CONVERGENCE OF THE NORMALIZED KÄHLER-RICCI FLOW ON KÄHLER-EINSTEIN FANO MANIFOLDS

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Abstract. We explain the proof of the following result due to Perelman: on a Kähler-Einstein Fano manifold X with no holomorphic vector field, the normalized Kähler-Ricci flow starting from an arbitrary Kähler form in $c_1(X)$ converges towards the unique Kähler-Einstein metric.

Contents

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Introduction

Let X be a Fano manifold, i.e. a compact (connected) complex projective algebraic manifold whose first Chern class $c_1(X)$ is positive, i.e. can be represented by a Kähler form. It has been an open question for decades to understand when such a manifold admits a Kähler-Einstein metric, i.e. if we can find a Kähler form $\omega_{KE} \in c_1(X)$ such that

$$\mathrm{Ric}(\omega_{KE}) = \omega_{KE}.$$

By comparison with the cases when $c_1(X) < 0$ (or $c_1(X) = 0$) treated in Song-Weinkove lecture notes [SW], there is neither existence nor uniqueness in general of Kähler-Einstein metrics in the Fano case.

After the spectacular progress in Ricci flow techniques, it has become a natural question to wonder whether the Ricci flow could help in understanding this problem. The goal of this series of lectures is to sketch the proof of an important result in this direction, which is due to Perelman:

Theorem. [Perelman, seminar talk at MIT, 2003]
Let X be a Fano manifold which admits a unique Kähler-Einstein metric ω_{KE}. Fix $\omega_0 \in c_1(X)$ an arbitrary Kähler form. Then the normalized Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \omega_t$$

converges, as $t \to +\infty$, in the $\mathcal{C}^\infty$-sense to ω_{KE}.

In other words, the normalized Kähler-Ricci flow detects the (unique) Kähler-Einstein metric if it exists.

This result has been generalized by Tian and Zhu [TZ07] to the case of Kähler-Ricci soliton. Other generalizations by Phong and his collaborators can be found in [PS10]. We follow here a slightly different path, using pluripotential techniques to establish a uniform $\mathcal{C}^0$-a priori estimate along the flow.

All proofs rely on deep estimates due to Perelman. These are explained in Cao's lectures [Cao], to which we refer the reader.

Nota Bene. These notes are written after the lectures the author delivered at the third ANR-MACK meeting (24-27 october 2011, Marrakech, Morocco). There is no claim of originality. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

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1. Background

1.1. **The Kähler-Einstein equation on Fano manifolds.** Let X be an n-dimensional Fano manifold and fix $\omega \in c_1(X)$ an arbitrary Kähler form. If we write locally

$$\omega = \sum \omega_{\alpha\beta} \frac{i}{\pi} dz_\alpha \wedge d\overline{z}_\beta,$$

then the Ricci form of ω is

$$\mathrm{Ric}(\omega) := -\sum \frac{\partial^2 \log(\det \omega_{pq})}{\partial z_\alpha \partial \overline{z}_\beta} \frac{i}{\pi} dz_\alpha \wedge d\overline{z}_\beta.$$

Observe that $\mathrm{Ric}(\omega)$ is a closed $(1,1)$-form on X such that for any other Kähler form ω' on X, the following holds globally:

$$\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega) - dd^c \left[\log \omega'^n / \omega^n\right].$$

Here $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/2i\pi$ are both real operators.

In particular $\mathrm{Ric}(\omega')$ and $\mathrm{Ric}(\omega)$ represents the same cohomology class, which turns out to be $c_1(X)$.

The associated complex Monge-Ampère equation. Since we have picked $\omega \in c_1(X)$, it follows from the dd^c-lemma that

$$\mathrm{Ric}(\omega) = \omega - dd^c h$$

for some smooth function $h \in \mathcal{C}^\infty(X, \mathbb{R})$ which is uniquely determined, up to an additive constant. We normalize h by asking for

$$\int_X e^{-h}\omega^n = V := \int_X \omega^n = c_1(X)^n.$$

We look for $\omega_{KE} = \omega + dd^c \varphi_{KE}$ a Kähler form such that $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$. Since $\mathrm{Ric}(\omega_{KE}) = \mathrm{Ric}(\omega) - dd^c \log(\omega_{KE}^n/\omega^n)$, an easy computation shows that

$$dd^c\{\log(\omega_{KE}^n/\omega^n) + \varphi_{KE} + h\} = 0.$$

Since pluriharmonic functions are constant on X (by the maximum principle), we infer

$$(MA) \qquad (\omega + dd^c\varphi_{KE})^n = e^{-\varphi_{KE}} e^{-h+C} \omega^n$$

for some normalizing constant $C \in \mathbb{R}$. Solving $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$ is thus equivalent to solving the above complex Monge-Ampère equation (MA).

Known results. When $n = 1$, X is the Riemann sphere $\mathbb{CP}^1$ and (a suitable multiple of) the Fubini-Study Kähler form is a Kähler-Einstein metric.

When $n = 2$ it is not always possible to solve (MA). In this case X is a *DelPezzo surface*, biholomorphic either to $\mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$ which both admit the (product) Fubini-Study metric as a Kähler-Einstein metric, or else to X_r, the blow up of $\mathbb{CP}^2$ at r points in general position, $1 \leq r \leq 8$. Various authors (notably Yau, Siu, Tian, Nadel) have studied the Kähler-Einstein problem on DelPezzo surfaces in the eighties. The final and difficult step was done by Tian who proved the following:

Theorem 1.1. [Tian90] *The DelPezzo surface X_r admits a Kähler-Einstein metric if and only if $r \neq 1, 2$.*

The interested reader will find an up-to-date proof of this result in [Tos12].

The situation becomes much more difficult and largely open in higher dimension. There is a finite but long list (105 families) of Fano threefolds[1]. It is

[1] The lecture at the workshop by S.Lamy was devoted to the classification of special weak-Fano threefolds, see [BL11].

unknown, for most of them, whether they admit or not a Kähler-Einstein metric. Among them, the Mukai-Umemura manifold is particularly interesting: this manifold admits a Kähler-Einstein metric as was shown by Donaldson [Don08], and there are arbitrary small deformations of it which do (resp. do not) admit a Kähler-Einstein metric as shown by Donaldson (resp. Tian)[2].

There are even more families in dimension $n \geq 4$. Those which are *toric* admit a Kähler-Einstein metric if and only if the Futaki invariant vanishes (see [WZ04]), the non-toric case is essentially open and has motivated an important conjecture of Yau-Tian-Donaldson (see [PS10]).

Uniqueness issue. Bando and Mabuchi have shown in [BM87] that any two Kähler-Einstein metrics on a Fano manifold can be connected by the holomorphic flow of a holomorphic vector field. This result has been generalized recently by Berndtsson [Bern11]. We shall make in the sequel the simplifying assumption that X does not admit non-zero holomorphic vector field, so that it admits a unique Kähler-Einstein metric, if any.

1.2. **The analytic criterion of Tian.** Given $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ an upper semi-continuous function, we say that φ is ω-plurisubharmonic (ω-psh for short) and write $\varphi \in PSH(X, \omega)$ if φ is locally given as the sum of a smooth and a plurisubharmonic function, and $\omega + dd^c\varphi \geq 0$ in the weak sense of currents. Set

$$E(\varphi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j}.$$

We let the reader check, by using Stokes formula, that

$$\frac{d}{dt} E(\varphi + tv)_{|t=0} = \int_X v\, MA(\varphi), \text{ where } MA(\varphi) := (\omega + dd^c\varphi)^n / V.$$

The functional E is thus a primitive of the complex Monge-Ampère operator, in particular $\varphi \mapsto E(\varphi)$ is non-decreasing since $E' = MA \geq 0$.

Definition 1.2. We set

$$\mathcal{F}(\varphi) := E(\varphi) + \log \left[\int_X e^{-\varphi - h} \omega^n \right].$$

The reader will check that φ is a critical point of the functional $\mathcal{F}$ if and only if

$$MA(\varphi) = \frac{e^{-\varphi-h}\omega^n}{\int_X e^{-\varphi-h}\omega^n}$$

so that $\omega + dd^c\varphi$ is Kähler-Einstein. Observe that $\mathcal{F}(\varphi + C) = \mathcal{F}(\varphi)$, for all $C \in \mathbb{R}$, thus $\mathcal{F}$ is actually a functional acting on the metrics $\omega_\varphi := \omega + dd^c\varphi$. It is natural to try and extremize the functional $\mathcal{F}$. This motivates the following:

Definition 1.3. We say that $\mathcal{F}$ is proper[3] if $\mathcal{F}(\varphi_j) \to -\infty$ whenever $\varphi_j \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ is such that $E(\varphi_j) \to -\infty$ and $\int_X \varphi_j \omega^n = 0$.

[2] This problem was addressed during the workshop in a series of lectures by A.Broustet and S.DiVerio who followed [Tian97, Don08].

[3] The properness of $\mathcal{F}$ is related to the so-called *Moser-Trudinger inequality* which was the subject of the lecture by Berndtsson at the workshop (see [BerBer11].

The importance of this notion was made clear in a series of works by Ding and Tian in the 90's, culminating with the following deep result[4] of [Tian97]:

Theorem 1.4 (Tian 97). *Let X be a Fano manifold with no holomorphic vector field. There exists a Kähler-Einstein metric if and only if $\mathcal{F}$ is proper.*

1.3. **The Kähler-Ricci flow approach.** The Ricci flow is the parabolic evolution equation

$$(KRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) \quad \text{with initial data } \omega_0.$$

When ω_0 is a Kähler form, so is ω_t, $t > 0$ hence it is called the Kähler-Ricci flow.

Long time existence. The short time existence is guaranteed by standard parabolic theory (see C.Imbert's lectures [Imbert]): in the Kähler context, this translates into a parabolic scalar equation as we explain below.

It is more convenient to analyze the long time existence by considering the normalized Kähler-Ricci flow, namely

$$(NKRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t.$$

One passes from (KRF) to (NKRF) by changing $\omega(t)$ in $e^t\omega(1-e^{-t})$. At the level of cohomology classes,

$$\frac{d\{\omega_t\}}{dt} = -c_1(X) + \{\omega_t\} \in H^{1,1}(X, \mathbb{R})$$

therefore $\{\omega_t\} \equiv c_1(X)$ is constant if we start from $\omega_0 \in c_1(X)$. This justifies the name (normalized KRF) since in this case

$$\mathrm{vol}_{\omega_t}(X) = \mathrm{vol}_{\omega_0}(X) = c_1(X)^n$$

is constant. Note that the volume blows up exponentially fast if $\{\omega_0\} > c_1(X)$.

Theorem 1.5 (Cao 85). *Let X be a Fano manifold and pick a Kähler form $\omega_0 \in c_1(X)$. Then the normalized Kähler-Ricci flow exists for all times $t > 0$.*

We will indicate the proof of this result, although it is already essentially contained in the lecture notes by Song and Weinkove [SW].

The main issue is then whether (ω_t) converges as $t \to +\infty$. Hopefully $\partial\omega_t/\partial t \to 0$ and $\omega_t \to \omega_{KE}$ such that $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$. We can now formulate Perelman's result as follows:

Theorem 1.6 (Perelman 03). *Let X be a Fano manifold and pick an arbitrary Kähler form $\omega_0 \in c_1(X)$. If $\mathcal{F}$ is proper, then the normalized Kähler-Ricci flow (ω_t) converges, as $t \to +\infty$, towards the unique Kähler-Einstein metric ω_{KE}.*

Remark 1.7. It turns out that the properness assumption insures that there can be no holomorphic vector field, hence the Kähler-Einstein metric (which exists by Tian's result) is unique (by Bando-Mabuchi's result).

