HOMOGENEOUS MONGE-AMPÈRE EQUATIONS AND FOLIATIONS BY HOLOMORPHIC DISKS

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Introduction

The goal of this lecture is to study the geometric properties of *maximal plurisubharmonic functions*. A plurisubharmonic function u defined in a domain $\Omega \subset \mathbf{C}^n$ is maximal if for all plurisubharmonic function v defined in $U \subset\subset \Omega$, the following holds:

$$v \leq u \text{ on } \partial U \Longrightarrow v \leq u \text{ in } U.$$

In dimension $n = 1$, a maximal psh function u is harmonic. In higher dimension $n \geq 2$, maximality is characterized by the non linear PDE $(dd^c u)^n = 0$, as explained in Zeriahi's lecture [?].

We seek here for a geometric understanding of the maximality property of a given psh function u. If one can fill out a small neighborhood of any point $x \in \Omega$ by holomorphic disks along which u is harmonic, then u is certainly maximal. When u is regular enough, this situation actually always happen, and follows from the Frobenius Integrability Theorem as we explain in *section 1*. In lower regularity, this is not anymore the case as shown by a striking construction due to the first author (see *section 4*). It seems that the critical regularity should be $\mathcal{C}^{1,1}$, as we will try to explain. In order to simplify the exposition, we restrict ourselves throughout this note to the two dimensional situation $n = 2$.

Nota Bene. These notes are written by Romain Dujardin and Vincent Guedj and grew up from a lecture delivered by Romain Dujardin in Marseille in march 2009. Most results are standard, except for the last section which briefly explains a recent result of the first author [?]. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

1. Monge-Ampère foliations

1.1. **Preliminaries on currents.** For this paragraph the reader is referred to Demailly's book [?]. Recall that a *current* of bidimension (p, q) in a domain $\Omega \subset \mathbf{C}^n$ is a continuous linear form on the space of smooth differential forms with compact support (test forms) of bidegree (p, q). It can be canonically identified with a differential form of bidegree $(n-p, n-q)$ with distribution coefficients. Since we are primarily interested in this case, *from now on we make the assumption that* $p = q = 1$ *and* $n = 2$, that is, we work with currents of bidegree $(1, 1)$ in a domain of $\mathbf{C}^2$.

Such a current T is *closed* if $\langle T, d\eta\rangle = 0$ for every test form of degree 1. The current T is *positive* if $\langle T, \theta\rangle \geq 0$ for every positive test form of bidegree $(1, 1)$. A test form is positive if it belongs to the closed convex set generated by forms of the type $\chi\omega$ where χ is a positive test function and ω is a Kähler form. Thus, by continuity, checking the positivity of a current T amounts to verify that $\langle T, \chi\omega\rangle \geq 0$, for all $\chi \geq 0$ and $\omega > 0$. A smooth current

$T = \sum T_{pq} i dz_p \wedge d\overline{z}_q$ is positive if and only if (T_{pq}) is a non negative hermitian matrix at every point.

Any positive closed current of bidegree $(1,1)$ is locally given as $T = dd^c u$, where u is a plurisubharmonic function, $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$ and the derivatives are taken in a weak sense.

Here are two fundamental examples of such currents:

Example 1.1. If V is a (closed) complex curve in a domain $\Omega \subset \mathbf{C}^2$, let us denote by $[V]$ the current of integration along V, defined by

$$\langle [V], \theta \rangle := \int_V \theta.$$

It is immediate to check that this is a well defined positive closed $(1,1)$-current in Ω.

A classical result of Lelong asserts that when V is merely an analytic subset of (complex) dimension one, it is still possible to consider $[V]$ by integrating along the regular points Reg(V) of V. This requires to show that the current of integration along Reg(V) has locally finite mass near Sing(V) and that the resulting current (extended by zero through Sing(V)) is still closed.

An important result of Siu asserts that these currents are extremal points of the convex cone of all positive closed $(1,1)$-currents. More precisely, any positive closed $(1,1)$-current supported on an irreducible complex curve V is a (positive) multiple of $[V]$.

