# Mathematical Programming Models for Portfolio Optimization Problem: A Review

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*Abstract*—Portfolio optimization problem has received a lot of attention from both researchers and practitioners over the last six decades. This paper provides an overview of the current state of research in portfolio optimization with the support of mathematical programming techniques. On top of that, this paper also surveys the solution algorithms for solving portfolio optimization models classifying them according to their nature in heuristic and exact methods. To serve these purposes, 40 related articles appearing in the international journal from 2003 to 2013 have been gathered and analyzed. Based on the literature review, it has been observed that stochastic programming and goal programming constitute the highest number of mathematical programming techniques employed to tackle the portfolio optimization problem. It is hoped that the paper can meet the needs of researchers and practitioners for easy references of portfolio optimization.

*Keywords*—Portfolio optimization, Mathematical programming, Multi-objective programming, Solution approaches.

## I. INTRODUCTION

PORTFOLIO selection problem has been one of the most important topics of research in modern finance. The problem is concerned with allocating capital over a number of available assets. The main goal of the portfolio selection is to select the best combination of assets that yields the highest expected returns, while at the same time ensuring an acceptable level of risk. Since the future returns of securities returns are unknown at the time of the investment decision is made, portfolio selection problem can be categorized as one of the decision-making under risk.

Over the last decades, several methods have been proposed to solve the portfolio selection problem. Tiryaki and Ahlatcioglu [1] for example proposed to construct a portfolio using analytical hierarchy process methodology whereas [2] and [3] performed a portfolio selection using data envelopment analysis. Outranking methods have also been employed to solve the problem. Some examples are PROMETHEE [4], and ELECTRE [5], [6].

The first portfolio selection model has been developed by Markowitz [7] based on mathematical programming. The so-called mean-variance model assumes that the total return of a portfolio can be described using the mean return of the assets and the variance of return between these assets. For a given level of risk, one can derive the maximum return by maximizing the expected return of a portfolio or alternatively for a given specific return one can derive the minimum risk by minimizing the variance of a portfolio.

After the introduction of mean-variance model, mathematical programming techniques have become essential tools to support financial decision making process and being increasingly applied in practice. Mathematical programming is one of the operations research techniques which seek to maximize or minimize a function of many variables subject to a set of constraints imposed by the nature of the problem being studied and integrality restrictions on some or all of the variables [8]. In contrast to other mathematical tools such as statistical models, forecasting, and simulation, mathematical programming models allow the decision maker to find the best or optimal solution.

In recent years, the development of new techniques in operations research and management science, as well as the progress in computer and information technologies gave rise to new mathematical programming techniques for modeling the portfolio problem. Many models based on mathematical programming have been developed to solve the current portfolio selection problems which involve a complex, yet realistic set of managing constraints. Thus, the purpose of this paper is to present the current state of research in portfolio optimization with the support of mathematical programming techniques by providing a comprehensive review of the existing literature in the field. This paper also surveys the solution algorithms proposed in the literature for solving portfolio optimization models classifying them according to their nature in heuristic and exact solution methods. Based on the literature review, some potential paths for future research within this area are also suggested.

The rest of the paper is organized as follows: Section II provides a literature review of the existing mathematical programming models for portfolio selection problem. Section III discusses the solution approaches used to solve the portfolio selection model. Finally, in Section IV the conclusions and the future direction of the study will be presented.

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## II. MATHEMATICAL PROGRAMMING MODELS

In this section, we present publication in which authors have used mathematical programming techniques to model portfolio optimization problem. For ease of presentation, the mathematical programming models will be divided into two

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main categories, namely, single objective portfolio optimization and multi-objective portfolio optimization.