[4] The technically involved proof of this result was explained at the workshop by S.Boucksom in a series of lectures, following [Tian97] and some refinements from [PSSW08].

The situation is much more delicate in the presence of holomorphic vector fields. The convergence of the (NKRF) for instance is unclear on the projective space $\mathbb{CP}^n$, $n \geq 2$ (for $n = 1$, the problem is already non-trivial and was settled by Hamilton [Ham86] and Chow [Chow91]).

Reduction to a scalar parabolic equation. Let $\omega = \omega_0 \in c_1(X)$ denote the initial data. Since ω_t is cohomologous to ω, we can find $\varphi_t \in PSH(X, \omega)$ a smooth function such that $\omega_t = \omega + dd^c\varphi_t$. The function φ_t is defined up to a time dependent additive constant. Then

$$\frac{d\{\omega_t\}}{dt} = dd^c\dot{\varphi}_t = -\mathrm{Ric}(\omega_t) + \omega + dd^c\varphi_t,$$

where $\dot{\varphi}_t := \partial\varphi_t/\partial t$. Let $h \in \mathcal{C}^\infty(X, \mathbb{R})$ be the unique function such that

$$\mathrm{Ric}(\omega) = \omega - dd^c h, \text{ normalized so that } \int_X e^{-h}\omega^n = V.$$

We also consider $h_t \in \mathcal{C}^\infty(X, \mathbb{R})$ the unique function such that

$$\mathrm{Ric}(\omega_t) = \omega_t - dd^c h_t, \text{ normalized so that } \int_X e^{-h_t}\omega_t^n = V.$$

It follows that $\mathrm{Ric}(\omega_t) = \omega - dd^c h - dd^c \log(\omega_t^n/\omega^n)$, hence

$$dd^c\left\{\log\left(\frac{\omega_t^n}{\omega^n}\right) + h + \varphi_t - \dot{\varphi}_t\right\} = 0,$$

therefore

$$(\omega + dd^c\varphi_t)^n = e^{\dot{\varphi}_t - \varphi_t - h + \beta(t)}\omega^n,$$

for some normalizing constant $\beta(t)$.

Observe also that $dd^c\dot{\varphi}_t = -\mathrm{Ric}(\omega_t) + \omega_t = dd^c h_t$ hence $\dot{\varphi}_t(x) = h_t(x) + \alpha(t)$ for some time dependent constant $\alpha(t)$. Our plan is to show the convergence of the metrics $\omega_t = \omega + dd^c\varphi_t$ by studying the properties of the potentials φ_t, so we should be very careful in the way we normalize the latter.

1.4. **Plan of the proof.**

Step 1: Choice of normalization. We will first explain two possible choices of normalizing constants. Chen and Tian have proposed in [CT02] a normalization which has been most commonly used up to now. We will emphasize an alternative normalization, which is most likely the one used by Perelman[5].

Step 2: Uniform $\mathcal{C}^0$-estimate. Once φ_t has been suitably normalized, we will use the properness assumption to show that there exists $C_0 > 0$ such that

$$|\varphi_t(x)| \leq C_0, \quad \text{for all } (x, t) \in X \times \mathbb{R}^+.$$

This $\mathcal{C}^0$-uniform estimate along the flow is the one that fails when there is no Kähler-Einstein metric. It is considered by experts as the core of the proof. We will indicate a less standard proof, using pluripotential techniques.

[5] In his seminar talk, Perelman apparently focused on his key estimates and did not say much about the remaining details.

Step 3: Uniform estimate for $\dot{\varphi}_t$. We will explain how to bound $|\dot{\varphi}_t|$ uniformly in finite time, i.e. on $X \times [0,T]$. To get a uniform bound for $|\dot{\varphi}_t|$ on $X \times \mathbb{R}^+$, one needs to invoke Perelman's deep estimates: the latter will not be explained here, but are sketched in Cao's lectures [Cao].

Step 4: Uniform $\mathcal{C}^2$-estimate. We will then show that $|\Delta_\omega \varphi_t| \leq C_2$ independent of $(x,t) \in X \times \mathbb{R}^+$, by a clever use of the maximum principle for the Heat operator $\frac{\partial}{\partial t} - \Delta_{\omega_t}$. This is a parabolic analogue of Yau's celebrated Laplacian estimate. The constant C_2 depends on uniform bounds for φ_t and $\dot{\varphi}_t$, hence on Steps 2,3.

Step 5: Higher order estimate. At this stage one can either establish a parabolic analogue of Calabi's $\mathcal{C}^3$-estimates (global reasoning, see [PSS07]), or a complex version of the parabolic Evans-Krylov theory (local arguments) to show that there exists $\alpha > 0$ and $C_{2,\alpha} > 0$ such that

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}(X \times \mathbb{R}+)} \leq C_{2,\alpha},$$

where the Sobolev norm has to be taken with respect to the parabolic distance

$$d\left((x,y),(t,s)\right) := \max\{D(x,y), \sqrt{|t-s|}\}.$$

We won't say a word about these estimates in these notes. The reader will find a neat treatment of the $\mathcal{C}^3$-estimates in Song-Weinkove lecture notes [SW], and an idea of the Evans-Krylov approach in the real setting in Imbert's lecture notes [Imbert] (see [Gill11, ShW11] for the complex case).

With these estimates in hands, one can try and estimate the derivatives of the curvature as in [SW] or simply invoke the parabolic Schauder theory to conclude (using a bootstrapping argument) that there exists $C_k > 0$ such that

$$\|\varphi_t\|_{\mathcal{C}^k(X \times \mathbb{R}+)} \leq C_k.$$

Step 6: Convergence of the flow. At this point, we know that (φ_t) is relatively compact in $\mathcal{C}^\infty$ and it remains to show that it converges.

For the first normalization, an easy argument shows that $\dot{\varphi}_t \to 0$. A differential Harnack inequality (à la Li-Yau) allows then to show that the flow converges exponentially fast towards a Kähler-Einstein potential, which is thus the unique cluster point by Bando-Mabuchi's result.

For Perelman's normalization, one can conclude by using the variational characterization of the Kähler-Einstein metric: it is the unique maximizer of $\mathcal{F}$.

2. Normalization of potentials

Recall that ω_t is a solution of the normalized Kähler-Ricci flow (NKRF),

$$(NKRF) \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t$$

with initial data $\omega = \omega_0 \in c_1(X)$. We let $\varphi_t \in PSH(X,\omega) \cap \mathcal{C}^\infty(X)$ denote a potential for ω_t, $\omega_t = \omega + dd^c \varphi_t$ which is uniquely determined up to a time dependent additive constant. It satisfies the complex parabolic Monge-Ampère flow

$$\dot{\varphi}_t := \frac{\partial \varphi_t}{\partial t} = \log\left(\omega_t^n/\omega^n\right) + \varphi_t + h - \beta(t)$$

for some normalizing constant $\beta(t) \in \mathbb{R}$.

2.1. **First normalization.** Observe that $dd^c\varphi_0 = \omega_0 - \omega = 0$, hence $\varphi_0(x) \equiv c_0$ is a constant. The choice of c_0 will turn out to be crucial.

It is somehow natural to adjust the normalization of φ_t so that $\beta(t) \equiv 0$. This amounts to replace φ_t by $\varphi_t + B(t)$, where B solves the ODE $B' - B = -\beta$. Now

$$\dot{\varphi}_t := \frac{\partial \varphi_t}{\partial t} = \log\left(\omega_t^n/\omega^n\right) + \varphi_t + h$$

with $\varphi_0(x) \equiv c_0' = c_0 + B(0)$. Since we can choose $B(0)$ arbitrarily without affecting this complex Monge-Ampère flow (in other words the transformation $\varphi_t \mapsto \varphi_t + B(0)e^t$ leaves the flow invariant), we can still choose the value of $c_0' \in \mathbb{R}$. This choice is now clearly crucial, since two different choices lead to a difference in potentials which blows up exponentially in time.

The Mabuchi functional. Recall that the scalar curvature of a Kähler form ω is the trace of the Ricci curvature,

$$\mathrm{Scal}(\omega) := n\frac{\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}.$$

Its mean value is denoted by

$$\overline{\mathrm{Scal}(\omega)} := V^{-1}\int_X \mathrm{Scal}(\omega)\omega^n = n\frac{c_1(X)\cdot\{\omega\}^{n-1}}{\{\omega\}^n}.$$

The *Mabuchi energy*[6] is defined by its derivative: if $\omega_t = \omega + dd^c\psi_t$ is any path of Kähler forms within the cohomology class $\{\omega\}$, then

$$\frac{d\mathcal{M}(\psi_t)}{dt} := V^{-1}\int_X \dot{\psi}_t\left[\mathrm{Scal}(\omega_t) - \overline{\mathrm{Scal}(\omega_t)}\right]\omega_t^n.$$

As we work here with $\omega \in c_1(X)$, we obtain $\overline{\mathrm{Scal}(\omega_t)} = n$. Since

$$\mathrm{Ric}(\omega_t) = \omega_t - dd^c h_t,$$

we observe that

$$\mathrm{Scal}(\omega_t) - \overline{\mathrm{Scal}(\omega_t)} = -\Delta_{\omega_t}h_t := -n\frac{dd^c h_t \wedge \omega_t^{n-1}}{\omega_t^n}.$$

Recall now that $dd^c\dot{\varphi}_t = dd^c h_t$. Therefore along the normalized Kähler-Ricci flow,

$$\frac{d\mathcal{M}(\varphi_t)}{dt} = -\frac{1}{V}\int_X \dot{\varphi}_t\Delta_{\omega_t}(\dot{\varphi}_t)\omega_t^n = +\frac{n}{V}\int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_t^{n-1} \geq 0.$$

We have thus proved the following important property:

Lemma 2.1. *The Mabuchi energy is non-decreasing along the normalized Kähler-Ricci flow. More precisely,*

$$\frac{d\mathcal{M}(\varphi_t)}{dt} = \frac{n}{V}\int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_t^{n-1} \geq 0.$$

[6] The Mabuchi energy is often denoted by K or ν in the literature; our sign convention is the opposite of the traditional one, so we call it here $\mathcal{M}$ to avoid any confusion.

We explain hereafter (see Proposition 2.5) that the Mabuchi functional is bounded from above if and only if the $\mathcal{F}$-functional introduced above is so. The previous computation therefore yields

$$\int_0^{+\infty} \|\nabla_t \dot{\varphi}_t\|^2_{L^2(X)} dt < +\infty.$$

One chooses c_0 so as to guarantee that

$$a(t) := \frac{1}{V} \int_X \dot{\varphi}_t \omega_t^n \overset{t\to+\infty}{\longrightarrow} 0.$$

This convergence will be necessary to show the convergence of the flow (see the discussion before Lemma 1 in [PSS07]).