Example 1.2. Let ω be a Kähler form in Ω, e.g. $\omega = dd^c \rho$ where ρ is a smooth strictly plurisubharmonic function in Ω. Then ω defines a positive closed current of bidegree $(1,1)$ by setting

$$\langle \omega, \theta \rangle := \int_\Omega \theta \wedge \omega.$$

Both families are dense in the cone of positive closed currents of bidegree $(1,1)$: one can regularize psh functions (using standard convolutions), add $\varepsilon ||z||^2$ and hence approximate any positive closed current by Kähler forms. Likewise, any plurisubharmonic function u is the limit in L^1_{loc} of rational multiples of $\log |f_j|$, f_j holomorphic functions, as follows for instance from Hörmander's L^2-estimates. Thus, the current $T = dd^c u$ is the weak limit of (rational multiples of) the currents of integration along the analytic sets $\{f_j = 0\}$.

One important aspect of $(1,1)$-positive closed currents is that it is often possible to wedge them. Our primary interest here is on *self-intersections.* If u is a psh function s.t. $\nabla u \in L^2_{\rm loc}$, then $(dd^c u)^2$ is a well-defined positive measure. Indeed $du \wedge d^c u$ is well defined by assumption, hence so is $u dd^c u$ (integrate by parts). One thus defines $(dd^c u)^2 := dd^c(u\, dd^c u)$. Blocki [**?**] has shown that the Monge-Ampère measure $(dd^c u)^2$ cannot be reasonably defined when $\nabla u \notin L^2_{loc}$.

Abusing terminology, we say that a $(1,1)$-positive closed current $T = dd^c u$ in $\mathbf{C}^2$, with $\nabla u \in L^2_{\rm loc}$, is *maximal* when $T \wedge T = (dd^c u)^2 = 0$. Of course this notion does not depend on the choice of the potential u.

1.2. **Foliated cycles.** Recall that a $\mathcal{C}^k$ *foliation* of a domain $\Omega \subset \mathbf{C}^2$ by complex leaves is given by a covering (Ω_α) by *coherent foliated charts* (or *flow boxes*), that is, each Ω_α is provided with a $\mathcal{C}^k$-diffeomorphism

$$\phi_\alpha : \mathbf{D}_z \times \mathbf{D}_w \longrightarrow \Omega_\alpha$$

which is holomorphic in the z coordinate. Coherence here means that the transition maps between charts preserve the "plaques" $\{w = C^{st}\}$. By definition, a connected immersed submanifold L is a *leaf* if for each α, $L \cap \Omega_\alpha$ is a union of plaques. A *transversal* to the foliation is a piece of submanifold which is transverse to the leaves.

All the problems we consider in these notes are local, so most often we restrict to a single foliated chart. We denote by $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha \in \mathbf{D}}$ the corresponding family of leaves. Abusing slightly, the notation $\mathcal{L}$ will also denote the foliation itself.

In this section we assume that $k \geq 1$ (while in the next section we focus on the case $k = 0$), in which case an alternative definition is that $\mathcal{L}$ is defined as the integral curves of the kernel of the differential form $\phi_*(dw \wedge d\bar{w})$.

Given μ a probability measure on $\mathbf{D}_w$, let us consider the current

$$T_\mu := \int [\mathcal{L}_\alpha] d\mu(\alpha).$$

This is a geometric current of bidegree $(1,1)$ which is *positive* and *closed*: it acts on a smooth form θ of bidegree $(1,1)$ by integrating along each leaf and averaging against μ,

$$\langle T, \theta \rangle = \int \left(\int_{\mathcal{L}_\alpha} \theta \right) d\mu(\alpha),$$

the result being nonnegative if θ is positive and zero if θ is exact.

We want to define a *foliated cycle* on a foliation as a positive closed current which is "locally of the above form". Passing from local to global here requires a little bit of care. An *invariant transverse measure* is the data of a locally finite measure on each transversal, which is invariant under holonomy (that is, transport along the leaves). We see that in a given coordinate chart $\mathbf{D}_z \times \mathbf{D}_w$, it is determined by its value μ on $\mathbf{D}_w$, and gives rise to a well-defined current T_μ as above.

Definition 1.3. *A positive closed current of bidegree $(1,1)$ in a domain $\Omega \subset \mathbf{C}^2$ is called a foliated cycle if there exists a foliation of Ω by complex curves and an invariant transverse measure whose T is the associated current.*

This concept was introduced by Sullivan [**?**] and has proved to be of fundamental importance in the theory of foliations.