### A. Single Objective Portfolio Optimization

The first application of mathematical programming for portfolio selection was due to Markowitz [7]. He built up a quadratic programming model for selecting a diversified portfolio of stocks which requires the use of complex non-linear numerical algorithms to solve the problem. In order to simplify the Markowitz model, several authors [9], [10] have proposed linear programming models with a different definition of the risk function. In [9], the authors proposed the mean absolute deviation (MAD) from the mean as the risk measure. The model is however equivalent to the Markowitz model when they possess a multivariate normal distribution of the returns. On the other hand, Young [10] introduced a model which maximizes the minimum return (Maximin) or minimizes the maximum loss (minimax). According to Young [10], the minimax formulation might be a more appropriate method compared to the mean-variance formulation when data are log-normally distributed.

In [11], the authors formulated two different linear programming models based on minimization of MAD and Maximin formulations. These models were then compared to the classical quadratic programming formulation to test to what extent all these formulations provide similar portfolios. The results from this study showed that the Maximin formulation yields the highest return and risk while the quadratic formulation provides the lowest risk. In addition, all the three formulations were found to outperform the top equity fund portfolios in Sweden and performed much better than the market portfolio.

Another linear programming model for the portfolio selection problem is presented by Kamil and Ibrahim [12]. In this study, the problem was modeled as a mean-risk bicriteria portfolio optimization problem with the mean absolute negative deviation of annual return from the average annual return is used as the downside risk. In order to evaluate the performance of the proposed model, the authors compared the results from the proposed model with the results from mean-variance model and MAD model. According to their results, the proposed model provides better returns than the mean-variance and MAD models.

Konno and Yamamoto [13] formulated a portfolio optimization problem with nonconvex transaction cost, minimal transaction unit constraints and cardinality constraints as a nonlinear integer programming problem. The aim of this study is to show that this class of problems can be successfully solved by the state-of-the-art integer programming approaches if absolute deviation is used as the risk measure. Computational experiments for medium size problem using CPLEX yield a good solution within a practical amount of time.

Chiodi et al [14] presented a model for the problem of selecting a portfolio of mutual funds when entering and management commissions are taken into account. This problem was formulated as a mixed integer linear programming (MILP) model using mean semi-absolute deviation. The authors have also designed some heuristic approaches to solve the portfolio problem. The results of the computational experiments proved that the problem can be solved using heuristics effectively and efficiently. The study, however, could be extended to consider leaving commissions which might become a relevant feature of the problem.

In [15], the authors proposed a new portfolio optimization problem based on an extension of the Markowitz model with value-at-risk replacing the variance on the objective function. This problem has been formulated as a MILP model. The authors show that the proposed model can be solved using CPLEX as a solver in a reasonable amount of time if the number of past observations or the number of assets involved in the study is low.

Two different MILP models for solving portfolio selection problem that takes into account minimum transaction lots, transaction costs and cardinality constraints were proposed by Angelelli et al. [16]. The first model is based on the maximization of the worst conditional expectation (CVaR) while the second model is based on the minimization of the MAD. Although from the computational experiments it was found that the CVaR portfolios had more stable returns compared to the MAD model, it required a huge computational time to solve the problem to optimality.

The portfolio optimization problem with real life features of financial market has also been studied in [17]. The authors extended the mean–variance model to include the minimum transaction lots, cardinality constraint, and sector capitalization constraint. As a consequence of considering these constraints, modeling a portfolio selection problem requires the use of mixed integer programming technique and thus, the model is classified as a mixed-integer non-linear programming model. To solve the model, genetic algorithm (GA) was utilized. Based on the computational results, it was found that the proposed model and the solution approach are applicable and reliable in real markets with large number of stocks.

Another study carried out by Golmakani and Fazel [18] was similar to [17], but they considered bounds on holdings constraint in the model. Since their model is a quadratic mixed-integer programming, the authors proposed a heuristic based on particle swarm optimization (PSO) method to tackle the complexity of the extended model. The authors also compared their approach with GA and the computational results clearly proved that the proposed PSO effectively outperforms GA especially in large-scale problems.