Lemma 2.2. *The function $a(t)$ converges to zero as $t \to +\infty$ iff we choose*

$$\varphi_0(x) \equiv c_0 := \int_0^{+\infty} \|\nabla_t \dot{\varphi}_t\|^2_{L^2(X)} e^{-t} dt - \frac{1}{V} \int_X h_0 \, \omega^n.$$

Proof. Observe that

$$a'(t) = \frac{1}{V} \int_X \ddot{\varphi}_t \omega_t^n + \frac{n}{V} \int_X \dot{\varphi}_t dd^c \dot{\varphi}_t \wedge \omega_t^{n-1} = a(t) - \frac{d\mathcal{M}(\varphi_t)}{dt}.$$

Indeed

$$\ddot{\varphi}_t = \frac{d}{dt} \left\{ \log\left(\frac{\omega_t^n}{\omega^n}\right) + \varphi_t + h_0 \right\} = \dot{\varphi}_t + \Delta_{\omega_t} \dot{\varphi}_t$$

hence $\int_X \ddot{\varphi}_t \omega_t^n = \int_X \dot{\varphi}_t \omega_t^n$. We can integrate this ODE and obtain

$$a(t) = \left[a_0 - \int_0^t k'(s) e^{-s} ds \right] e^t,$$

where $k(s) := \mathcal{M}(\varphi_s)$. Since k is non decreasing and bounded from above, the function $k'(s)e^{-s}$ is integrable on $\mathbb{R}^+$ and $a(t) \to 0$ as $t \to +\infty$ if and only if

$$a(0) = \int_0^{+\infty} k'(s) e^{-s} ds.$$

Now $a(0) = V^{-1} \int_X \dot{\varphi}_0 \omega^n = c_0 + V^{-1} \int_X h_0 \omega^n$. The result follows. $\square$

Conclusion. The first normalization amounts to considering the parabolic flow of potentials

$$\dot{\varphi}_t := \frac{\partial \varphi_t}{\partial t} = \log\left(\omega_t^n / \omega^n\right) + \varphi_t + h_0$$

with constant initial potential

$$\varphi_0(x) \equiv c_0 := \int_0^{+\infty} \|\nabla_t \dot{\varphi}_t\|^2_{L^2(X)} e^{-t} dt - \frac{1}{V} \int_X h_0 \, \omega^n.$$

This choice of initial potential being possible only when the Mabuchi functional $\mathcal{M}$ is bounded from above, which is the case under our assumptions.

2.2. **Perelman's normalization ?** There is another choice of normalization which is perhaps more natural from a variational point of view. Namely we choose

$$\beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right]$$

so that

$$\dot{\varphi}_t = \log\left[\frac{MA(\varphi_t)}{\mu_t}\right],$$

where $MA(\varphi_t) = (\omega + dd^c\varphi_t)^n/V$ and

$$\mu_t := \frac{e^{-\varphi_t - h_0}\omega^n}{\int_X e^{-\varphi_t - h_0}\omega^n}$$

are both probability measures. This is the normalization used in [BBEGZ11].

Observe that changing further $\varphi_t(x)$ in $\varphi_t(x) + B(t)$ leaves both $MA(\varphi_t)$ and μ_t unchanged, but modifies $\dot{\varphi}_t(x)$ into $\dot{\varphi}_t(x) + B'(t)$. Thus we can only afford replacing φ_t by $\varphi_t - c_0$ so that $\varphi_0 \equiv 0$.

The Ricci deviation. Recall that we have set $\mathrm{Ric}(\omega_t) = \omega_t - dd^c h_t$, with

$$\frac{1}{V}\int_X e^{-h_t}\omega_t^n = 1.$$

We have observed that $\dot{\varphi}_t(x)$ and $h_t(x)$ only differ by a constant (in space). Now

$$\frac{1}{V}\int_X e^{-\dot{\varphi}_t}\omega_t^n = \int_X e^{-\dot{\varphi}_t} MA(\varphi_t) = \mu_t(X) = 1,$$

so that $\dot{\varphi}_t \equiv h_t$ with this choice of normalization. As we recall below, Perelman has succeeded in getting uniform estimates on the Ricci deviations h_t, these estimates therefore apply immediately to the function $\dot{\varphi}_t$ with our present choice of normalization.

Monotonicity of the functionals along the flow. We have observed previously that the Mabuchi functional is non-decreasing along the normalized Kähler-Ricci flow. Since this functional acts on metrics (rather than on potentials), this property is independent of the chosen normalization. The same holds true for the $\mathcal{F}$ functional:

Lemma 2.3. *The $\mathcal{F}$ functional is non-decreasing along the normalized Kähler-Ricci flow. More precisely,*

$$\frac{d\mathcal{F}(\varphi_t)}{dt} = H_{MA(\varphi_t)}(\mu_t) + H_{\mu_t}(MA(\varphi_t)) \geq 0.$$

Here $H_\mu(\nu)$ denotes the relative entropy of the probability measure ν with respect to the probability measure μ. It is defined by

$$H_\mu(\nu) = \int_X \log\left(\frac{\nu}{\mu}\right) d\nu$$

if ν is absolutely continuous with respect to μ, and $H_\mu(\nu) = +\infty$ otherwise. It follows from the concavity of the logarithm that

$$H_\mu(\nu) = -\int_X \log\left(\frac{\mu}{\nu}\right) d\nu \geq -\log\left(\mu(X)\right) = 0,$$

with strict inequality unless $\nu = \mu$.

Proof. Recall that $\mathcal{F}(\varphi) = E(\varphi) + \log\left[\int_X e^{-\varphi-h_0}\,\omega^n\right]$, where E is a primitive of the complex Monge-Ampère operator. We thus obtain along the NKRF,

$$\frac{dE(\varphi_t)}{dt} = \int_X \dot{\varphi}_t MA(\varphi_t) = \int_X \log\left(\frac{MA(\varphi_t)}{\mu_t}\right) MA(\varphi_t) = H_{\mu_t}(MA(\varphi_t)),$$

while

$$\frac{d\log\left[\int_X e^{-\varphi_t-h_0}\,\omega^n\right]}{dt} = -\int_X \dot{\varphi}_t d\mu_t = H_{MA(\varphi_t)}(\mu_t).$$

This proves the lemma. □

Recall that in the first normalization, the initial constant c_0 has been chosen so that

$$a(t) := \frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n = \int_X \dot{\varphi}_t MA(\varphi_t)$$

converges to zero as $t \to +\infty$. We relate this quantity to the above functionals:

Lemma 2.4. *Along the normalized Kähler-Ricci flow, one has*

$$\frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n = \frac{1}{V}\int_X h_0\,\omega^n + \mathcal{F}(\varphi_t) - \mathcal{M}(\varphi_t).$$

Observe that the right hand side only depends on ω_t, while the left hand side depends on the choice of normalization for φ_t. It is understood here that this identity holds under the Perelman normalization.

Proof. Recall that

$$\dot{\varphi}_t = \log(\omega_t^n/\omega^n) + \varphi_t + h_0 + \beta(t), \text{ with } \beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right].$$

We let $a(t) = \int_X \dot{\varphi}_t MA(\varphi_t)$ denote the left hand side and compute

$$a'(t) = \int_X \ddot{\varphi}_t MA(\varphi_t) - \frac{d\mathcal{M}(\varphi_t)}{dt},$$

where

$$\ddot{\varphi}_t = \Delta_{\omega_t}\dot{\varphi}_t + \dot{\varphi}_t + \beta'(t).$$

Therefore

$$a'(t) = a(t) + \beta'(t) - \frac{d\mathcal{M}(\varphi_t)}{dt} = \frac{d}{dt}\left\{\mathcal{F}(\varphi_t) - \mathcal{M}(\varphi_t)\right\},$$

noting that $a(t) = \frac{dE(\varphi_t)}{dt}$.

The conclusion follows since $a(0) = V^{-1}\int_X h_0\,\omega^n$ while $\mathcal{F}(0) = \mathcal{M}(0) = 0$. □

Mabuchi vs $\mathcal{F}$. We now show that the Mabuchi energy and the $\mathcal{F}$ functional are bounded from above simultaneously. This seems to have been noticed only recently (see [Li08, CLW09]).

Proposition 2.5. *Let X be a Fano manifold. The Mabuchi functional $\mathcal{M}$ is bounded from above if and only if the $\mathcal{F}$ functional is so. If such is the case, then*

$$\sup \mathcal{M} = \sup \mathcal{F} + \int_X h_0 \frac{\omega^n}{V}.$$

Proof. We have noticed in previous lemma, using Perelman' normalization, that

$$\mathcal{M}(\varphi_t) + \frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n = \mathcal{F}(\varphi_t) + \frac{1}{V}\int_X h_0\, \omega^n.$$

It follows from Perelman's estimates that $\dot{\varphi}_t$ is uniformly bounded along the flow. Thus $\mathcal{M}(\varphi_t)$ is bounded if and only if $\mathcal{F}(\varphi_t)$ is so. We assume such is the case.

The error term $a(t) = \frac{1}{V}\int_X \dot{\varphi}_t \omega_t^n$ is non-negative, with

$$0 \leq a(t) = \frac{dE(\varphi_t)}{dt}.$$

Since $\mathcal{F}(\varphi_t) = E(\varphi_t) + \beta(t)$ is bounded from above and $t \mapsto \beta(t)$ is increasing, the energies $t \mapsto E(\varphi_t)$ are bounded from above as well. Thus $\int^{+\infty} a(t)dt < +\infty$, hence there exists $t_j \to +\infty$ such that $a(t_j) \to 0$. We infer

$$\sup_{t>0} \mathcal{M}(\varphi_t) = \sup_{t>0} \mathcal{F}(\varphi_t) + \int_X h_0 \frac{\omega^n}{V}.$$

$\square$

Conclusion. The Perelman normalization amounts to consider the parabolic flow of potentials

$$\dot{\varphi}_t := \log\left(\omega_t^n/\omega^n\right) + \varphi_t + h_0 + \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right],$$

with initial potential $\varphi_0 \equiv 0$. Our plan is to show that when $\mathcal{F}$ is proper and $H^0(X, TX) = 0$, then $\tilde{\varphi}_t := \varphi_t - V^{-1}\int_X \varphi_t \omega^n$ converges, in the $\mathcal{C}^\infty$-sense, towards the unique function φ_{KE} such that

$$MA(\varphi_{KE}) = \frac{e^{-\varphi_{KE} - h}\omega^n}{\int_X e^{-\varphi_{KE} - h}\omega^n}$$

and $\int_X \varphi_{KE}\omega^n = 0$. This will imply that ω_t smoothly converges towards the unique Kähler-Einstein metric $\omega_{KE} = \omega + dd^c\varphi_{KE}$.

2.3. **Perelman's estimates.** We first explain how a uniform control on $|\varphi_t(x)|$ in finite time easily yields a uniform control in finite time on $|\dot{\varphi}_t(x)|$:

Proposition 2.6. *Assume $\varphi_t \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ satisfies*

$$\dot{\varphi}_t = \log\left(\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right) + \varphi_t + h_0 + \beta(t),$$

with $\varphi_0 = 0$, $\beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right]$. *Then* $\forall (x,t) \in X \times [0,T]$,

$$e^{2T}\inf_X h_0 \leq \dot{\varphi}_t(x) \leq \mathrm{Osc_X}(\varphi_t) + (\mathrm{n}+1)\mathrm{T} + \sup_{\mathrm{X}} \mathrm{h}_0.$$

Proof. Consider

$$H(x,t) := \dot{\varphi}_t(x) - \varphi_t(x) - (n+1)t - \beta(t),$$

and let $(x_0, t_0) \in X \times [0,T]$ be a point at which H realizes its maximum.