Observe that foliated cycles interpolate between smooth differential forms and currents of integration: if $\mathcal{L}$ is the foliation by horizontal lines $\{w = cst\}$ and μ is the Dirac mass at the origin, then $T_\mu = [w = 0]$ is the current of integration along the complex line $(w = 0)$, while if μ is the Lebesgue measure then $T_\mu = cidw \wedge d\overline{w}$ is smooth.

It is an easy fact that foliated cycles are maximal currents:

Proposition 1.4. *If T is a $\mathcal{C}^1$-smooth foliated cycle, then $T \wedge T = 0$.*

Proof. Working in a foliated chart, we can assume that $\mathcal{L}_\alpha = \{w = \alpha\}$. The current of integration along $\mathcal{L}_\alpha$ can be formally written as $[\mathcal{L}_\alpha] = \delta_\alpha idw \wedge d\overline{w}$ so that $T_\mu = \chi idw \wedge d\overline{w}$ for some $\mathcal{C}^1$-smooth function χ. Elementary calculus on differential forms then shows that $T \wedge T = 0$. □

It turns out that the converse is also true, as follows from the Frobenius Integrability Theorem.

Theorem 1.5. *Let T be a $\mathcal{C}^1$-smooth positive closed differential form of bidegree $(1,1)$, satisfying the equation $T \wedge T = 0$. Then there exists a foliation by complex curves on the interior of* $\operatorname{supp}(T)$ *and T is a foliation cycle associated to this foliation.*

The foliation by complex curves induced by T in the interior of its support is called a *Monge-Ampère foliation*. The reader is referred to [?] for a thorough discussion on the connections between Monge-Ampère equations and foliations.

Proof. Since T is a positive differential form of bidegree $(1,1)$, it can be decomposed as

$$T = \sum_{p,q=1}^{2} T_{pq} \frac{i}{2} dz_p \wedge d\overline{z}_q,$$

where (T_{pq}) is a nonnegative hermitian matrix at every point. Now

$$T \wedge T = c \det(T_{pq})(idz_1 \wedge d\overline{z}_1) \wedge (idz_2 \wedge d\overline{z}_2) = 0$$

hence (T_{pq}) has complex rank ≤ 1 in general and exactly 1 at interior points of the support of T. This shows that $\ker T$ defines a $\mathcal{C}^1$ distribution of complex lines on the interior of $\operatorname{supp}(T)$. Notice that, having continuous coefficients, T gives zero mass to the boundary of its support.

Let Ω be an open subset of the support of T. We would like to show that the complex lines defined by $\ker T$ are tangent to a $\mathcal{C}^1$-foliation in Ω. By the Frobenius theorem, we need to check that the distribution $\ker T$ is *involutive*, i.e. for every pair of smooth vector fields X, Y belonging to $\ker T$, then $[X,Y] \in \ker T$. Let Z be any vector field. By standard calculus on differential forms (see e.g. [?, p.44]), and using the fact that T is closed we get that

$$\begin{aligned} 0 &= dT(X,Y,Z) \\ &= X(T(Y,Z)) - Y(T(X,Z)) + Z(T(X,Y)) - T([X,Y],Z) - T(Y,[X,Z]) + T(X,[Y,Z]), \end{aligned}$$

whence $T([X,Y],Z) = 0$, so $[X,Y] \in \ker T$, which is the desired result.

The leaves are then automatically complex curves because their tangent space is complex at every point. Let us denote these curves by $\mathcal{L}_\alpha$.

Let us now show that T is an average of currents of integration along the complex curves $\mathcal{L}_\alpha$. We provide a proof which is not the most simple, but carries over to a wider context (see e.g. [?]).

Working in a single flow box again, the space of leaves is compact for the topology of currents. By the Choquet Integral Representation Theorem, it is enough to prove that T belongs to the closed convex cone generated by the $\mathcal{L}_\alpha$'s.

Assume this is not the case. It then follows from the Hahn-Banach theorem that there exists a test form θ such that $\langle T, \theta \rangle > 0$ while $\langle \mathcal{L}_\alpha, \theta \rangle \leq -1$ for all α.

Fix a transversal τ to the foliation and let $\{\chi_i\}$ be a partition of unity subordinate to an open covering of τ by open sets of diameter $\leq 1/2$. We extend the functions χ_i to Ω by making them constant along each leaf. Since $T = \sum \chi_i T$, there exists i_0 such that $\langle \chi_{i_0} T, \theta \rangle > 0$. Set $T_1 := \chi_{i_0} T / ||\chi_{i_0} T||$.