Ibrahim et al. [19] proposed single-stage and two-stage stochastic programming models with the objective of the models are to minimize the maximum downside deviation from the expected return. The purpose of this study is to compare the optimal portfolio of the two models. The results showed that the two stage model outperforms the single stage model in both out-of-sample and in-sample analysis. However, the authors noticed that the models had lost the trend information due to the use of original historical data treated as future return scenarios.

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*B. Multi-Objective Portfolio Optimization*

Based on the goal programming (GP) approach, Pendaraki et al. [20] proposed a methodology for the construction of equity mutual fund portfolio. The proposed methodology was conducted in two stages. In the first stage, the UTADIS classification method was used to evaluate and select a limited set of the mutual funds. In the second stage, a goal programming model was employed to determine the proportion of the selected mutual funds in the final portfolios. The proposed methodology has been applied on data of Greek equity mutual funds with promising results. The method however, could also be extended to consider other types of mutual funds.

In order to construct an optimal mutual fund portfolio for an investor, Sharma and Sharma [21] employed lexicographic goal programming approach with specific parameters such as standard deviation, portfolio beta, expected annual return and expense ratio were taken into account. The objective of the GP model is to minimize a weighted sum of deviations from the target goals. In this study, the distances of all possible solutions from the ideal solution were measured by using Euclidean distance method. Although the model is flexible enough to accommodate other constraints, the performance of the model depends on the appropriate weights in a priority structure.

Based on Sharpe's single index model, Bilbao-Terol et al. [22] have formulated a new model for portfolio selection. In this work, imprecise future beta of each asset was represented through a fuzzy trapezoidal numbers constructed on the basis of statistical data and the relevant knowledge of experts. The authors have modeled the problem using fuzzy compromise programming and introduced the fuzzy ideal solution concept. The main feature of this model, as pointed out by the authors is its sensitivity to the analyst's opinion as well as to the investors' preferences.

Based on the combination of chance constrained programming and compromise programming, Ben Abdelaziz et al. [23] proposed a chance constrained compromise programming to convert the multi-objective stochastic programming portfolio model into a deterministic one. This study assumes that the parameters associated with the objectives are random and normally distributed. A numerical example was carried out to illustrate that the proposed model could be effectively and efficiently used in practice.

In Masmoudi and Ben Abdelaziz [24], the authors addressed a problem of portfolio selection where the cost of not achieving an acceptable expected rate of return was minimized. The problem was modeled as a bi-objective stochastic programming where the first objective function was to maximize the return and the second objective function was to optimize the risk. The certainty equivalent program was obtained through a combination of a GP and recourse approach. The model results were illustrated through a case study using data from S&P100 securities.

By taking into account stochastic and fuzzy uncertainties, Messaoudi and Rebai [25] developed a novel fuzzy goal programming model for solving a stochastic multi-objective portfolio selection problem. In this model, fuzzy chance-constrained goals were described along with the imprecise importance relations among them. The proposed model was then utilized to build a new portfolio selection model that considered the tradeoff between expected return, Value-at-Risk the price earnings ratio and the flexibility of investor's preferences. However, the applicability of the proposed model on real world data had not been tested in this study.

Xidonas et al. [26] developed a multi-objective MILP model for equity portfolio construction and selection. In order to generate Pareto optimal portfolios, the authors utilized the novel version of the ε-constraint method. Additionally, an interactive filtering process was also proposed to guide the decision maker in selecting among a number of Pareto optimal portfolios his/her most preferred. The proposed methodology could be a useful tool in helping the investors to construct and design their portfolios.

Stoyan and Kwon [27] presented a complex Stochastic-Goal Mixed-Integer Programming (SGMIP) approach for an integrated stock and bond portfolio problem. The portfolio model integrates uncertainty in security prices and involves several real-world trading constraints as well as other important portfolio elements such as liquidity, management costs, portfolio size and diversity. An algorithm to solve the model that consists of a decomposition, warm-start, and iterative procedure has also been proposed. This study contributes a significant finding as the proposed algorithm is able to solve the problem of practical size in an efficient manner.

