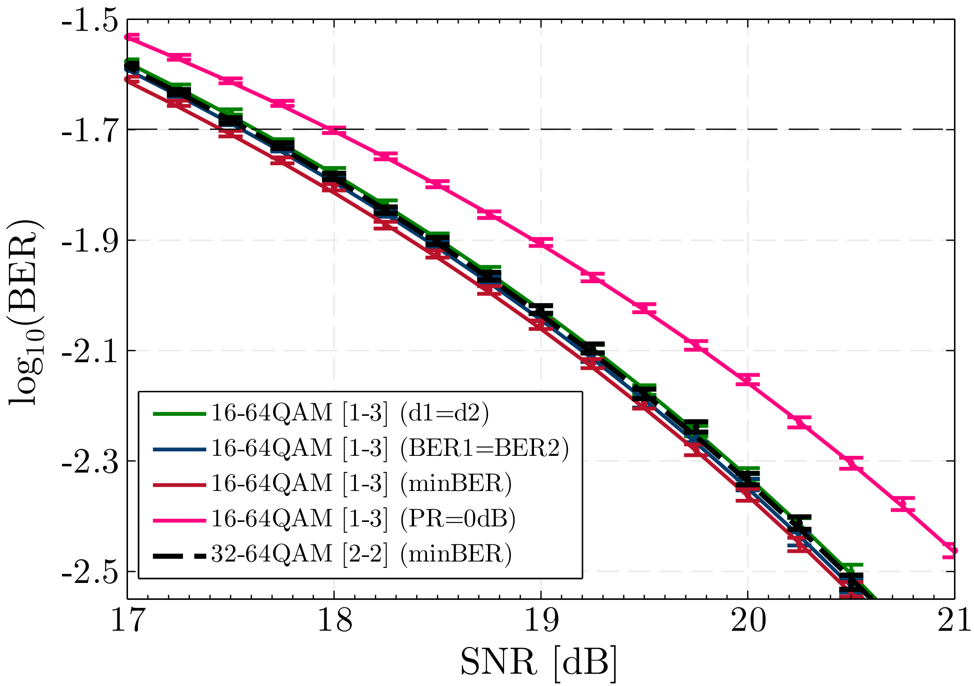
- Performance of Hybrid 16-64QAM with **50%-50%** power ratio (**5 bits per symbol**):



- Hybrid 16-64QAM (50%-50%) is a slightly better solution than 32QAM;
- Most probably due to the sub-optimum performance of rectangular 32QAM.

Hybrid 16/64QAM: Numerical Results

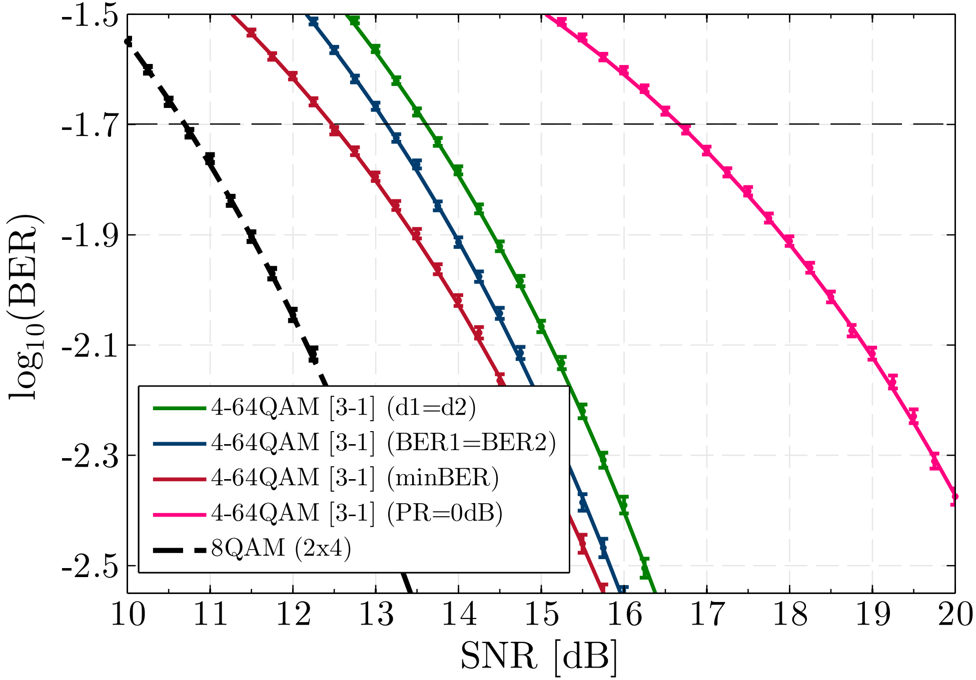
- Performance of Hybrid 16-64QAM with **25%-75%** power ratio (**5.5 bits per symbol**):