Observe that if χ is any function which is constant along the leaves, then χT is closed. Indeed there are C^1 real coordinates[1] (x_1, x_2, x_3, x_4) in which the leaves are defined by the equations $\{x_3 = a_3,\ x_4 = a_4\}$, and T is a multiple of $dx_3 \wedge dx_4$. It is then clear that if χ depends only on (x_3, x_4), $d\chi \wedge T = 0$. In particular T_1 is closed.

[1]Caution is in order here because that positivity makes no sense in these coordinates.

Now we repeat the above procedure, that is we consider a covering of supp (χ_{i_0}) by open sets of diameter $\leq 1/4$ and an associated partition of unity, and build a current $T_2 = \chi_{1,i_1}T_1/||\chi_{1,i_1}T_1||$ of mass 1 such that

$$\langle T_2, \theta\rangle > 0 \text{ and } \langle \mathcal{L}_\alpha, \theta\rangle < 0 \text{ for all } \alpha.$$

We thus inductively obtain a sequence of positive closed currents T_n of bidegree $(1,1)$ and mass 1 whose support is contained in an arbitrarily small neighborhood of a leaf $\mathcal{L}_{\alpha_n}$, and satisfying $\langle T_n, \theta\rangle > 0$. Any cluster point σ of (T_n) is supported on a leaf $\mathcal{L}_{\alpha_0}$ hence coincides with the current of integration along $\mathcal{L}_{\alpha_0}$. By definition of θ, $\langle \sigma, \theta\rangle \leq -1$, which is a contradiction. □

1.3. **Geometric maximality.** Our main interest in these series of lectures is the understanding of various Dirichlet problems for complex Monge-Ampère operators. Recall from [**?**] that a continuous psh function u is maximal in a domain $\Omega \subset \mathbf{C}^2$ if and only if it satisfies $(dd^c u)^2 = 0$. In dimension 1, maximal functions are harmonic hence very regular. This is not necessarily the case in higher dimension, however the expectation is that they should be somehow harmonic along a foliation by complex curves. We can reformulate Theorem **??** in this spirit:

Theorem 1.6. *Let u be a maximal plurisubharmonic function in some domain $\Omega \subset \mathbf{C}^2$. If $u \in \mathcal{C}^3(\Omega)$ then $T = dd^c u$ is a foliated cycle in the interior of* Supp T *and u is harmonic along the leaves of the Monge-Ampère foliation.*

A natural question is therefore to understand what happens when u is less regular. This will be the subject of the remaining sections. Notice that this type of question was already raised at the very beginnings of pluripotential theory (see e.g. [**?**]).

2. Laminations and laminar currents

It is natural to expect that a result similar to Theorem **??** should hold under weaker regularity assumptions on u. This leads to the concepts of uniformly laminar and laminar currents. Although it is not clear how to extend the Frobenius Theorem to weaker regularity, it is still interesting to explore the properties of these objects, which have recently played an important role in the context of complex dynamics.

2.1. **Laminations and holomorphic motions.** An *embedded lamination* by complex curves is given by a covering of a closed subset $X \subset \Omega$ by a coherent system of flow boxes

$$\phi_\alpha : \mathbf{D}_z \times K \longrightarrow \Omega_\alpha$$

which are holomorphic along the leaves (i.e. in the z coordinate) and only continuous in the transverse direction (the w coordinate). Here K denotes a compact subset of $\mathbf{D}_w$.

It follows from a celebrated result of Mane-Sad-Sullivan [**?**] that the assumption that K is closed is superfluous, and that ϕ_α is automatically Hölder continuous.

Proposition 2.1. *Fix $K \subset \mathbf{D}$ an arbitrary subset and let $\{\mathcal{L}_\alpha\}_{\alpha\in K}$ be a bounded family of disjoint graphs over K, $\mathcal{L}_\alpha := \{(z,w) \in \mathbf{D}^2 \,/\, w = f_\alpha(z)\}$, where f_α is a holomorphic function in $\mathbf{D}$ with $f_\alpha(0) = \alpha$. Then*

1) $\{\mathcal{L}_\alpha\}_{\alpha\in K}$ extends uniquely to a family of disjoint graphs parameterized by $\overline{K}$;

2) the collection $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha\in\overline{K}}$ form a lamination, in the sense that the holonomy map $\{0\} \times \mathbf{D} \to \{z\} \times \mathbf{D}$ is automatically continuous.

