A Study on the Least Squares Reduced Parameter Approximation of FIR Digital Filters

S. Seyedtabaii, and E. Seyedtabaii

Abstract**—**Rounding of coefficients is a common practice in hardware implementation of digital filters. Where some coefficients are very close to zero or one, as assumed in this paper, this rounding action also leads to some computation reduction. Furthermore, if the discarded coefficient is of high order, a reduced order filter is obtained, otherwise the order does not change but computation is reduced. In this paper, the Least Squares approximation to rounded (or discarded) coefficient FIR filter is investigated. The result also succinctly extended to general type of FIR filters.

Keywords**—**Digital filter, filter approximation, least squares, model order reduction.

I. Introduction

An FIR filter has desirable characteristics such as linear phase response, finite duration of startup transients, and perpetual stability. In applications where distortion-free transmission of waveforms in the passband is required, FIR filters are attractive because the linear phase or constant group delay requirement can be easily satisfied. This is one of the main advantages of FIR filters over IIR designs. However, from the view of the practitioner's landscape, FIR filters may be difficult to implement because of their potentially much higher order than IIR filters satisfying the same magnitude specifications [1].

In a model order reduction problem, it is desired to approximate a system of a relatively high order, with a lower order model. A linear-phase FIR filter of high order, by applying model reduction techniques, can be transformed to a lower order IIR filter that meets the original magnitude specifications while maintaining a linear-phase response in the passband [2]. Various model reduction techniques have been introduced for the filter approximation. Among them, the most popular technique is based on the well-known balanced truncation (BT) method [3]. Other model reduction techniques that are suitable for filter approximation include the Impulse Response Gramian (IRG) technique [4] and the Hankel norm approximation method [5] [6].

An indirect linear-phase IIR filter design technique based on a reduction of linear-phase FIR filters is discussed in [7] that the desired filter is obtained by minimizing the L2 norm of the difference between the original FIR filter and the lower order IIR filter

Model/filter order reduction with frequency weighting is an interesting problem with practical significance as in most of applications we are more interested in signals within certain frequency ranges. Enns [8] extended the balanced truncation method to the frequency weighted case which was applied to the design of IIR filters with linear phase characteristics.

Order reduction technique in digital filter design based on the utilization of the Chebyshev polynomial given in [9] renders a reduced order FIR filter.

The main objective of this work is to exploit the Least Squares optimal Linear Phase FIR approximation to a rounded coefficient linear phase FIR filter, where some coefficients are rounded to either 1 or zero. Furthermore, the coefficients are not limited necessarily to be of the high order.

This paper is organized as follows. Section 2 presents the problem statement and formulation. Section 3 gives the result of the case preparation simulation. Theoretical analysis is given in section 4 and finally, conclusion comes in section 5.

II. Problem Statement and Formulation

Consider a symmetric (antisymmetric) linear phase FIR filter,

$$H(z)=\sum_{n=0}^{N} b_n z^{-n} \qquad b_n=\pm b_{N-n} \qquad b_n \in R$$

where some coefficients are assumed to have a relatively small value, or to be too much close to 1. The problem is to exploit an optimal least squares linear phase FIR filter approximation to the original filter where small value coefficients are removed and or "close to one" coefficients are rounded to 1. In other word, we are looking for an optimal filter where its amplitude and phase deviation from the original one is minimized, while low value coefficient computation burden is avoided. To formulate the case, filter transfer function is written as,

$$H(z)=z^{-M}\sum_{n=0}^{L} a_n\left(z^{M-n}\pm z^{-(M-n)}\right)$$

where

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Saeed Seyedtabaii is with the Shahed University, Tehran, Iran (corresponding author e-mail: stabaii@shahed.ac.ir).