- Hybrid 16-64QAM (25%-75%) is a slightly better solution than Hybrid 32-64QAM (50%-50%);
- Most probably due to the sub-optimum performance of rectangular 32QAM.

Hybrid 4/64QAM: Numerical Results

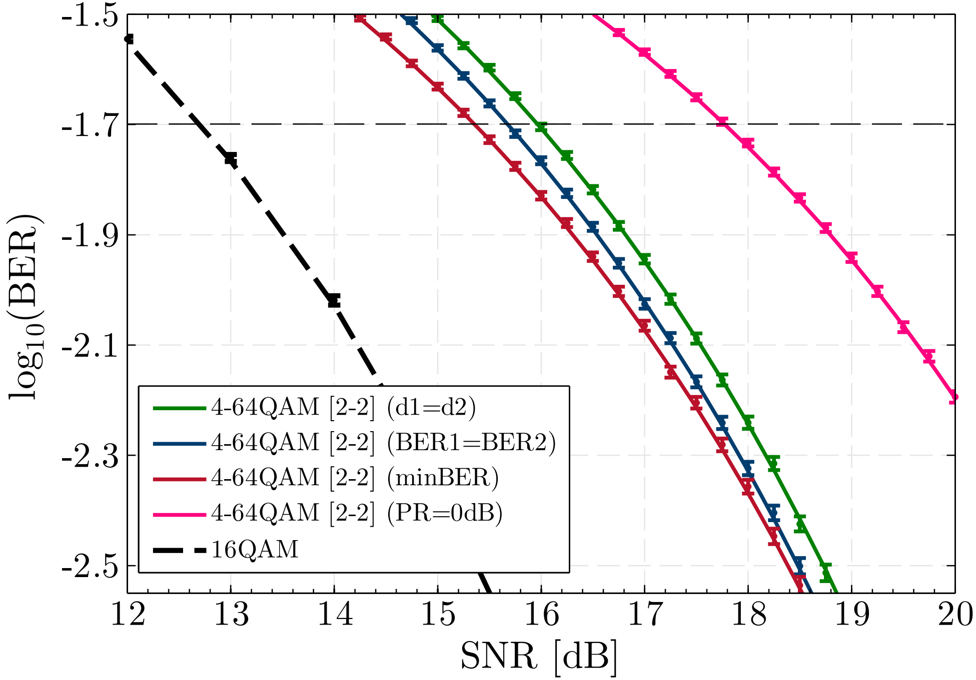
- Performance of Hybrid 4-64QAM with **75%-25%** power ratio (**3 bits per symbol**):



- Hybrid 4-64QAM (75%-25%) is a much worse solution than 8QAM (∼2.5 dB penalty);
- It seems that hybridizing formats with very different constellation size is not advantageous.