Laminations by graphs over the unit disk are often called *holomorphic motions*

Proof. Assume without loss of generality that $|f_\alpha| < 1$ Then for $\alpha \neq \beta$, the function $h = h_{\alpha,\beta} := -\log(|f_\alpha(z) - f_\beta(z)|/2)$ is harmonic and positive in the unit disk $\mathbf{D}$. It therefore follows from the Harnack inequality that

$$\frac{1-|z|}{1+|z|} h(0) \leq h(z), \text{ for all } z \in \mathbf{D}.$$

Since $f_\alpha(0) = \alpha$, we infer that

$$|f_\alpha(z) - f_\beta(z)| \leq 2\left(\frac{|\alpha-\beta|}{2}\right)^{\frac{1-|z|}{1+|z|}}. \tag{1}$$

This shows that the holonomy map $\{0\}\times\mathbf{D} \to \{z\}\times\mathbf{D}$ is locally uniformly Hölder continuous.

Since the family (f_α) is bounded, it thus uniquely extends to $\overline{K}$. For $\alpha \in \overline{K}$ we simply set $f_\alpha = \lim f_{\alpha_n}$, where α_n is any sequence converging to α (by the previous Hölder estimate, the limiting map f_α is independent of the choice both of the cluster point of f_{α_n} and of the sequence (α_n)). Note finally that by Rouché's Theorem, the extended family is still a family of disjoint graphs. □

Remark 2.2. Concluding of long series of previous works, Słodkowski [?] proved that any holomorphic motion of a compact subset $K \subset \mathbf{C}$ can be extended to a holomorphic motion of $\mathbf{C}$. By [?] the holonomy is then quasiconformal. Conversely any such quasiconformal map can be realized as the holonomy map (at time $t = 1$) of a lamination by disjoint graphs. To find such a holomorphic motion, one simply takes the Beltrami coefficient of this quasiconformal map and multiplies it by the complex parameter t. The interested reader will find more information on the topic in [?].

2.2. **Uniformly laminar currents.** Uniformly laminar current are the natural generalization of foliated cycles to laminations:

Definition 2.3. *A positive closed $(1,1)$-current T in a domain $\Omega \subset \mathbf{C}^2$ is called uniformly laminar if there exists an embedded lamination $\mathcal{L}$ in Ω such that in any flow box $\{\mathcal{L}_\alpha\}_{\alpha\in K}$, T has the form*

$$T = \int_{\alpha\in K} [\mathcal{L}_\alpha]\, d\mu(\alpha).$$

for some probability measure μ on K.

We say that T is *subordinate* to the lamination $\mathcal{L}$. Globally speaking, a uniformly laminar current induces an invariant transverse measure on the lamination to which it is subordinate. The following result of Demailly [?] provides an interesting class of uniformly laminar currents.

Proposition 2.4. *Let T be a positive closed current supported on a C^1 Levi-flat hypersurface in $\Omega \subset \mathbf{C}^2$. Then T is uniformly laminar.*

Recall that in $\mathbf{C}^2$ a Levi flat hypersurface is a real hypersurface which can be defined as $\Re(z) = 0$ in some holomorphic system of coordinates (z, w). It is naturally foliated by the curves $\{z = iy, y \in \mathbf{R}\}$. If a Levi-flat hypersurface carries a positive closed current, then its underlying foliation posseses an invariant transverse measure.

Proof. We work locally and assume that the hypersurface has the form $\Re(z) = 0$. Write $z = x + iy$. Then since $x = 0$ on $\operatorname{supp}(T)$ and T has measure coefficients, we infer that $xT = 0$. Since T is ∂ and $\overline{\partial}$ closed, we infer that $\partial(xT) = dz \wedge T = 0$ and $\overline{\partial}(xT) = d\overline{z} \wedge T = 0$.

Now if χ is a test function which is constant along the leaves, that is, which depends only on z along supp (T), we infer that χT is closed. We conclude as in Theorem **??**. □

We now prove that the potentials of uniformly laminar currents are maximal.