Set $\Delta_t := \Delta_{\omega_t}$. Observe that $\ddot{\varphi}_t = \dot{\varphi}_t + \Delta_t \dot{\varphi}_t + \beta'(t)$ and estimate

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H = \Delta_t \varphi_t - (n+1) \leq -1,$$

where the latter inequality comes from the identity

$$\Delta_t \varphi_t = n - n\frac{\omega \wedge \omega_t^{n-1}}{\omega_t^n} \leq n.$$

We infer that $t_0 = 0$, hence for all $(x,t) \in X \times [0,T]$,

$$H(x,t) \leq H(x_0, 0) = h_0(x_0) \leq \sup_X h_0,$$

thus

$$\dot{\varphi}_t(x) \leq [\sup_X \varphi_t + \beta(t)] + (n+1)T + \sup_X h_0.$$

The desired upper-bound follows by observing that $\beta(t) \leq -\inf_X \varphi_t$.

We use a similar reasoning to obtain the lower-bound, using the minimum principle for the Heat operator $\frac{\partial}{\partial t} - \Delta_t$, instead of the maximum principle. Indeed observe that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\dot{\varphi}_t) = \dot{\varphi}_t + \beta'(t) \geq \dot{\varphi}_t,$$

hence

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(e^{-2t}\dot{\varphi}_t) \geq -e^{-2t}\dot{\varphi}_t.$$

Let $(x, 0, t_0) \in X \times [0,T]$ be a point where $e^{-2t}\dot{\varphi}_t(x)$ realizes its minimum. If $t_0 > 0$, then

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta_t\right)(e^{-2t}\dot{\varphi}_t)_{|(x_0,t_0)} \geq -e^{-2t_0}\dot{\varphi}_{t_0}(x_0)$$

hence $\dot{\varphi}_t(x) \geq 0$ for all (x,t). If $t_0 = 0$, then

$$e^{-2t}\dot{\varphi}_t(x) \geq \dot{\varphi}_0(x_0) = \inf_X h_0 + \beta(0) = \inf_X h_0.$$

The desired lower-bound follows, as $\inf_X h_0 \leq 0$ since $\int_X e^{-h_0}\omega^n = V$. □

We let the reader check that similar bounds can be obtained for the first normalization. These bounds are sufficient to prove Cao's result [Cao85] (the normalized Kähler-Ricci flow exists in infinite time), however they blow up as $t \to +\infty$ hence are too weak to study the convergence of the NKRF.

By using the monotonicity of his $\mathcal{W}$-functional, together with a non-collapsing argument, Perelman was able to prove the following deep estimate:

Theorem 2.7. *There exists $C_1 > 0$ such that for all $(x,t) \in X \times \mathbb{R}^+$,*

$$|\dot{\varphi}_t(x)| \leq C_1.$$

We refer the reader to [SeT08] for a detailed proof. A sketchy proof is also provided in the Appendix of [TZ07], and more information can be found in [Cao].

3. $\mathcal{C}^0$-estimate

The main purpose of this section is to explain how to derive a uniform estimate on $|\varphi_t(x)|$. We first show that this is an elementary task in finite time, and then use the properness assumption and pluripotential tools to derive a uniform estimate on $X \times \mathbb{R}^+$. The latter estimate can not hold on Fano manifolds which do not admit a Kähler-Einstein metric.

3.1. **Control in finite time.**

Proposition 3.1. *Assume $\varphi_t \in PSH(X,\omega) \cap \mathcal{C}^\infty(X)$ satisfies*

$$\dot{\varphi}_t = \log\left(\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right) + \varphi_t + h_0 + \beta(t),$$

with $\varphi_0 = 0$, $\beta(t) = \log\left[\frac{1}{V}\int_X e^{-\varphi_t - h_0}\omega^n\right]$. Then $\forall (x,t) \in X \times [0,T]$,

$$e^{2T}\inf_X h_0 \leq \varphi_t(x) \leq e^{4T}\mathrm{Osc_X(h_0)}.$$

Proof. Let $(x_0,t_0) \in X \times [0,T]$ be a point at which the function $(x,t) \mapsto F(x,t) = e^{-2t}\varphi_t(x)$ realizes its maximum. If $t_0 = 0$, we obtain

$$e^{-2t}\varphi_t(x) \leq \varphi_0(x_0) = 0, \text{ hence } \varphi_t(x) \leq 0.$$

If $t_0 > 0$, then at (x_0,t_0) we have $dd^cF = e^{-2t_0}dd^c\varphi_{t_0}(x_0) \leq 0$ hence

$$\dot{\varphi}_{t_0}(x_0) \leq \varphi_{t_0}(x_0) + \sup_X h_0 + \beta(t_0),$$

while

$$0 \leq \frac{\partial F}{\partial t} = e^{-2t_0}\left\{\dot{\varphi}_{t_0}(x_0) - 2\varphi_{t_0}(x_0)\right\} \leq e^{-2t_0}\left\{\sup_X h_0 + \beta(t_0) - \varphi_{t_0}(x_0)\right\}.$$

The upper-bound follows by recalling that β is non-decreasing and

$$\beta(T) \leq -\inf_X \varphi_T \leq e^{2T}(-\inf_X h_0),$$

assuming the lower-bound holds true.

The latter is proved along the same lines: looking at the point where F realizes its minimum, we end up with a lower-bound

$$\varphi_t(x) \geq e^{2T}\inf_X h_0 + \beta(0) = e^{2T}\inf_X h_0,$$

since β vanishes at the origin. $\square$

3.2. **Uniform bound in infinite time.**

Theorem 3.2. *Let X be a Fano manifold such that the functional $\mathcal{F}$ is proper. Let $\omega_t := \omega + dd^c\psi_t$ be the solution of the normalized Kähler-Ricci flow with initial data $\omega \in c_1(X)$, where $\psi_t \in PSH(X,\omega)$ is normalized so that $\int_X \psi_t\, \omega^n = 0$. There exists $C_0 > 0$ such that*

$$\forall (x,t) \in X \times \mathbb{R}^+,\ |\psi_t(x)| \leq C_0.$$

Proof. Observe that $\psi_t = \varphi_t - \int_X \varphi_t\, \omega^n$, where φ_t satisfies

$$\dot{\varphi}_t = \log\left(\frac{MA(\varphi_t)}{\mu_t}\right),$$

with

$$MA(\varphi_t) = \frac{(\omega + dd^c\varphi_t)^n}{V} \text{ and } \mu_t = \frac{e^{-\varphi_t - h_0}\omega^n}{\int_X e^{-\varphi_t - h_0}\omega^n}.$$

We have observed that the functional $\mathcal{F}$ is translation invariant and non-decreasing along the NKRF. Since it is proper, we infer that the energies $t \mapsto E(\psi_t)$ are uniformly bounded below. Now

$$E(\psi_t) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \psi_t(\omega + dd^c\psi_t)^j \wedge \omega^{n-j} \leq \sup_X \psi_t \leq C_\omega,$$

since $\int_X \psi_t\omega^n = 0$ (see [GZ05, Proposition 1.7]), thus the energies $(E(\psi_t))$ are uniformly bounded along the flow. It follows that the functions ψ_t belong to

$$\mathcal{E}^1_C(X,\omega) := \{u \in PSH(X,\omega)\,|\, u \leq C \text{ and } E(u) \geq -C\},$$

for $C > 0$ large enough. This is a compact set (for the L^1-topology) of functions which have zero Lelong numbers at all points (see below). It follows therefore from Skoda's uniform integrability theorem that there exists $A > 0$ such that

$$\sup_{t \geq 0}\int_X e^{-2\psi_t - 2h_0}\omega^n \leq A.$$

Note that $\int_X e^{-\psi_t - h_0}\omega^n \geq Ve^{-\sup_X \psi_t} \geq \delta_0 > 0$ and recall that $\dot{\varphi}_t(x) \leq C_1$ by Perelman's fundamental estimate to conclude that

$$MA(\psi_t) = e^{\dot{\varphi}_t}\frac{e^{-\psi_t - h_0}\omega^n}{\int_X e^{-\psi_t - h_0}\omega^n} = f_t\omega^n,$$

where the densities $0 \leq f_t$ are uniformly in $L^2(X)$, $\|f_t\|_{L^2(\omega^n)} \leq A'$. It follows therefore from Theorem 3.8 that ψ_t is uniformly bounded. □

Remark 3.3. The reader will find a rather different approach in [TZ07, PSS07, PS10], using the first normalization, Moser iterative process and a uniform Sobolev inequality along the flow. It takes some efforts to check that the two normalizations are uniformly comparable along the flow, give it a try !

3.3. **Pluripotential tools.**

We explain here some of the pluripotential tools that have been used in the above proof.

3.3.1. *Finite energy classes.* Recall that X is an n-dimensional Fano manifold, ω is a fixed Kähler form in $c_1(X)$, and $V = c_1(X)^n = \int_X \omega^n$. The energy $E(\psi)$ of a smooth ω-plurisubharmonic function,

$$E(\psi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \psi(\omega + dd^c\psi)^j \wedge \omega^{n-j},$$

is non-decreasing in ψ. It can thus be extended to any $\varphi \in PSH(X,\omega)$ by setting

$$E(\varphi) := \inf_{\psi \geq \varphi} E(\psi),$$

where the infimum runs over all smooth ω-psh functions ψ that lie above φ.

Definition 3.4. We set

$$\mathcal{E}^1(X,\omega) := \{\varphi \in PSH(X,\omega) \,|\, E(\varphi) > -\infty\}.$$

and

$$\mathcal{E}^1_C(X,\omega) := \{\varphi \in \mathcal{E}^1(X,\omega) \,|\, E(\varphi) \geq -C \text{ and } \varphi \leq C\}.$$

The following properties are established in [GZ07, BEGZ10]:

- the complex Monge-Ampère operator $MA(\cdot)$ is well defined on the class $\mathcal{E}^1(X,\omega)$, since the Monge-Ampère measure of a function $\varphi \in \mathcal{E}^1(X,\omega)$ is very well approximated (in the Borel sense) by the Monge-Ampère measures $MA(\varphi_j)$ of its canonical approximants $\varphi_j := \max(\varphi, -j)$;
- the maximum and comparison principles hold, namely if $\varphi, \psi \in \mathcal{E}^1(X,\omega)$,
$$1_{\{\varphi>\psi\}} MA(\max(\varphi,\psi)) = 1_{\{\varphi>\psi\}} MA(\varphi)$$
and
$$\int_{\{\varphi<\psi\}} MA(\psi) \leq \int_{\{\varphi<\psi\}} MA(\varphi).$$
- the functions with finite energy have zero Lelong number at all points, as follows by observing that the class $\mathcal{E}^1(X,\omega)$ is stable under the max-operation, while $\chi \log \mathrm{dist}(\cdot, \mathrm{x})$ is ω-plurisubharmonic and does not belong to $\mathcal{E}^1(X,\omega)$ for a suitable cut-off function χ;
- the sets $\mathcal{E}^1_C(X,\omega)$ are compact subsets of $L^1(X)$: this easily follows from the upper semi-continuity property of the energy, together with the fact that the set
$$\{\varphi \in PSH(X,\omega) \,|\, -C' \leq \sup_X \varphi \leq C\}$$
is compact in $L^1(X)$.