Hybrid 4/64QAM: Numerical Results

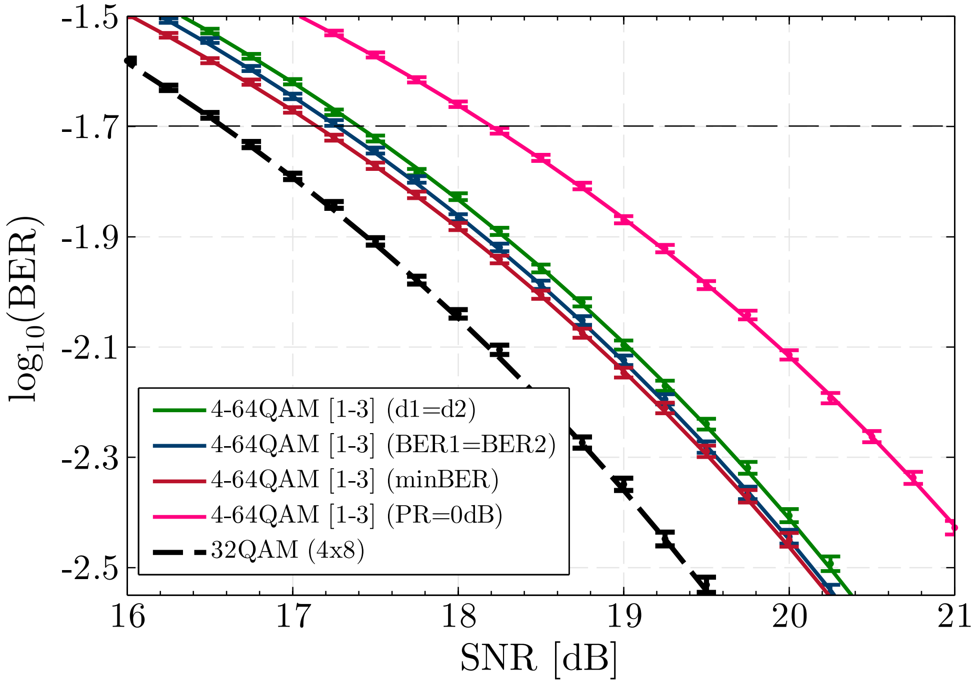
- Performance of Hybrid 4-64QAM with **50%-50%** power ratio (**4 bits per symbol**):



- Hybrid 4-64QAM (50%-50%) is a much worse solution than 16QAM (∼2.5 dB penalty);
- It seems that hybridizing formats with very different constellation size is not advantageous.

Hybrid 4/64QAM: Numerical Results

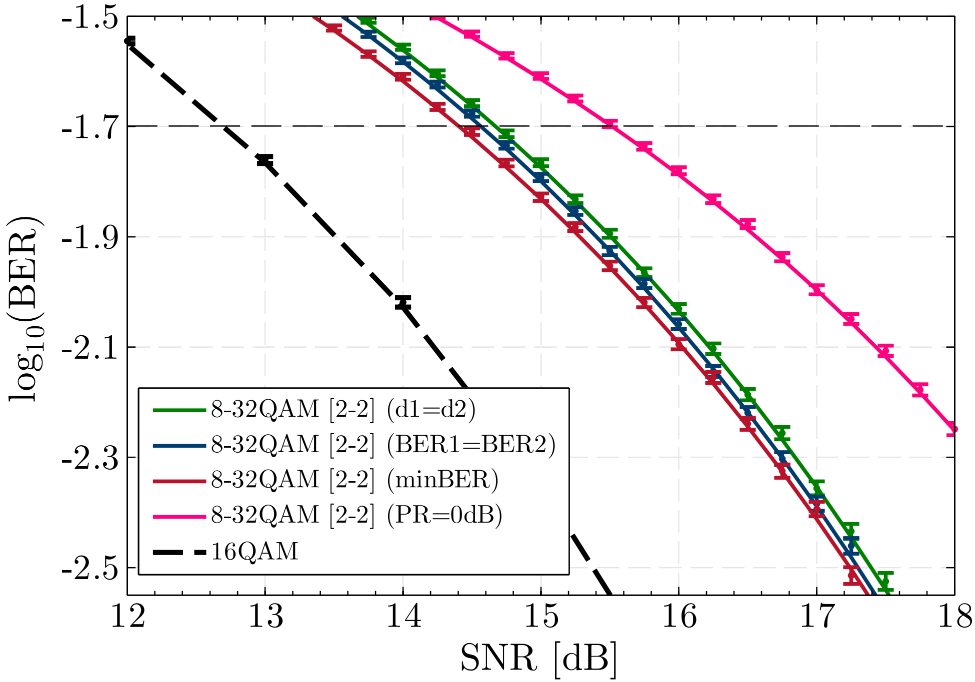
- Performance of Hybrid 4-64QAM with **25%-75%** power ratio (**5 bits per symbol**):



- Hybrid 4-64QAM (25%-75%) yields a penalty of ~ 0.6 dB over 32QAM;
- Surprisingly, the penalty is significantly lower with this configuration.

Hybrid 8/32QAM: Numerical Results

- Performance of Hybrid 8-32QAM with **50%-50%** power ratio (**4 bits per symbol**):



- Hybrid 8-32QAM (50%-50%) yields a penalty of ~ 1.5 dB over 16QAM;
- Maybe in this case the penalty is mostly due the sub-optimum performance of 8QAM and 32QAM.