A very recent study by Tamiz et al. [28] investigated the problem of portfolio selection for international mutual funds. The authors employed three variants of GP, namely, Weighted, Lexicographic and MinMax approach to model the portfolio problem. Seven factors were considered to be treated as objectives in the GP models in which three were specific to mutual funds, three were taken from macroeconomics and one factor represented regional and country preferences. The results of this study, although very promising were not globally conclusive as they were based on certain factors such as target values, priority levels and other sets of penalized unwanted deviational variables.

Considering the increasing importance of investment in financial portfolios, Amiri et al. [29] developed a new model called Nadir Compromising Programming (NCP) model by using an extended of Compromise Programming (CP). This model which can be used to optimize multi-objective problems was formulated on the basis of the nadir values of each objective. In order to compare the performance of the CP method and the proposed method, the authors conducted a case study by selecting a portfolio with 35 stocks from the Iran stock exchange. The results obtained confirmed that in spite of being feasible and optimal, the NCP model was more consistent with decision maker purposes.

Kırış and Ustun [30] built a multi objective portfolio optimization model which combined Markowitz's model with the objective of the expected performance value of portfolio and cardinality constraints.

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TABLE I
MATHEMATICAL PROGRAMMING APPLIED TO PORTFOLIO OPTIMIZATION

| | Authors | Real-life constraints | Mathematical Programming | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | LP | IP | MIP | MIQP | SP | GP | CP | SCP | SGP |
| Multi-objective | Tamiz et al. [28] | No | | | | | | ✓ | | | |
| | Messaoudi and Rebai [25] | No | | | | | | | | | ✓ |
| | Kırış and Ustun [30] | No | | | | | | ✓ | | | |
| | Masmoudi and Ben Abdelaziz [24] | No | | | | | | | | | ✓ |
| | Stoyan and Kwon [27] | Yes | | | ✓ | | | | | | ✓ |
| | Amiri et al. [29] | No | | | | | | | ✓ | | |
| | Xidonas et al. [26] | No | | | ✓ | | | | | | |
| | Ben Abdelaziz et al. [23] | No | | | | | | | | ✓ | |
| | Bilbao-Terol et al. [22] | No | | | | | | | ✓ | | |
| | Sharma and Sharma [21] | No | | | | | | ✓ | | | |
| | Pendaraki et al. [20] | No | | | | | | ✓ | | | |
| Single objective | Golmakani and Fazel [18] | Yes | | | | ✓ | | | | | |
| | Soleimani et al. [17] | Yes | | | | ✓ | | | | | |
| | Ibrahim et al. [19] | No | | | | | ✓ | | | | |
| | Angelelli et al. [16] | Yes | | | ✓ | | | | | | |
| | Kamil and Ibrahim [12] | No | ✓ | | | | | | | | |
| | Benati and Rizzi [15] | No | | | ✓ | | | | | | |
| | Konno and Yamamoto [13] | Yes | | ✓ | | | | | | | |
| | Papahristodoulou and Dotzauer [11] | No | ✓ | | | | | | | | |
| | Chiodi et al. [14] | Yes | | | ✓ | | | | | | |

Note: LP-Linear Programming, IP-Integer Programming, MIP-Mixed Integer Programming, MIQP-Mixed Integer Quadratic Programming, SP-Stochastic Programming, GP-Goal programming, CP-Compromise Programming, SCP-Stochastic Compromise Programming, SGP, Stochastic Goal Programming

This model is classified as a multi-objective mixed integer nonlinear programming. The proposed model was solved by utilizing reservation level driven Tchebycheff procedure.