Recall now the following uniform version of Skoda's integrability theorem [Zer01]:

Theorem 3.5. *Let $\mathcal{B} \subset PSH(X,\omega)$ be a compact family of ω-psh functions, set*

$$\nu(\mathcal{B}) := \sup\{\nu(\varphi, x) \,|\, x \in X \text{ and } \varphi \in \mathcal{B}\}.$$

For every $A < 2/\nu(\mathcal{B})$, there exists $C_A > 0$ such that

$$\forall \varphi \in \mathcal{B}, \quad \int_X e^{-A\varphi}\, \omega^n \leq C_A.$$

It follows from this result that functions from $\mathcal{E}^1_C(X,\omega)$ satisfy such a uniform integrability property with $A > 0$ as large as we like.

3.3.2. *Capacities and volume.* For a Borel set $K \subset X$, we consider

$$M_\omega(K) := \sup_X V^*_{K,\omega} \in [0, +\infty], \tag{3.1}$$

where

$$V_{K,\omega} := \sup\{\varphi \in PSH(X,\omega) \,|\, \varphi \leq 0 \text{ on } K\}.$$

One checks that $M_\omega(K) = +\infty$ if and only if K is pluripolar. We also set

$$\mathrm{Cap}(K) := \sup\left\{\int_K MA(u) \,|\, 0 \leq u \leq 1\right\}.$$

This is the Monge-Ampère capacity. It vanishes on pluripolar sets.

Lemma 3.6. *For every non-pluripolar compact subset K of X, we have*

$$1 \leq \mathrm{Cap}(K)^{-1/n} \leq \max(1, M_\omega(K)).$$

Proof. The left-hand inequality is trivial. In order to prove the right-hand inequality we consider two cases. If $M_\omega(K) \leq 1$, then $V^*_{K,\omega}$ is a candidate in the definition of $\mathrm{Cap}(K)$. One checks that $\mathrm{MA}(V^*_{K,\omega})$ is supported on K, thus

$$\mathrm{Cap}(K) \geq \int_K \mathrm{MA}(V^*_{K,\omega}) = \int_X \mathrm{MA}(V^*_{K,\omega}) = 1$$

and the desired inequality holds in that case.

On the other hand if $M := M_\omega(K) \geq 1$ we have $0 \leq M^{-1}V^*_{K,\omega} \leq 1$ and it follows by definition of the capacity again that

$$\mathrm{Cap}(K) \geq \int_K \mathrm{MA}(M^{-1}V^*_{K,\omega}).$$

Since $\mathrm{MA}(M^{-1}V^*_{K,\omega}) \geq M^{-n}\mathrm{MA}(V^*_{K,\omega})$ we deduce that

$$\int_K \mathrm{MA}(M^{-1}V^*_{K,\omega}) \geq M^{-n} \int_X \mathrm{MA}(V^*_{K,\omega}) = M^{-n}$$

and the result follows. □

Proposition 3.7. *Let $\mu = fdV$ be a positive measure with L^p density with respect to Lebesgue measure, with $p > 1$. Then there exists $C > 0$ such that*

$$\mu(B) \leq C \cdot \mathrm{Cap}(B)^2$$

for all Borelian $B \subset X$, where $C := (p-1)^{-2n} A\|f\|_{L^{1+\varepsilon}(dV)}$, and $A = A(\omega, dV)$.

Proof. It is enough to consider the case where $B = K$ is compact. We can also assume that K is non-pluripolar since $\mu(K) = 0$ otherwise and the inequality is then trivial. Set

$$\nu(X) := \sup_{T,x} \nu(T, x) \tag{3.2}$$

the supremum ranging over all positive currents $T \in c_1(X)$ and all $x \in X$, and $\nu(T, x)$ denoting the Lelong number of T at x. Since all Lelong numbers of

$\nu(X)^{-1}T$ are < 2 for each positive current $T \in c_1(X)$, Skoda's uniform integrability theorem yields $C_\omega > 0$ only depending on dV and ω such that

$$\int_X \exp(-\nu(X)^{-1}\psi)dV \leq C_\omega$$

for all ω-psh functions ψ normalized by $\sup_X \psi = 0$. Applying this to $\psi = V^*_{K,\omega} - M_\omega(K)$ (which has the right normalization by (3.1)) we get

$$\int_X \exp(-\nu(X)^{-1}V^*_{K,\omega})dV \leq C_\omega \exp(-\nu(X)^{-1}M_\omega(K)).$$

On the other hand $V^*_{K,\omega} \leq 0$ on K a.e. with respect to Lebesgue measure, hence

$$\mathrm{vol}(K) \leq C_\omega \exp(-\nu(X)^{-1}M_\omega(K)). \tag{3.3}$$

Now Hölder's inequality yields

$$\mu(K) \leq \|f\|_{L^p(dV)} \,\mathrm{vol}(K)^{1/q}, \tag{3.4}$$

where q denotes the conjugate exponent. We may also assume that $M_\omega(K) \geq 1$. Otherwise Lemma 3.6 implies $\mathrm{Cap}(K) = 1$, and the result is thus clear in that case. By Lemma 3.6, (3.3) and (3.4) together we thus get

$$\mu(K) \leq C_\omega^{1/q}\|f\|_{L^p(dV)} \exp\left(-\frac{1}{q\nu(X)}\,\mathrm{Cap}(K)^{-1/n}\right)$$

and the result follows since $\exp(-t^{-1/n}) = O(t^2)$ when $t \to 0_+$. □

3.3.3. *Kolodziej's uniform a priori estimate.* We are now ready to prove the following celebrated result of Kolodziej [Koł98]:

Theorem 3.8. *Let $\mu = MA(\varphi) = fdV$ be a probability Monge-Ampère measure with density $f \in L^p$, $p > 1$. Then*

$$Osc_X\varphi \leq C$$

where C only depends on $\omega, dV, ||f||_{L^p}$.

Proof. We can assume φ is normalized so that $\sup_X \varphi = 0$. Consider

$$g(t) := (\mathrm{Cap}\{\varphi < -t\})^{1/n}\,.$$

Our goal is to show that $g(M) = 0$ for some M under control. Indeed we will then have $\varphi \geq -M$ on $X \setminus P$ for some Borel subset P such that $\mathrm{Cap}(P) = 0$. It then follows from Proposition 3.7 (applied to the Lebesgue measure itself) that P has Lebesgue measure zero hence $\varphi \geq -M$ will hold everywhere.

Since $\mathrm{MA}(\varphi) = \mu$ it follows from Proposition 3.7 and Lemma 3.9 that

$$g(t+\delta) \leq \frac{C^{1/n}}{\delta} g(t)^2 \quad \text{for all } t > 0 \text{ and } 0 < \delta < 1.$$

We can thus apply Lemma 3.10 below which yields $g(M) = 0$ for $M := t_0 + 5C^{1/n}$. Here $t_0 > 0$ has to be chosen so that

$$g(t_0) < \frac{1}{5C^{1/n}}.$$

Now Lemma 3.9 (with $\delta = 1$) implies that

$$g(t)^n \leq \mu\{\varphi < -t+1\} \leq \frac{1}{t-1}\int_X |\varphi| f dV \leq \frac{1}{t-1}\|f\|_{L^p(dV)}\|\varphi\|_{L^q(dV)}$$

by Hölder's inequality. Since φ belongs to the compact set of ω-psh functions normalized by $\sup_X \varphi = 0$, its $L^q(dV)$-norm is bounded by a constant C_2 only depending on ω, dV and p. It is thus enough to take

$$t_0 > 1 + 5^{n-1} C_2 C \|f\|_{L^p(dV)}.$$

□

Lemma 3.9. *Fix $\varphi \in \mathcal{E}^1(X, \omega)$. Then for all $t > 0$ and $0 < \delta < 1$ we have*

$$\mathrm{Cap}\{\varphi < -t-\delta\} \leq \delta^{-n} \int_{\{\varphi < -t\}} \mathrm{MA}(\varphi).$$

Proof. Let ψ be a ω-psh function such that $0 \leq \psi \leq 1$. We then have

$$\{\varphi < -t-\delta\} \subset \{\varphi < \delta\psi - t - \delta\} \subset \{\varphi < -t\}.$$

Since $\delta^n \mathrm{MA}(\psi) \leq \mathrm{MA}(\delta\psi)$ and $\varphi \in \mathcal{E}^1(X, \omega)$ it follows from the comparison principle that

$$\begin{aligned}\delta^n \int_{\{\varphi < -t-\delta\}} \mathrm{MA}(\psi) &\leq \int_{\{\varphi < \delta\psi - t - \delta\}} \mathrm{MA}(\delta\psi) \\ &\leq \int_{\{\varphi < \delta\psi - t - \delta\}} \mathrm{MA}(\varphi) \leq \int_{\{\varphi < -t\}} \mathrm{MA}(\varphi)\end{aligned}$$

and the proof is complete. □

Lemma 3.10. *Let $g : \mathbb{R}^+ \to [0, 1]$ be a decreasing function such that $g(+\infty) = 0$ and*

$$g(t+\delta) \leq \frac{C^{1/n}}{\delta} g(t)^2 \quad \textit{for all } t > 0 \textit{ and } 0 < \delta < 1.$$

Then $g(t) = 0$ for all $t \geq t_0 + 5C^{1/n}$, where

$$t_0 = \inf\{s > 0 \mid g(s) \leq e^{-1} C^{-1/n}\}.$$

Proof. Set $f(t) = -\log g(t)$ so that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing with

$$f(t+\delta) \geq 2f(t) - \log\left(\delta / C^{1/n}\right).$$

By induction we define an increasing sequence t_j such that

$$t_{j+1} = t_j + \delta_j, \text{ with } \delta_j = eC^{1/n} \exp(-f(t_j)) = eC^{1/n} g(t_j).$$

Observe that $0 < \delta_0$ is smaller than 1 if we choose t_0 as indicated. Since (t_j) is increasing and g is decreasing, this insures that δ_j is smaller than 1 for all $j \in \mathbb{N}$. We can thus use the growth estimate and obtain

$$f(t_{j+1}) = f(t_j + \delta_j) \geq f(t_j) + 1.$$

Since $f \geq 0$, we infer $f(t_j) \geq j$ for all j. Now

$$t_\infty := t_0 + \sum_{j \geq 0}(t_{j+1} - t_j) \leq t_0 + C^{1/n} e \sum_{j \geq 0} \exp(-j) \leq t_0 + 5C^{1/n}.$$

The proof is thus complete since $f(t) \geq f(t_\infty) = +\infty$ for all $t \geq t_\infty$. $\square$

4. Higher order estimates

4.1. **Preliminaries.** We shall need two auxiliary results.