Appendix A - Average Constellation Power

Average Constellation Power of Rectangular QAM

- Any rectangular QAM constellation (if centered in the origin) is fully symmetric over the 4 quadrants of the complex plane;
- To analyze the average power of a general rectangular QAM constellation, we may double-fold it over one of its quadrants:

$$\begin{aligned}
\overline{P} &= \frac{4}{M_\mathrm{I}\cdot M_\mathrm{Q}} \sum_{Q=1}^{M_\mathrm{Q}/2} \sum_{I=1}^{M_\mathrm{I}/2} \left[(2I-1)^2 + (2Q-1)^2\right] \\
&= \frac{4}{M_\mathrm{I}\cdot M_\mathrm{Q}} \sum_{Q=1}^{M_\mathrm{Q}/2} \left[\frac{M_\mathrm{I}}{2}(2Q-1)^2 + \sum_{I=1}^{M_\mathrm{I}/2}(2I-1)^2\right] \\
&= \frac{4}{M_\mathrm{I}\cdot M_\mathrm{Q}} \sum_{Q=1}^{M_\mathrm{Q}/2} \left[\frac{M_\mathrm{I}}{2}(2Q-1)^2 + \frac{M_\mathrm{I}^3 - M_\mathrm{I}}{6}\right] \\
&= \frac{4}{M_\mathrm{I}\cdot M_\mathrm{Q}} \frac{M_\mathrm{I}}{2} \sum_{Q=1}^{M_\mathrm{Q}/2} (2Q-1)^2 + \frac{M_\mathrm{Q}}{2}\frac{M_\mathrm{I}^3 - M_\mathrm{I}}{6} \\
&= \frac{4}{M_\mathrm{I}\cdot M_\mathrm{Q}} \frac{M_\mathrm{I}(M_\mathrm{Q}^3 - M_\mathrm{Q}) + M_\mathrm{Q}(M_\mathrm{I}^3 - M_\mathrm{I})}{12} = \frac{M_\mathrm{I}^2 + M_\mathrm{Q}^2 - 2}{3}
\end{aligned} \tag{32}$$

Average Constellation Power of Rectangular QAM

- If the I and Q components are divided by general factors C_I and C_Q, respectively, expression (32) can be generalized as:

$$\overline{P} = \frac{M_\mathrm{I}^2 - 1}{3C_\mathrm{I}^2} + \frac{M_\mathrm{Q}^2 - 1}{3C_\mathrm{Q}^2} \tag{33}$$

Average Constellation Power of Rectangular QAM

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- Now, let us consider that the I and Q components are normalized to unit power (in the complex plane) and then scaled according to the power ratio defined in the $\mathbf{P}$ matrix. The corresponding mean constellation power is given by:

$$\begin{aligned}\overline{P} &= \frac{M_{\mathrm{I}}^2 - 1}{3\left(\frac{\sqrt{\frac{2M_{\mathrm{I}}^2-2}{3}}}{\sqrt{P_{\mathrm{I}}}}\right)^2} + \frac{M_{\mathrm{Q}}^2 - 1}{3\left(\frac{\sqrt{\frac{2M_{\mathrm{Q}}^2-2}{3}}}{\sqrt{P_{\mathrm{Q}}}}\right)^2} \\ &= P_{\mathrm{I}}\frac{M_{\mathrm{I}}^2 - 1}{2M_{\mathrm{I}}^2 - 2} + P_{\mathrm{Q}}\frac{M_{\mathrm{Q}}^2 - 1}{2M_{\mathrm{Q}}^2 - 2} \\ &= \frac{P_{\mathrm{I}} + P_{\mathrm{Q}}}{2}\end{aligned} \tag{34}$$

Appendix B - Analytical Attempt to Determine the Optimum SNR for Minimum BER Operation