In summary, it has been observed from this literature review that the majority of the research conducted on single objective optimization for portfolio selection problems employed mixed integer programming techniques. On the other hand, for multi-objective optimization, goal programming is the most utilized approach. Table I presents the summary of past studies that use mathematical programming techniques in portfolio optimization model.

### III. SOLUTION APPROACHES

In recent years, the research community has made significant advances in portfolio optimization problem. One of the focuses has been put on the identification of efficient and effective solution approaches for solving the mathematical programming model of portfolio optimization problems. The approaches can be classified into two main categories: exact methods and heuristics methods.

#### A. Exact Methods

A number of exact approaches have been proposed to solve portfolio optimization model. For instance, Mansini and Separanza [31] presented an exact algorithm for MILP model. Their method is based on the partition of the initial problem into two sub-problems and the use of local search heuristic to obtain an initial solution. The computational results showed that the solution of the first subproblem alone could be effectively used as heuristic, indeed the authors showed that in all the instances this subproblem can find an optimal solution;

therefore, it is very likely that it can achieve a very good performance in general.

In the study of Li et al. [32], proposed a solution for cardinality constrained mean-variance model under concave transaction costs and minimum transaction lots constraints. The proposed method is based on a Lagrangian relaxation scheme and contour-domain cut branching rule. However they performed the computational experiments with only one data set containing 30 assets which was too small both to take into account the size of real world portfolios and the computational behavior of algorithms as the problem size grew.

A branch and bound algorithm for mixed integer nonlinear programs in portfolio selection problem was presented in the work of Bonami and Lejeune [33]. The algorithm features two new branching rules which are the idiosyncratic risk and portfolio risk branching rule. The computational results showed that the proposed algorithm was effective to solve to optimality with up to 200 assets.

Veilma et al. [34] presented a branch-and-bound algorithm for the exact solution of the cardinality constrained portfolio optimization problem based on a lifted polyhedral relaxation of conic quadratic constraints. Computational results were presented for problems drawn from real-world data.

Shaw et al. [35] investigated a branch-and-bound method for portfolio optimization problem, where the asset returns are driven by a factor model. Subgradient method was employed to compute the lagrangian bound of each subproblem in the branch and bound process. The authors reported that the proposed algorithm could produce optimal solutions with up to 250 assets in a reasonable time.

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Another branch and bound method was proposed by Sun et al. [36] who implemented a branch and bound procedure based on Lagrangian relaxation to solve the cardinality constrained portfolio optimization problem. The numerical results for test problems using real-world data up to 150 securities demonstrated that the proposed method was capable of solving the portfolio problem. However, the authors did not report solving the problems to optimality.

Bertsimas and Shioda [37] presented an approach for the exact solution of the cardinality constrained portfolio optimization problem. The authors utilized Lemke's method to optimize the convex quadratic programming at each node. Computational results were presented for their approach as well as for CPLEX on problems involving up to 500 assets. Although the proposed algorithm appeared to have advantage over generalized solver, it failed to find the optimal solution within the computational time limit.

Gulpinar et al. [38] proposed an exact solution method based on difference of convex function's algorithm to solve cardinality constrained portfolio optimization model. The authors selected a portfolio with respect to the worst-case associated with specified scenarios. The computational results for test problem up to 98 assets showed that the method outperformed in almost all cases the commercial solver CPLEX.

Utilizing a new Lagrangian decomposition scheme, a convex relaxation and a mixed integer quadratically constrained quadratic program reformulation were derived in Cui et al. [39] for cardinality constrained portfolio problem. The numerical results have shown that the dual problem obtained from the decomposition scheme can be reduced to a second-order cone program problem which is tighter than or at least as tight as the continuous relaxation of the standard reformulation.

In a recent development, Gao and Li [40] developed a branch and bound method based on a novel geometric approach to solve cardinality constrained mean-variance portfolio selection problem. Computational results on test problems of portfolio selection demonstrated that the method was promising in finding good quality solution.