Elaheh Seyedtabaii is with the Shahed University, Tehran, Iran (e-mail: e.tabaii@gmail.com).

$$L = M = \frac{N}{2}, \begin{cases} \alpha_{n,n\neq M} = b_n \\ \alpha_M = \frac{b_M}{2} \end{cases} \qquad N \quad even$$

$$\alpha_n = b_n, L = \frac{N-1}{2}, M = \frac{N}{2} \qquad N \quad odd$$

In the symmetric case, for example, the filter amplitude is,

$$\left|H(\omega)\right| = \sum_{n=0}^{L} 2a_n \cos[(M-n)\omega] \tag{1}$$

After discarding a small value coefficient (or rounding to one) of (1), it yields,

$$\left|H'(\omega)\right| = \sum_{n=0,n\neq m}^{L} 2a_n \cos[(M-n)\omega] \qquad a_m = 0,(or \quad 1)$$

Now an optimal approximation to (1), is sought that can be written as,

$$\left|\hat{H}(\omega)\right| = \sum_{n=0,n\neq m}^{M} 2\hat{a}_n \cos[(M-n)\omega] \qquad a_m = 0,(or \quad 1)$$

where optimal $\hat{a}$'s have to be calculated.

III. Math

To develop the idea, the FIR filter given in [9],

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + 2 * a_4 z^{-4} + a_3 z^{-5} + a_2 z^{-6} + a_1 z^{-7} + a_0 z^{-8} \tag{2}$$

with

$a_0 = -0.0299 \quad a_1 = -0.0438 \quad a_2 = 0.0785 \quad a_3 = 0.2904 \quad a_4 = 0.2$

is simulated. The filter amplitude is

$$\left|H(\omega)\right| = \sum_{n=0}^{4} 2a_n \cos((4-n)\omega)$$

Discarding, for example, the small value coefficient α_0, leads to

$$\left|H(\omega)\right| = \sum_{n=1}^{4} 2a_n \cos((4-n)\omega)$$

where can be viewed as a type of approximation to (1). While α_0 is discarded, another filter approximation can be formulated, that its coefficients are calculated optimally, to best match the amplitude of the original filter

$$\left|\hat{H}(\omega)\right| = \sum_{n=1}^{4} 2\hat{a}_n \cos((4-n)\omega)$$

To obtain the optimal Least Squares filter coefficients, the following error equation is minimized

$$\sum_{i=0}^{K} e(\omega_i)^2 = \sum_{i=0}^{K} \left[\left|H(\omega_i)\right| - \left|\hat{H}(\omega_i)\right|\right]^2 \qquad \omega_i = \frac{\pi}{k} * i$$

The optimal coefficients obtained experimentally for K=10 are

$\hat{a}_1 = -0.0438 \quad \hat{a}_2 = 0.0739 \quad \hat{a}_3 = 0.2904 \quad \hat{a}_4 = 0.1977$

while the results for k=50 are

$\hat{a}_1 = -0.0438 \quad \hat{a}_2 = 0.0774 \quad \hat{a}_3 = 0.2904 \quad \hat{a}_4 = 0.1994$

The results look different, but as the number of points in K is increased, the obtained numbers will tend to match the coefficients of the discarded coefficient original filter, $H_1(\omega)$. Fig. 1 shows the original filter and the optimally approximated discarded coefficient amplitudes.

Theoretically investigating this case gives interesting results coming out of the Fourier basis functions orthogonality. In the next section, its proof is presented.

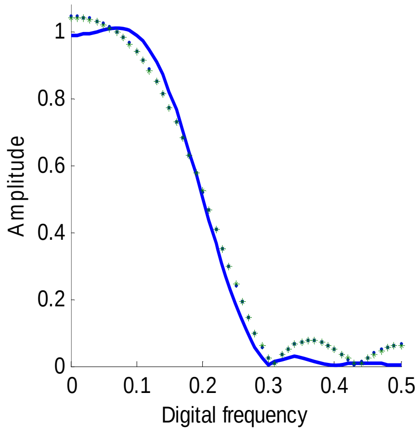

Fig. 1 Solid line: The original filter frequency response, Dotted line: the optimally approximated and the discarded coefficient filter frequency response