Lemma 4.1. *Let α, β be positive $(1,1)$-forms. Then*

$$n \left(\frac{\alpha^n}{\beta^n} \right)^{\frac{1}{n}} \leq Tr_\beta(\alpha) \leq n \left(\frac{\alpha^n}{\beta^n} \right) \cdot (Tr_\alpha(\beta))^{n-1}.$$

Proof. At a given point $x \in X$, we can assume that the quadratic forms defined by $\alpha(x), \beta(x)$ are co-reduced so that $\beta(x)$ is the identity matrix while $\alpha(x)$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. It follows then from the concavity of the logarithm that

$$\frac{1}{n} \log \Pi_j \lambda_j = \frac{1}{n} \sum \log \lambda_j \leq \log \left(\frac{1}{n} \sum \lambda_j \right),$$

which yields the first inequality. The second one follows by observing that

$$\sum_j \lambda_j \leq (n-1) \sum_j \lambda_j \leq (\Pi_j \lambda_j) \left(\sum_j \frac{1}{\lambda_j} \right)^{n-1},$$

when $n \geq 2$. There is actually equality when $n = 1$. $\square$

Applying these inequalities to $\alpha = \omega_t := \omega + dd^c \varphi_t$ and $\beta = \omega$, we obtain:

Corollary 4.2. *There exists $C > 0$ which only depends on $\|\dot{\varphi}_t\|_{L^\infty}$ such that*

$$\frac{1}{C} \leq Tr_\omega(\omega_t) \leq C[Tr_{\omega_t}(\omega)]^{n-1}.$$

The second result we need is the following estimate which goes back to the work of Aubin [Aub78] and Yau [Yau78]; in this form it is due to Siu [Siu87].

Lemma 4.3. *Let ω, ω' be arbitrary Kähler forms. Let $-B \in \mathbb{R}$ be a lower bound on the holomorphic bisectional curvature of (X, ω). Then*

$$\Delta_{\omega'} \log Tr_\omega(\omega') \geq -\frac{Tr_\omega(Ric(\omega'))}{Tr_\omega(\omega')} - B\, Tr_{\omega'}(\omega).$$

We include a proof for the reader's convenience.

Proof. Since this is a local differential inequality, we can use a local system of normal coordinates so that near the origin (which is the point to be considered), $\omega = \sum \omega_{pq} \sqrt{-1} dz_p \wedge d\overline{z}_q$ with

$$\omega_{pq} = \delta_{pq} - \sum_{i,j} R_{ijpq} z_i \overline{z}_j + O(\|z\|^3).$$

Here R_{ijpq} denotes the curvature tensor of ω and δ_{pq} stands for the Kronecker symbol. We can also impose that

$$\omega'_{pq} = \lambda_p \delta_{pq} + O(\|z\|), \text{ with } \lambda_1 \leq \ldots \leq \lambda_n.$$

Observe that the inverse matrix $(\omega^{pq}) = (\omega_{pq})^{-1}$ satisfies

$$\omega^{pq} = \delta_{pq} + \sum_{i,j} R_{ijpq} z_i \overline{z}_j + O(\|z\|^3).$$

We also recall that the curvature tensor of ω' is given by

$$R'_{ijpq} = \sum_k \frac{1}{\lambda_k} \partial_i \omega_{pk} \overline{\partial}_j \omega'_{pq} - \partial_i \overline{\partial}_j \omega'_{pq}.$$

Set $u := Tr_\omega(\omega')$ and observe that $dd^c \log u = \frac{dd^c u}{u} - \frac{du \wedge d^c u}{u^2}$, thus

$$\Delta_{\omega'} \log u = \frac{\Delta_{\omega'} u}{u} - \frac{Tr_{\omega'}(du \wedge d^c u)}{u^2}.$$

Now $\Delta_{\omega'} u = \sum_{ik} \lambda_i^{-1} \partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk})$ and $\partial_i \overline{\partial}_i (\omega^{kk} \omega'_{kk}) = \lambda_k R_{iikk} + \partial_i \overline{\partial}_i \omega'_{kk}$ hence

$$\Delta_{\omega'} u = \sum_{ik} \lambda_k \lambda_i^{-1} R_{iikk} + \sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk}.$$

Our assumption on ω is $R_{iikk} \geq -B$. Therefore the first term above is bounded below by $\sum_{ik} \lambda_k \lambda_i^{-1} R_{iikk} \geq -B(\sum_i \lambda_i^{-1})(\sum_k \lambda_k) \geq -B Tr_{\omega'}(\omega) Tr_\omega(\omega')$, thus

$$\frac{\sum_{ik} \lambda_k \lambda_i^{-1} R_{iikk}}{u} \geq -B Tr_{\omega'}(\omega).$$

We now take care of the remaining terms. By definition of R'_{ijpq},

$$\sum_{i,k} \lambda_i^{-1} \partial_i \overline{\partial}_i \omega'_{kk} = \sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2 - \sum_{i,k} \lambda_i^{-1} R'_{iikk}.$$

Note that $\sum_{i,k} \lambda_i^{-1} R'_{iikk} = Tr_\omega(Ric(\omega'))$, while

$$\begin{aligned} \sum_{i,k,p} \lambda_i^{-1} \lambda_p^{-1} |\partial_i \omega'_{kp}|^2 &\geq \sum_{i,k} \lambda_i^{-1} \lambda_k^{-1} |\partial_i \omega'_{kk}|^2 \\ &\geq \frac{\sum_{i,k,l} \lambda_i^{-1} \partial_i \omega'_{kk} \partial_i \omega'_{ll}}{\sum_i \lambda_i} = \frac{Tr_{\omega'}(du \wedge d^u)}{u}, \end{aligned}$$

where the last estimate follows from the Cauchy-Schwarz inequality. Altogether we obtain

$$\frac{\Delta_{\omega'} u}{u} \geq -\frac{Tr_\omega(Ric(\omega'))}{Tr_\omega(\omega')} - B\, Tr_{\omega'}(\omega) + \frac{Tr_{\omega'}(du \wedge d^c u)}{u^2},$$

which yields the claim. $\square$

4.2. $\mathcal{C}^2$-estimate.

Theorem 4.4. *Let X be a Fano manifold such that $\mathcal{F}$ is proper. Let ω_t be the solution of the normalized Kähler-Ricci flow with initial data $\omega \in c_1(X)$. There exists $C_2 > 0$ such that for all $(x,t) \in X \times \mathbb{R}^+$,*

$$0 \leq \mathrm{tr}_\omega(\omega_t) \leq C_2.$$

Proof. Set $\alpha(x,t) := \log \operatorname{tr}_\omega(\omega_t) - (B+1)\varphi_t$, were $-B$ denotes a lower bound on the holomorphic bisectional curvature of (X,ω) (as in Lemma 4.3). Fix $T > 0$ and let $(x_0, t_0) \in X \times [0,T]$ be a point at which α realizes its maximum.

Either $t_0 = 0$, in which case $\alpha(x,t) \leq \alpha(x_0, 0) = \log n$ yields

$$\operatorname{tr}_\omega(\omega_t)(x) \leq n \exp([B+1]\varphi_t(x)) \leq C_2' = n \exp([B+1]C_0),$$

since φ_t is uniformly bounded from above.

Or $t_0 > 0$. In this case it follows from Lemma 4.5 that at point (x_0, t_0),

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t\right) \alpha \leq -\operatorname{tr}_{\omega_{t_0}}(\omega)(x_0) + \kappa$$

so that

$$\operatorname{tr}_\omega(\omega_t)(x) \leq C_2'' = \kappa \exp(2[B+1]C_0).$$

The conclusion follows since both C_2' and C_2'' are independent of T. $\square$

Lemma 4.5. *Set $\alpha(x,t) := \log \operatorname{tr}_\omega(\omega_t) - (B+1)\varphi_t$. There exists $\kappa > 0$ such that*

$$\forall (x,t) \in X \times \mathbb{R}^+, \quad \left(\frac{\partial}{\partial t} - \Delta_t\right) \alpha \leq -\operatorname{tr}_{\omega_t}(\omega) + \kappa.$$

Here $-B$ denotes a lower bound on the holomorphic bisectional curvature of (X,ω) (as in Lemma 4.3).

Proof. It follows from Perelman's estimate that

$$\frac{\partial \alpha}{\partial t} = \frac{\Delta_\omega \dot{\varphi}_t}{\operatorname{tr}_\omega(\omega_t)} - (B+1)\dot{\varphi}_t \leq \frac{\Delta_\omega \dot{\varphi}_t}{\operatorname{tr}_\omega(\omega_t)} + C.$$

Now $\dot{\varphi}_t = \log(\omega_t^n/\omega^n) + \varphi_t + h_0 + \beta(t)$ thus

$$\begin{aligned} \Delta_\omega \dot{\varphi}_t &= \Delta_\omega \log\left(\frac{\omega_t^n}{\omega^n}\right) + \operatorname{tr}_\omega(\omega_t) - n + \Delta_\omega h_0 \\ &\leq \Delta_\omega \log\left(\frac{\omega_t^n}{\omega^n}\right) + \operatorname{tr}_\omega(\omega_t) + C'. \end{aligned}$$

Since $dd^c \log\left(\frac{\omega_t^n}{\omega^n}\right) = \operatorname{Ric}(\omega) - \operatorname{Ric}(\omega_t)$, we infer

$$\Delta_\omega \dot{\varphi}_t \leq -\operatorname{tr}_\omega(\operatorname{Ric}(\omega_t)) + \operatorname{tr}_\omega(\omega_t) + C'',$$

hence

$$\frac{\partial \alpha}{\partial t} \leq -\frac{\operatorname{tr}_\omega(\operatorname{Ric}(\omega_t))}{\operatorname{tr}_\omega(\omega_t)} + \frac{C''}{\operatorname{tr}_\omega(\omega_t)} + C + 1.$$

We now estimate $\Delta_{\omega_t}\alpha = \Delta_t \alpha$ from below. It follows from Lemma 4.3 that

$$\begin{aligned} \Delta_t \alpha &= \Delta_t \log \operatorname{tr}_\omega(\omega_t) - (B+1)[n - \operatorname{tr}_{\omega_t}(\omega)] \\ &\geq -\frac{\operatorname{tr}_\omega(\operatorname{Ric}(\omega_t))}{\operatorname{tr}_\omega(\omega_t)} + \operatorname{tr}_{\omega_t}(\omega) - n(B+1). \end{aligned}$$

Therefore

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)(\alpha) \leq -\operatorname{tr}_{\omega_t}(\omega) + \frac{C''}{\operatorname{tr}_\omega(\omega_t)} + C'''.$$

The conclusion follows since $\operatorname{tr}_\omega(\omega_t)$ is uniformly bounded from below away from zero, as we have observed in the preliminaries. $\square$

Remark 4.6. The reader can go through the above proof and realize that one can obtain similarly a uniform upper bound for $\mathrm{tr}_\omega(\omega_t)$ on any finite interval of time, without assuming the properness of the functional $\mathcal{F}$.

4.3. **Complex parabolic Evans-Krylov theory and Schauder estimates.** At this stage, it follows from local arguments that one can obtain higher order uniform a priori estimates. We won't dwell on these techniques here and rather refer the reader to the lecture notes by C.Imbert [Imbert] for the real theory. The latter can not be directly applied in the complex setting, but the technique can be adapted as was done for instance in [Gill11].

5. Convergence of the flow

5.1. **Asymptotic of the time-derivatives.**

Proposition 5.1. *The time-derivatives $\dot{\psi}_t$ converge to zero in $\mathcal{C}^\infty(X)$.*

Proof. Note that $\int_X \psi_t\,\omega^n = 0$ hence $\int_X \dot{\psi}_t\,\omega^n = 0$ in the Perelman normalization, while for the first normalization, φ_t has been so normalized that

$$\int_X \dot{\varphi}_t\,\omega_t^n \stackrel{t\to+\infty}{\longrightarrow} 0.$$

It therefore suffices to check that

$$\int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_t^{n-1} \longrightarrow 0,$$

since ω_t and ω are uniformly equivalent, by Theorem 4.4.