Analytical Determination of Optimum SNR

- Determining the optimum SNR for the first modulation format, $\mathrm{SNR}(1)$, in a frame composed of 2 modulation formats ($N_{\mathrm{PAM}} = 2$) involves solving the following differential equation:

$$\frac{\partial \Psi_{\mathrm{TDHMF}}\left[\mathrm{SNR}(1), \frac{\overline{\mathrm{SNR}}-F(1)\mathrm{SNR}(1)}{F(2)}\right]}{\partial \mathrm{SNR1}} = 0$$

$$\Leftrightarrow F(1)\log_2\left(M_{\mathrm{I}}(1)M_{\mathrm{Q}}(1)\right)\frac{\partial \Psi\left[\mathrm{SNR}(1)\right]}{\partial \mathrm{SNR1}} = -F(2)\log_2\left(M_{\mathrm{I}}(2)M_{\mathrm{Q}}(2)\right)\frac{\partial \Psi\left[\frac{\overline{\mathrm{SNR}}-F(1)\mathrm{SNR}(1)}{F(2)}\right]}{\partial \mathrm{SNR1}}, \quad (35)$$

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$$\Leftrightarrow F(1)\log_2\left(M_{\mathrm{I}}(1)M_{\mathrm{Q}}(1)\right)\frac{\partial \Psi\left[\mathrm{SNR}(1)\right]}{\partial \mathrm{SNR1}} = -F(2)\log_2\left(M_{\mathrm{I}}(2)M_{\mathrm{Q}}(2)\right)\frac{\partial \Psi\left[\frac{\overline{\mathrm{SNR}} - F(1)\mathrm{SNR}(1)}{F(2)}\right]}{\partial \mathrm{SNR1}}, \quad (35)$$

- The derivative of the $\mathrm{erfc}(\cdot)$ function is defined and the following derivative pairs can be used to solve (35):

$$\frac{\partial}{\partial x}\mathrm{erfc}\left(\sqrt{c_1 x}\right) = -\frac{c_1}{\sqrt{c_1 \pi x}}e^{-c_1 x}, \quad (36)$$

$$\frac{\partial}{\partial x}\mathrm{erfc}\left(\sqrt{c_2 - c_1 x}\right) = \frac{c_1}{\sqrt{\pi(c_2 - c_1 x)}}e^{-c_2 + c_1 x}, \quad (37)$$

Analytical Determination of Optimum SNR

- To simplify the notation, let us define the following constants:
  - $A_1 = F(1)\left(\frac{M_{\mathrm{I}}(1)-1}{M_{\mathrm{I}}(1)} + \frac{M_{\mathrm{Q}}(1)-1}{M_{\mathrm{Q}}(1)}\right);$
  - $A_2 = F(2)\left(\frac{M_{\mathrm{I}}(2)-1}{M_{\mathrm{I}}(2)} + \frac{M_{\mathrm{Q}}(2)-1}{M_{\mathrm{Q}}(2)}\right);$
  - $B_2 = 3\frac{\overline{\mathrm{SNR}}}{F(2)\left(M_{\mathrm{I}}(2)^2 + M_{\mathrm{Q}}(2)^2 - 2\right)};$
  - $C_1 = \frac{3}{\left(M_{\mathrm{I}}(1)^2 + M_{\mathrm{Q}}(1)^2 - 2\right)};$
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$$-\frac{A_1 C_1}{\sqrt{C_1 \pi \mathrm{SNR}(1)}} e^{-C_1 \mathrm{SNR}(1)} = \frac{A_2 C_2}{\sqrt{\pi(B_2 - C_2 \mathrm{SNR}(1))}} e^{-B_2 + C_2 \mathrm{SNR}(1)} \quad (38)$$

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- Since $\mathrm{SNR}(1)$ and all constants in (38) are supposed to be positive, we can apply the $(\cdot)^2$ operation to both sides of the equation without loss of generality, yielding:

$$\left[\frac{e^{2B_2}B_2}{\mathrm{SNR}(1)} - e^{2B_2}C_2\right]e^{-2(C_1+C_2)\mathrm{SNR}(1)} = \frac{A_2^2C_2^2}{A_1^2C_1} \tag{39}$$

Analytical Determination of Optimum SNR

- Equation (39) is a transcendental equation of the form

$$\left[\frac{\alpha}{x} - \beta\right] e^{\kappa x} = \gamma, \qquad (40)$$

with

- $x = \mathrm{SNR}(1)$;
- $\alpha = e^{2B_2} B_2$;
- $\beta = e^{2B_2} C_2$;
- $\kappa = -2(C_1 + C_2)$;
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- Anyway, equation (39) can be solved numerically, in a similar way as described in (28), with the advantage of not requiring the computation of $\mathrm{erfc}(\cdot)$ functions.

Thanks for your attention!

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