Exercise 2.5. Let $\mathcal{L} = \{\mathcal{L}_\alpha\}_{\alpha\in K} \subset \Omega$ be a lamination by disjoint graphs, $\mathcal{L}_\alpha = \{(z,w) \in \mathbf{D}\times\mathbf{C} \,/\, w = f_\alpha(z)\}$. Fix a probability measure μ on $K \subset \mathbf{C}$ and consider $T = T_\mu$ the corresponding uniformly laminar current. Check that $T = dd^c u$, where

$$u(z,w) = \int_{\alpha\in K} \log|w - f_\alpha(z)| d\mu(\alpha).$$

Proposition 2.6. *If $T = dd^c u$ is a uniformly laminar current such that $\nabla u \in L^2_{loc}$ then*

$$T\wedge T = (dd^c u)^2 = 0.$$

Proof. We work in a flow box. With notation as in **??**, fix $\beta \in K$. Note that μ has no atom since $\nabla u \in L^2_{loc}$ thus

$$u(z,w) = \int_{\alpha\in K\setminus\{\beta\}} \log|w - f_\alpha(z)| d\mu(\alpha)$$

is harmonic on $\mathcal{L}_\beta$ as an average of harmonic functions, or identically $-\infty$. Since u is locally integrable with respect to $dd^c u$, $u_{|\mathcal{L}_\beta}$ is harmonic on μ-almost every leaf. We infer that $T\wedge[\mathcal{L}_\beta] = 0$, whence

$$T\wedge T = \int_{\beta\in K} T\wedge[\mathcal{L}_\beta]\, d\mu(\beta) = 0.$$

□

Laminar currents which are made up of graphs over the unit disk have the following important compactness property (this precise version is taken from [**?**]).

Proposition 2.7. *Let T_n be a sequence of uniformly laminar currents in $\mathbf{D}^2$, respectively subordinate to a sequence of laminations $\mathcal{L}_n$ by graphs over the unit disk. Assume that T_n converges to T. Then $(\mathcal{L}_n)$ converges to a limit lamination $\mathcal{L}$ and T is subordinate to $\mathcal{L}$.*

What we mean by convergence for the sequence of laminations $\mathcal{L}$ is the following. Fix the transversal $\{0\}\times\mathbf{D}$, and denote by $\mathcal{L}_{n,\alpha}$ the leaf of $\mathcal{L}_n$ through $(0,\alpha)$. Let $K_n = \mathcal{L}_n \cap (\{0\}\times\mathbf{D})$. We say that $\mathcal{L}_n$ converges to $\mathcal{L}$ if for every $\alpha \in \limsup K_n$ there exists a unique graph $\mathcal{L}_\alpha$ through $(0,\alpha)$ such that if $\alpha_n \in K_n$ is any sequence converging to α, $\mathcal{L}_{n,\alpha}$ converges to $\mathcal{L}_\alpha$.

Proof. Of course the family of graphs with norm at most 1 over the unit disk is compact for the compact-open topology. Write

$$T_n = \int [\mathcal{L}_{n,\alpha}] d\mu_n(\alpha),$$

with μ_n a positive measure supported in $\{0\}\times\mathbf{D}$. Likewise, denote by $\mu_n(z)$ the measure induced by holonomy on $\{z\}\times\mathbf{D}$. It can also be expressed as $\mu_n = T_n \wedge [\{z\}\times\mathbf{D}]$. Since the currents T_n have locally uniformly bounded mass, so do the μ_n. Restricting μ_n to $\{0\}\times D(0,1-\varepsilon)$ if necessary, we can always assume that its mass is uniformly bounded, say by 1.

Since $T_n \to T$ it is a basic consequence of Slicing Theory that for Lebesgue a.e. z, $\mu_n(z)$ converges to some $\mu(z)$. Now for every test function $\varphi(w)$, by the Hölder continuity property (**??**) the family $\int \varphi d\mu_n(z)$ is equicontinuous in z so we get that $\mu_n(z)$ converges for all z, in particular at $z = 0$.

It remains to prove that the laminations $\mathcal{L}_n$ converge in the previous sense to some lamination $\mathcal{L}$. It will then be clear that the T_n converge to $T = \int [\mathcal{L}_\alpha] d\mu(\alpha)$

Let $\mathcal{L}$ be the set of graphs over $\mathbf{D}$ which are cluster values of $\mathcal{L}_{n_j,\alpha_j}$ for some subsequence (n_j). We need to show that the graphs of $\mathcal{L}$ do not intersect. Then by Proposition **??**, they will form a lamination. Notice that this is more subtle than just applying the Hurwitz Theorem because the leaves of different $\mathcal{L}_n$ can intersect. We use the convergence of currents instead.