IV. Analysis of the LS Approximation Method

The Amplitude of a symmetric (antisymetric) FIR filter is

$$\left|H(\omega)\right| = \sum_{n=0}^{L} 2a_n \cos((M-n)\omega))$$

After discarding, for example, a near zero coefficient α_m, the problem gets finding an optimal cosine series to compensate for the cancelled part. In other word, we have to determine optimal parameters for the following equation,

$$cos(m\omega_i) = \sum_{m\neq n, n=0}^{L} a_n \, cos(n\omega_i) \quad \omega_i = \tfrac{2\pi}{K} i \quad i = 0,...,K-1 \qquad (3)$$

where m is the index of the discarded coefficient and K is the number of the selected frequency points for minimization. The approximation error then becomes

$$e(\omega_i) = cos(m\omega_i) - \sum_{m\neq n, n=0}^{L} a_n \, cos(n\omega_i)$$

Hence, its sum squares is

$$\sum_{i=0}^{K-1} e(\omega_i)^2 = \sum_{i=0}^{K-1}\left[\mathbf{cos(} m\omega_i \mathbf{)} - \sum_{m\neq n, n=0}^{L} a_n \, \mathbf{cos(} n\omega_i \mathbf{)} \right]^2$$

$$J = \sum_{i=0}^{K-1} e(\omega_i)^2 =$$

$$\sum_{i=0}^{K-1} cos(m\omega_i)^2 + \sum_{i=0}^{K-1}\left[\sum_{m\neq n, n=0}^{L} a_n \, cos(n\omega_i) \right]^2$$

$$-2 \sum_{m\neq n, n=0}^{L} \sum_{i=0}^{K-1} a_m \, cos(n\omega_i) cos(m\omega_i)$$

Due to the orthogonality of cosine terms, the third part is zero and the optimal values for α's are

$$\frac{\partial J}{\partial \alpha} = \frac{\partial \left(\sum_{i=0}^{K-1} \left[\sum_{m\neq n, n=0}^{L} a_n \, cos(n\omega_i) \right]^2 \right)}{\partial \alpha} = 0 \quad \Rightarrow \alpha_n = 0 \quad n = 0 \to N$$

This result indicates that the discarded term filter is also the LS optimal approximation to the original filter.

This method similarly can be applied to an exponential polynomial such as general Discrete Fourier Transform, for example, FIR filter

$$H(e^{j\omega}) = a_0 + a_1 e^{-j\omega} + a_2 e^{-2j\omega} + a_3 e^{-3j\omega} + a_4 e^{-4j\omega} + ...$$

To obtain an optimal approximation to a rounded coefficient (for example α_n) filter, the rounded part is expressed in term of the other remaining exponential functions that still exist in the filter structure, as follows:

$$e^{jm\omega_i} = \sum_{m\neq n} a_n e^{jn\omega_i} \qquad \omega_i = \tfrac{2\pi}{K} i \quad i = 0, ..., K-1$$

Then the error in between is minimized

$$J = \sum_{i=0}^{K-1} e(\omega_i) e(\omega_i)^* =$$

$$\sum_{i=0}^{K-1} \left[e^{jm\omega_i} - \sum_{m\neq n} a_m e^{jn\omega_i} \right] \left[e^{jm\omega_i} - \sum_{m\neq n} a_m e^{jn\omega_i} \right]^* \Rightarrow$$

$$\frac{\partial}{\partial \vec{\alpha}} \left[K + \sum_{i=0}^{K-1} \sum_{l\neq n} \sum_{m\neq n} a_m a_l e^{j(l-m)\omega_i} - 2 \sum_{m\neq n} a_m \sum_{i=0}^{K-1} e^{j(n-m)\omega_i} \right] = 0$$

The third term, again according to the orthogonality, is zero, and the differentiation with respect to $\vec{\alpha}$ renders $\vec{\alpha}$ =0.

V. Conclusion

In this paper shown that a rounded (discarded) coefficient FIR filter, is an optimal approximation to the original filter. The result was proved based on the LS algorithm. Obviously, this argument does not necessarily mean that a good approximation is also achieved, since good approximation depends on the amount of discarded (or rounded) value.

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