To check the latter convergence, we follow some arguments by Phong and Sturm [PS06]. Set

$$Y(t) := \int_X |\nabla_t\dot{\varphi}_t|^2\omega_t^n = n\int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega_t^{n-1}.$$

Recall that the Mabuchi functional $\mathcal{M}$ is bounded from above and increasing along the flow, with

$$\frac{d\mathcal{M}(\varphi_t)}{dt} = Y(t) \geq 0, \text{ thus } \int_0^{+\infty} Y(t)dt < +\infty.$$

We cannot of course immediately deduce that $Y(t) \to 0$ as $t \to +\infty$, however Phong-Sturm succeed, by using a Bochner-Kodaira type formula and a uniform control of the curvatures along the flow, in showing that $Y' \leq CY$ for some uniform positive constant $C > 0$.

The reader will easily check that this further estimate allows to conclude. We refer to [SW] for the controls on the curvatures along the flow, and to [PS06] for the remaining details. We propose in Lemma 5.2 a slightly weaker, but economical control that is also sufficient, as the reader will check. □

Lemma 5.2. *Set*

$$Z(t) := n\int_X d\dot{\varphi}_t \wedge d^c\dot{\varphi}_t \wedge \omega^{n-1}.$$

Then $Z'(t) \leq 2Z(t) + C$ for some uniform constant $C > 0$.

Proof. Observe that

$$Z'(t) = -2n \int_X \ddot{\varphi}_t dd^c \dot{\varphi}_t \wedge \omega^{n-1} \text{ with } \dot{\varphi}_t = \log\left(\frac{\omega_t^n}{\omega^n}\right) + \varphi_t + h_0.$$

We use here the first normalization, this clearly does not affect the value of $Z(t)$. Since $\ddot{\varphi}_t = \Delta_t \dot{\varphi}_t + \dot{\varphi}_t$, we infer

$$Z'(t) = 2Z(t) - 2\int_X \Delta_t \dot{\varphi}_t \Delta_\omega \dot{\varphi}_t \omega^n \leq 2Z(t) + C,$$

since the latter quantities are uniformly bounded along the flow. □

5.2. **Conclusion.** We are now in position to conclude.

First normalization. It follows from previous sections that the family (φ_t) is relatively compact in $\mathcal{C}^\infty(X \times [0, +\infty])$. Let $\varphi_\infty = \lim_{j\to+\infty} \varphi_{t_j}$ be a cluster point of $(\varphi_t)_{t>0}$. It follows from Proposition 5.1 that $\dot{\varphi}_{t_j} \to 0$ hence

$$(\dagger) \qquad (\omega + dd^c\varphi_\infty)^n = e^{-\varphi_\infty} e^{-h_0} \omega^n,$$

hence $\omega + dd^c\varphi_\infty$ is a Kähler-Einstein metric. Since we have assumed that X has no holomorphic vector field, it follows from Bando-Mabuchi's uniqueness result [BM87] that φ_∞ coincides with the Kähler-Einstein potential φ_{KE}, which is the unique solution of (†). There is thus a unique cluster point for (φ_t) as $t \to +\infty$, hence the whole family converges in the $\mathcal{C}^\infty$-sense towards φ_{KE}.

It turns out that the above convergence holds at an exponential speed. We refer the interested reader to [PSSW08, PS10] for a proof of this fact.

Perelman normalization. A similar argument could be used for the potentials $\psi_t = \varphi_t - V^{-1}\int_X \varphi_t \omega^n$ if we could show the convergence of $\int_X \varphi_t \omega^n$ as $t \to +\infty$. To get around this difficulty, we can proceed as follows: let $\mathcal{K}$ denote the set of cluster values of $(\omega_t)_{t>0}$. Observe that $\mathcal{K}$ is invariant under the normalized Kähler-Ricci flow and the functional $\mathcal{F}$ is constant on $\mathcal{K}$.

It follows now from Lemma 2.3 that $\mathcal{F}$ is strictly increasing along the NKRF, unless we start from a fixed point ω_0. Thus $\mathcal{K}$ consists in fixed points for the NKRF. There is only one such fixed point, the unique Kähler-Einstein metric. Therefore ω_t converges to ω_{KE} and ψ_t converges to the unique Kähler-Einstein potential ψ_{KE} such that $\omega_{KE} = \omega + dd^c\psi_{KE}$ and $\int_X \psi_{KE}\, \omega^n = 0$.

6. An alternative approach

We finally briefly mention an alternative approach to the weak convergence of the normalized Kähler-Ricci flow, as recently proposed in [BBEGZ11].

The convergence of ω_t towards ω_{KE} is only proved in the weak sense of (positive) currents, but without using Perelman's deep estimates: this allows us in [BBEGZ11] to extend Perelman's convergence result to singular settings (weak Fano varieties and pairs), where these estimates are not available.

6.1. **The variational characterization of Kähler-Einstein currents.** The alternative approach we propose in [BBEGZ11] relies on the variational characterization of Kähler-Einstein currents established in [BBGZ09].

Let X be a Fano manifold and fix $\omega \in c_1(X)$ a Kähler form. A positive current $T = \omega + dd^c\psi \in c_1(X)$ is said to have finite energy if $E(\psi) > -\infty$. We set then

$$E(T) := E(\psi) - \frac{1}{V}\int_X \psi\omega^n.$$

We let $\mathcal{E}^1(c_1(X))$ denote the set of currents with finite energy in $c_1(X)$ and

$$\mathcal{E}^1_C(c_1(X)) := \{T \in \mathcal{E}^1(c_1(X)) \,|\, E(T) \geq -C\}$$

the compact convex set of those positive closed currents in $c_1(X)$ whose energy is uniformly bounded from below by C.

A combination of [BM87, Tian97] and [BBGZ09, Theorems D,E] yields the following criterion:

Theorem 6.1. *Let X be a Fano manifold with $H^0(X, TX) = 0$. Let T be a positive closed current in $c_1(X)$ with finite energy. The following are equivalent:*

1. *T maximizes the functional $\mathcal{F}$;*
2. *T is a Kähler-Einstein current;*
3. *T is the unique Kähler-Einstein metric;*
4. *the functional $\mathcal{F}$ is proper.*

We say here that a current $T = \omega + dd^c\varphi \in \mathcal{E}^1(c_1(X))$ is Kähler-Einstein if it satisfies $T^n = e^{-\varphi-h_0}\omega^n$, where as previously $\mathrm{Ric}(\omega) = \omega - dd^c h_0$.

The equivalence of the last two items is due to Tian [Tian97] (and Bando-Mabuchi [BM87] for the uniqueness). It was also realized by Ding-Tian [DT92] that the Kähler-Einstein metric is the unique Kähler metric maximizing $\mathcal{F}$. This result being extended to the class of finite energy currents allows to use the soft compacity criteria available in these Sobolev-like spaces:

Corollary 6.2. *Let X be a Fano manifold such that $\mathcal{F}$ is proper. If $\omega_t \in c_1(X)$ are Kähler forms with uniformly bounded energies such that $\mathcal{F}(\omega_t) \nearrow \sup \mathcal{F}$, then*

$$\omega_t \longrightarrow \omega_{KE}$$

in the weak sense of (positive) currents.

6.2. **Maximizing subsequences.** We let the potential $\varphi_t \in PSH(X, \omega) \cap \mathcal{C}^\infty(X)$ evolve according to the complex Monge-Ampère flow,

$$\dot{\varphi}_t = \log\left(\frac{MA(\varphi_t)}{\mu_t}\right) = \log\left(\frac{\omega_t^n}{\omega^n}\right) + \varphi_t + h_0 + \beta(t),$$

where

$$\beta(t) = \log\left[\int_X e^{-\varphi_t - h_0}\,\omega^n\right],$$

with initial condition $\varphi_0 = 0$. We set $\psi_t := \varphi_t - \int_X \varphi_t\,\omega^n$.

Recall that the functional $\mathcal{F}$ is non-decreasing along this flow. It follows more precisely from Lemma 2.3 and Pinsker's inequality (see [Villani, Remark 22.12]) that for all $0 < s < t$,

$$(P) \qquad \mathcal{F}(\varphi_t) - \mathcal{F}(\varphi_s) \geq \int_s^t ||MA(\varphi_r) - \mu_r\|^2 \, dr,$$

where $\|\nu - \mu\|$ denotes the total variation of the signed measure $\nu - \mu$.

Since $\mathcal{F}$ is assumed to be proper, it follows from the monotonicity property that the ψ_t's have uniformly bounded energies hence form a relatively compact family. Let ψ_∞ be any cluster point. If we could show that

$$\mathcal{F}(\varphi_t) \nearrow \sup_{\mathcal{E}^1(X,\omega)} \mathcal{F},$$

it would follow from the upper-semi continuity of $\mathcal{F}$ that $\mathcal{F}(\psi_\infty) = \sup \mathcal{F}$, hence ψ_∞ is the only maximizer of $\mathcal{F}$, the Kähler-Einstein potential normalized by $\int_X \psi_\infty \, \omega^n = 0$. Thus the whole family $(\psi_t)_{t>0}$ actually converges towards ψ_∞ (see Corollary 6.2). Note that this convergence is easy when (ψ_t) is known to be relatively compact in $\mathcal{C}^\infty$. The delicate point here is that we only have weak compactness.

It thus remains to check that $\mathcal{F}(\varphi_t) \nearrow \sup_{\mathcal{E}^1(X,\omega)} \mathcal{F}$. By (P), we can find $r_j \to +\infty$ such that

$$MA(\varphi_{r_j}) - \mu_{r_j} \longrightarrow 0,$$

since $\mathcal{F}$ is bounded from above. By compactness we can further assume that $\psi_{r_j} \to \psi_\infty$ in $L^1(X)$, almost everywhere, and *in energy* (see below), so that

$$MA(\psi_\infty) = \mu(\psi_\infty).$$

Thus $\omega + dd^c\psi_\infty$ is a Kähler-Einstein current. It follows again from the variational characterization that it maximizes $\mathcal{F}$, hence the upper-bound along the flow is actually the absolute upper-bound and we are done.

6.3. **Convergence in energy.** As explained above, the last step to be justified is that $\mathcal{F}(\varphi_t)$ increases towards the absolute maximum of $\mathcal{F}$ when ω_t evolves along the normalized Kähler-Ricci flow, without assuming high order a priori estimates.

We already know that the normalized potentials $\omega + dd^c\psi_t = \omega_t$, $\int_X \psi_t \, \omega^n = 0$, have uniformly bounded energies hence form a relatively compact family. Using (P) we have selected a special subsequence $\psi_{t_j} \to \psi_\infty$ (convergence in L^1 and almost everywhere) such that

$$MA(\psi_{t_j}) \longrightarrow \mu(\psi_\infty) = \frac{e^{-\psi_\infty} \mu}{\int_X e^{-\psi_\infty} \, d\mu}, \text{ where } \mu = e^{-h_0} \omega^n / V$$

We would be done if we could justify that $MA(\psi_{t_j}) \to MA(\psi_\infty)$.