It suffices to show the following fact : "if a sequence $\mathcal{L}_{n_j,\alpha_j}$ satisfies $\alpha_j \to \alpha \in \operatorname{supp}(\mu)$ then the sequence converges". Suppose this is not the case: then there exists two subsequences $\mathcal{L}_{n_j^i,\alpha_j^i} \to \mathcal{L}_\alpha^i$, $i = 1, 2$, and $\mathcal{L}_\alpha^1 \neq \mathcal{L}_\alpha^2$. If α is not an atom of μ, we can assume that $\mathcal{L}_\alpha^1$ and $\mathcal{L}_\alpha^2$ are transverse at $(0,\alpha)$: if not we can move the $\mathcal{L}_{n_j^1,\alpha_j^1}$ slightly so that they will converge to a $\mathcal{L}_\beta^1$ close to $\mathcal{L}_\alpha^1$ and disjoint from it, by the Hurwitz Theorem. Then $\mathcal{L}_\beta^1$ and $\mathcal{L}_\alpha^2$ intersect transversely (see [**?**, Lemma 6.4]).

Now all the graphs near $\mathcal{L}_{n_j^i,\alpha_j^i}$ have slope close to that of the limiting graph $\mathcal{L}_\alpha^i$. Since $\alpha \in \operatorname{supp}(\mu)$ and $\mathcal{L}_\alpha^1$ and $\mathcal{L}_\alpha^2$ are transverse at $(0,\alpha)$, this contradicts the convergence of currents.

The case where μ has an atom at α is similar and we leave it to the reader. □

2.3. **Laminar currents.** Laminar currents are a generalization of uniformly laminar currents, suitable for dynamical applications, which were introduced by Bedford, Lyubich and Smillie [**?**].

Definition 2.8. *A positive closed $(1,1)$-current T in a domain $\Omega \subset \mathbf{C}^2$ is called* **laminar** *if for every $\varepsilon > 0$ there exists a locally uniformly laminar current T_ε in a subdomain $\Omega_\varepsilon \subset \Omega$ such that $0 \leq T_\varepsilon \leq T$ and $||T - T_\varepsilon|| \leq \varepsilon$.*

It may not seem obvious at first glance why this definition should be so different from Definition **??**. The following example is very illustrative.

Example 2.9. Let $T = dd^c \log \max(|z|, |w|, 1)$ in $\mathbf{C}^2$. The structure of this current was studied thoroughly in [**?**] where it was proved to be the first example of extremal positive closed current not supported on an irreducible subvariety. This was a negative answer to a conjecture of Lelong's [**?**]. We claim that T is laminar but not uniformly laminar.

We first show that T is laminar. The support of T can be decomposed as

$$\begin{aligned}\operatorname{supp}(T) &= \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \mathbb{T} \\ &= \{|z| < 1,\ |w| = 1\} \cup \{|w| < 1,\ |z| = 1\} \cup \{|z| = |w| > 1\} \cup \{|z| = |w| = 1\}\end{aligned}$$

Each of the Σ_i is Levi-flat, so we infer that T is uniformly laminar outside the unit torus $\mathbb{T} = \{|z| = |w| = 1\}$. Furthermore, Demailly proves that T gives zero mass to $\mathbb{T}$ (this is a variation on Proposition **??**). So we conclude that T is laminar, even in a very strong form, since it is uniformly laminar outside a fixed closed subset. Of course it is clear that T is not locally uniformly laminar near any point of $\mathbb{T}$.

On the other hand it is easy to prove that $T \wedge T$ is, up to normalization, the Lebesgue measure on $\mathbb{T}$, in particular it is not zero. Hence this example shows that the potentials of laminar currents are not maximal in general.

Example 2.10. Complex dynamics is a source of interesting examples of maximal psh functions. For instance, the invariant currents of polynomial automorphisms of $\mathbf{C}^2$ are laminar currents with continuous potentials and $T \wedge T = 0$, but in general they are not *uniformly* laminar (examples are provided e.g. by mappings with indifferent periodic points). This shows that it is too much to expect for a continuous maximal psh function u that $dd^c u$ is uniformly laminar. On the other hand it is unclear at this point whether $dd^c u$ should be expected to be laminar in general. We give an answer to this problem in section **??** below.