*B. Metaheuristics Approaches*

There are many studies applying metaheuristics methods to solve the problem of portfolio optimization. One of the studies was carried out by Crama and Schyns [41] who applied heuristic technique based on simulated annealing (SA) to an extended version of the mean-variance model with trading and turnover constraints. Computational results for problems with up to 151 stocks seem to show that the approach is promising for medium size problems.

Maringer and Kellerer [42] considered the mean–variance model as extended to include the cardinality constraints. Then, a hybrid local search algorithm that combines principles of SA and evolutionary strategies was applied to solve the resulting mixed-integer quadratic programming model. The effectiveness and applicability of the technique were demonstrated via computational experiment for two data sets involving 30 and 96 assets.

Another interesting solution approach was presented by Fernandez and Gomez in [43] who employed Hopfield neural networks to solve the mean-variance model with cardinality and bounding constraints. The authors also compared the approach with genetic algorithm (GA), tabu search (TS) as well as SA and performed the computational experiments using five sets of benchmark data that have been used in [44]. Although the results showed that none of the four has clearly outperformed the others, when dealing with problem demanding portfolios with low investment risk, the proposed method provides better solutions than the other heuristics.

Cura [45] developed an approach based on particle swarm optimization (PSO) to solve the same portfolio problem as in [43]. In order to evaluate the performance of the approach, it was compared to GA, TS and SA. The numerical tests were conducted employing the same benchmark datasets used in [44]. The results indicated that none of the heuristic approaches outperformed the others.

Chang et al. [46] introduced a heuristic approach based on GA for solving portfolio optimization problems in different risk measures and compared its performance to mean–variance model in cardinality constrained efficient frontier. The authors showed that the problems could be solved effectively by GA if mean–variance, semivariance, mean absolute deviation, and variance with skewness were used as the measures of risk. They conducted empirical tests in order to prove the robustness of their heuristic method.

In Deng and Lin [47], the authors proposed ant colony optimization (ACO) for solving mean-variance model with cardinality and bounding constraints which is a mixed integer quadratic programming problem. According to the computational results obtained on benchmark data sets, the proposed ACO has shown to be more robust and effective than PSO especially for low risk investment portfolios.

A new hybrid solution approach combining an improved PSO and SA was proposed by Mozafari et al. [48] to address the problem of portfolio optimization presented in [42]. The effectiveness of the proposed algorithm was tested on benchmark data with up to 225 assets and the results indicated that it could generate good solutions within acceptable computing times.

Zhu et al. [49] suggested a meta-heuristic approach to portfolio optimization problem using particle swarm optimization (PSO) technique. The objective functions and the constraints are based on the Markowitz and Sharp Ratio model. Computational experiments were carried out on various restricted and unrestricted risky investment portfolios and the results obtained were very encouraging.

Woodside-Oriakhi et al. [50] studied the application of GA, TS and SA approaches to find the solution of the cardinality constrained efficient frontier. The authors considered the extended mean–variance model under the discrete restrictions of cardinality and bounding constraints. The authors carried out numerical experiments on test problems consisting up to

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1318 assets and the results showed that the proposed heuristics were effective and quite efficient.

A heuristic framework based upon kernel search is presented in [51]. This new framework is proposed to solve portfolio optimization problem with real features that is modeled as a MILP problem. A computational test was performed using different data sets involving 400 stocks. The results demonstrated that the proposed heuristics were very effective and applicable to a variety of combinatorial problems.