The delicate problem is that the complex Monge-Ampère operator is not continuous for the L^1-topology. A slightly stronger notion of convergence (convergence in energy) is necessary. We refer the reader to [BBEGZ11] for its precise definition, suffices to say here that it is equivalent to checking that

$$\int_X |\psi_{t_j} - \psi_\infty| \, MA(\psi_{t_j}) \longrightarrow 0.$$

Set

$$f_t := e^{\dot{\varphi}_t} \frac{e^{-\varphi_t}}{\int_X e^{-\varphi_t}\, d\mu} \quad \text{so that } MA(\psi_t) = f_t\, \mu.$$

If the densities f_t were uniformly in L^p for some $p > 1$, we could conclude by using Hölder inequality, since

$$\int_X \left|\psi_{t_j} - \psi_\infty\right| MA(\psi_{t_j}) \leq \|f_{t_j}\|_{L^p(\mu)} \cdot \|\psi_{t_j} - \psi_\infty\|_{L^q(\mu)}.$$

We cannot prove such a strong uniform bound in general, however our next lemma provides us with a weaker bound that turns out to be sufficient:

Lemma 6.3. *Set $\mu := e^{-h_0}\omega^n/V$. Then*

$$\mathcal{M}(\psi_t) = E(\psi_t) - H_\mu(MA(\psi_t)) - \int_X \psi_t MA(\psi_t) + \int_X h_0 \frac{\omega^n}{V}.$$

Therefore there exists $C > 0$ such that for all $t > 0$,

$$0 \leq \int_X f_t \log f_t\, d\mu \leq C.$$

Proof. Recall that $\psi_t = \varphi_t - \int_X \varphi_t \omega^n/V$. It follows from Lemma 2.4 that

$$\begin{aligned} \mathcal{M}(\varphi_t) &= \mathcal{F}(\varphi_t) - \int_X \dot{\varphi}_t MA(\varphi_t) + \int_X h_0 \omega^n/V \\ &= E(\varphi_t) + \beta(t) - \int_X \dot{\varphi}_t MA(\varphi_t) + \int_X h_0 \omega^n/V, \end{aligned}$$

where $\beta(t) = \log\left[\int_X e^{-\varphi_t} d\mu\right]$, while

$$H_\mu(MA(\psi_t)) = \int_X \log\left(\frac{MA(\varphi_t)}{\mu}\right) MA(\varphi_t) = \int_X \dot{\varphi}_t MA(\varphi_t) - \int_X \varphi_t MA(\varphi_t) - \beta(t).$$

The equality follows.

Recal now that the Mabuchi functional $\mathcal{M}$ is bounded along the flow, as well as the energies $E(\psi_t)$. Since the latter are uniformly comparable to $\int_X \psi_t MA(\psi_t)$, we infer that the entropies $H_\mu(MA(\psi_t))$ are uniformly bounded, i.e.

$$0 \leq H_\mu(MA(\psi_t)) = \int_X f_t \log f_t\, d\mu \leq C.$$

$\square$

We can thus use the Hölder-Young inequality to deduce that

$$\int_X \left|\psi_{t_j} - \psi_\infty\right| f_{t_j}\, d\mu \leq C' \left\|\psi_{t_j} - \psi_\infty\right\|_{L^\chi(\mu)},$$

where $\chi : t \in \mathbb{R}^+ \mapsto e^t - t - 1 \in \mathbb{R}^+$ denotes the convex weight conjugate to the weight $t \in \mathbb{R}^+ \mapsto (t+1)\log(t+1) - t \in \mathbb{R}^+$ naturally associated to the entropy, and $\|\cdot\|_{L^\chi(\mu)}$ denotes the Luxembourg norm on $L^\chi(\mu)$,

$$\|g\|_{L^\chi(\mu)} := \inf\left\{\alpha > 0 \,|\, \int_X \chi\left(\alpha^{-1}|g|\right) d\mu \leq 1\right\}.$$

There remains to check that $\left\|\psi_{t_j} - \psi_\infty\right\|_{L^\chi(\mu)} \to 0$. By definition, this amounts to verifying that for all $\alpha > 0$,

$$\int_X \chi\left(\alpha^{-1}|\psi_{t_j} - \psi_\infty|\right) d\mu \longrightarrow 0.$$

Since $\chi(t) \leq te^t$ and the functions (ψ_{t_j}) have uniformly bounded energies, the latter convergence follows from Hölder's inequality and Skoda's uniform integrability theorem.

References

[Aub78] T. Aubin: Equation de type Monge-Ampère sur les variétés kählériennes compactes. Bull. Sci. Math. **102** (1978), 63–95.

[BM87] S. Bando, T. Mabuchi: Uniqueness of Einstein Kähler metrics modulo connected group actions, in *Algebraic geometry, Sendai, 1985* (T. Oda, Ed.), Adv. Stud. Pure Math. **10**, Kinokuniya, 1987, 11-40.

[BT82] E. Bedford, B. A. Taylor: A new capacity for plurisubharmonic functions. Acta Math. **149** (1982), no. 1-2, 1–40.

[BerBer11] R. Berman, B. Berndtsson: Moser-Trudinger inequalities. Preprint (2011) arxiv:1109.1263.

[BBGZ09] R. Berman, S. Boucksom, V. Guedj, A. Zeriahi: A variational approach to complex Monge-Ampère equations. Preprint (2009) arXiv:0907.4490.

[BBEGZ11] R. Berman, S. Boucksom, P.Eyssidieux, V. Guedj, A. Zeriahi: Kähler-Ricci flow and Ricci iteration on log-Fano varieties. Preprint arXiv (2011).

[Bern11] B. Berndtsson: A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem. Preprint (2011) arXiv:1103.0923.

[BL11] J.Blanc, S.Lamy: Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links. Preprint arXiv (2011).

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi: Monge-Ampère equations in big cohomology classes. Acta Math. **205** (2010), 199–262.

[Cao85] H.D.Cao: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. Invent. Math. **81** (1985), no. 2, 359-372.

[Cao] H.D.Cao: The Kähler-Ricci flow on Fano manifolds: An introduction (this volume).

[CLW09] X. X. Chen, H.Li, B.Wang: Kähler-Ricci flow with small initial energy. Geom. Funct. Anal. **18** (2009), no. 5, 15251563.

[CT02] X. X. Chen, G.Tian: Ricci flow on Kähler-Einstein surfaces. Invent. Math. **147** (2002), no. 3, 487-544.

[Chow91] B.Chow: The Ricci flow on the 2-sphere. J. Differential Geom. **33** (1991), no. 2, 325-334.

[DT92] W.-Y. Ding, G. Tian: Kähler-Einstein metrics and the generalized Futaki invariant. Invent. Math. **110** (1992), no. 2, 315-335

[Don08] S. K. Donaldson: Kähler geometry on toric manifolds, and some other manifolds with large symmetry. Handbook of geometric analysis. No. 1, 29-75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

[EGZ09] P. Eyssidieux, V. Guedj, A. Zeriahi: Singular Kähler-Einstein metrics. J. Amer. Math. Soc. **22** (2009), 607-639.

[Gill11] M.Gill: Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds. Comm. Anal. Geom. **19** (2011), no. 2, 277-303.

[GZ05] V. Guedj, A. Zeriahi: Intrinsic capacities on compact Kähler manifolds. J. Geom. Anal. **15** (2005), no. 4, 607-639.

[GZ07] V. Guedj, A. Zeriahi: The weighted Monge-Ampère energy of quasiplurisubharmonic functions. J. Funct. An. **250** (2007), 442-482.

[Ham86] R.Hamilton: The Ricci flow on surfaces. Mathematics and general relativity (Santa Cruz, CA, 1986), 237262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[Imbert] C.Imbert: Introduction to fully non-linear parabolic equations. Lecture Notes (this volume).

[Koł98] S. Kołodziej: The complex Monge-Ampère equation. Acta Math. **180** (1998), no. 1, 69–117.

[Li08] H.Li: On the lower bound of the K-energy and F-functional. Osaka J. Math. **45** (2008), no. 1, 253-264.

[Pau08] M. Păun: Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds. Chin. Ann. Math. Ser. B 29 (2008), no. 6, 623-630.

[PSS07] D. H. Phong, N.Sesum, J. Sturm: Multiplier ideal sheaves and the Kähler-Ricci flow, Comm. Anal. Geom. **15** (2007), no. 3, 613-632.

[PSSW08] D. H. Phong, J. Song, J. Sturm, B. Weinkove: The Moser-Trudinger inequality on Kähler-Einstein manifolds. Amer. J. Math. **130** (2008), no. 4, 1067–1085.

[PS06] D. H. Phong, J. Sturm: On stability and the convergence of the Kähler-Ricci flow. J. Differential Geom. **72** (2006), no. 1, 149-168.

[PS10] D. H. Phong, J. Sturm: Lectures on stability and constant scalar curvature. Handbook of geometric analysis, No. 3, 357436, Adv. Lect. Math. (ALM), 14, Int. Press, Somerville, MA (2010).

[SeT08] N.Sesum, G.Tian: Bounding scalar curvature and diameter along the Kähler-Ricci flow (after Perelman). J. Inst. Math. Jussieu **7** (2008), no. 3, 575-587.

[ShW11] M.Sherman, B.Weinkove: Interior derivative estimates for the Kähler-Ricci flow. Preprint arXiv (2011).

[Siu87] Y. T. Siu: Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics. DMV Seminar, 8. Birkhäuser Verlag, Basel, 1987.

[Sko72] H. Skoda: Sous-ensembles analytiques d'ordre fini ou infini dans $\mathbb{C}^n$. Bull. Soc. Math. France **100** (1972), 353–408.

[ST08] J. Song, G. Tian: Canonical measures and Kähler-Ricci flow. Preprint (2008) arXiv:0802.2570.

[ST09] J. Song, G. Tian: The Kähler-Ricci flow through singularities. Preprint (2009) arXiv:0909.4898.

[SW] J.Song, B.Weinkove: Lecture notes on the Kähler-Ricci flow (this volume).

[Tian90] G.Tian: On Calabi's conjecture for complex surfaces with positive first Chern class. Invent. Math. **101** (1990), no. 1, 101-172.

[Tian97] G. Tian: Kähler-Einstein metrics with positive scalar curvature. Inv. Math. **130** (1997), 239–265.

[Tian] G. Tian: Canonical metrics in Kähler geometry. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2000).

[TZ07] G. Tian, X. Zhu: Convergence of Kähler-Ricci flow. J. Amer. Math. Soc. **20** (2007), no. 3, 675–699.

[Tos12] V.Tosatti: Kähler-Einstein metrics on Fano surfaces Expo. Math. **30** (2012), no.1, 11-31.

[Tru84] N. S. Trudinger: Regularity of solutions of fully nonlinear elliptic equations. Boll. Un. Mat. Ital. A (6) 3 (1984), no. 3, 421–430.

[Villani] C.Villani: Optimal transport. Old and new. Grundlehren der Mathematischen Wissenschaften, **338**. Springer-Verlag, Berlin, (2009). xxii+973 pp.

[WZ04] X.J.Wang, X.Zhu: Kähler-Ricci solitons on toric manifolds with positive first Chern class. Adv. Math. **188** (2004), no. 1, 87-103.

[Yau78] S. T. Yau: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Zer01] A. Zeriahi: Volume and capacity of sublevel sets of a Lelong class of psh functions. Indiana Univ. Math. J. **50** (2001), no. 1, 671–703.

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