TABLE II
SOLUTION APPROACHES FOR PORTFOLIO OPTIMIZATION MODEL

| Authors | Mathematical Programming | Exact Methods | | | | | Metaheuristics methods | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | BB | LR | PM | LDS | DCA | PSO | GA | ACO | SA | NN | TS | KS |
| Gao and Li [40] | MIQP | ✓ | | | | | | | | | | | |
| Cui et al. [39] | MIQP | | | | ✓ | | | | | | | | |
| Angelelli et al. [51] | MILP | | | | | | | | | | | | ✓ |
| Mozafari et al. [48] | MIQP | | | | | | ✓ | | | ✓ | | | |
| Woodside-Oriakhi et al. [50] | MIQP | | | | | | | ✓ | | ✓ | | ✓ | |
| Zhu et al. [49] | MIQP | | | | | | ✓ | | | | | | |
| Deng and Lin [47] | MIQP | | | | | | | | ✓ | | | | |
| Gulpinar et al. [38] | MIQP | | | | | ✓ | | | | | | | |
| Chang et al. [46] | MINLP | | | | | | | ✓ | | | | | |
| Sun et al. [36] | MIQP | ✓ | | | | | | | | | | | |
| Cura [45] | MIQP | | | | | | ✓ | | | | | | |
| Bonami and Lejeune [33] | MINLP | ✓ | | | | | | | | | | | |
| Bertsimas and Shioda [37] | MIQP | ✓ | | | | | | | | | | | |
| Fernandez and Gomez [43] | MIQP | | | | | | | | | | ✓ | | |
| Shaw et al. [35] | MIQP | ✓ | | | | | | | | | | | |
| Veilma et al. [34] | MIQP | ✓ | | | | | | | | | | | |
| Li et al. [32] | NLIP | | ✓ | | | | | | | | | | |
| Mansini and Speranza [31] | MILP | | | ✓ | | | | | | | | | |
| Maringer and Kellerer [42] | MIQP | | | | | | | | | ✓ | | | |
| Crama and Schyns [41] | MIQP | | | | | | | | | ✓ | | | |

Note: BB-Branch and Bound, LR-Lagrangian Relaxation, PM-Partitioning Method, LDS-Lagrangian Decomposition Scheme, DCA-Difference of Convex Algorithm, PSO-Particle Swarm Optimization, GA-Genetic Algorithm, ACO-Ant Colony Optimization, SA- Simulated Annealing, NN-Neural Network, TS-Tabu Search, KS-Kernel Search.

Table II presents the summary of past studies along with the solution approaches used to solve the portfolio optimization model. It can be seen that metaheuristics methods, particularly simulated annealing are the most popular approaches used to solve the mathematical programming model of the portfolio problem. For the exact solution methods, 60% of the publications employed branch and bound and the largest size of problems solved by this method was 250 assets.

## IV. DISCUSSION AND FUTURE RESEARCH

The review has shown that mathematical programming techniques have been applied successfully to formulate the portfolio optimization problems in the past decade. Most of the studies on single objective portfolio optimization used mixed integer programming techniques to formulate the problem. This is due to the incorporation of non-negligible aspects of real-world trading constraints.

The analysis of the literature also shows that, there has been an increasing interest in the design of multi-objective programming techniques to handle the problem of portfolio optimization. However, most of the studies reviewed employed goal programming technique to formulate the problem. In future research it might be possible to use different multi-objective optimization techniques such as compromise programming. In addition, there is only one study on multi-objective portfolio optimization incorporated real-world trading constraints in the model. Hence, drawing attention to multi-objective portfolio optimization problem with the consideration of these constraints seems quite worthwhile and practical.

Another important observation is that many studies reviewed have chosen heuristics or metaheuristics method rather than exact solution approaches in solving portfolio optimization problem with real-life constraints. However, most of these studies have focused on a sole metaheuristics technique which is often not sufficient to achieve results meeting practical requirements. Thus, there exists the need to design hybrid metaheuristics techniques.

## V. CONCLUSION

This paper has presented a comprehensive review of literature on the application of mathematical programming techniques in portfolio optimization problems. For this purpose, 40 papers from scholarly journals were gathered and analyzed. This overview has also enlightens some potential areas for future research. Finally, it is hoped that this paper gives a clear overview of the application of mathematical programming as a support tool in the portfolio optimization problem.

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