2.4. $\mathcal{C}^2$ **maximal psh functions.** In view of the results of §**??** it is natural to ask:

Question 2.11. *Assume $u \in PSH \cap \mathcal{C}^2(\Omega)$ is maximal. Is $T = dd^c u$ uniformly laminar?*

The assumption that u is C^2 is natural for $dd^c u$ then determines a *continuous* field of complex lines in the tangent bundle, which we can hope could be integrated into a lamination. A positive answer to this question has been given by Kruzhilin [**?**] when u has rotational symmetry in z, i.e. $u(z, w) = u(|z|, w)$, using some properties of solutions to real Monge Ampère equations. The general case remains open.

3. Polynomial hulls

The notion of polynomial hull is a central concept in analysis in several complex variables. We briefly recall its definition and the connection made by Bremermann with the Dirichlet problem for the complex Monge-Ampère operator. We then indicate a construction due to Stolzenberg [**?**] and Wermer [**?**] of a polynomial hull without complex structure (i.e. containing no holomorphic disk). A variation of this construction will be used in section 4 to exhibit maximal currents without complex structure.

3.1. The Bremermann construction.

Definition 3.1. *Let $K \subset \mathbf{C}^n$ be a compact subset. The polynomial hull $\hat{K}$ of K is the set*

$$\hat{K} := \left\{ z \in \mathbf{C}^n \,/\, |P(z)| \leq \sup_K |P| \text{ for all polynomials } P \text{ on } \mathbf{C}^n \right\}.$$

In dimension $n = 1$, the hull $\hat{K}$ is easy to understand by using the maximum principle: $\hat{K}$ is the union of K and the bounded connected components of $\mathbf{C} \setminus K$. This is a much more subtle notion in higher dimension which is not invariant by biholomorphic change of coordinates. The reader will easily convince himself that the following tori have very different polynomial hulls,

$$K_1 := \{(e^{i\theta}, 0) \in \mathbf{C}^2 \,/\, \theta \in \mathbf{R}\} \text{ and } K_2 := \{(e^{i\theta}, e^{-i\theta}) \in \mathbf{C}^2 \,/\, \theta \in \mathbf{R}\}.$$

Indeed K_2 is polynomially convex (that is $\hat{K_2} = K_2$) while $\hat{K_1} = \overline{\mathbf{D}} \times \{0\}$.

The latter examples are very particular. It is in general very difficult (if not impossible) to determine the polynomial hull of a given compact set. As any plurisubharmonic function in $\mathbf{C}^n$ can be (well) approximated by rational multiples of $\log |P|$, P polynomial, an alternative definition of the polynomial hull is

$$\hat{K} := \{z \in \mathbf{C}^n \,/\, \varphi(z) \leq \sup_K \varphi \text{ for all } \varphi \in PSH(\mathbf{C}^n)\}.$$

Bremermann made in [**?**] an interesting connection between the construction of certain polynomial hulls and the Dirichlet problem for the complex Monge-Ampère operator. Let $\Omega = \{\rho < 0\} \subset \mathbf{C}^n$ be a smoothly bounded strictly pseudoconvex domain and Φ be a smooth function on $\partial\Omega$. We let

$$u(z) := \sup\{v(z) \,/\, v \in PSH(\Omega) \text{ with } \limsup v \leq \Phi \text{ on } \partial\Omega\}$$

denote the Perron-Bremermann envelope (which Bremermann introduced for this purpose). This is a maximal plurisubharmonic function in Ω which is continuous up to the boundary, with boundary values $u_{|\partial\Omega} = \Phi$. We refer the reader to [?] for an up-to-date discussion of such envelopes. Consider

$$K := \{(z,w) \in \partial\Omega \times \mathbf{C} \,/\, |w| \leq \exp(-\Phi(z))\}.$$

Proposition 3.2. *The polynomial hull of K is*

$$\hat{K} = \{(z,w) \in \overline{\Omega} \times \mathbf{C} \,/\, |w| \leq \exp(-u(z))\}.$$

Proof. Set $F := \{(z,w) \in \overline{\Omega} \times \mathbf{C} \,/\, |w| \leq \exp(-u(z))\}$. Note that ρ admits a plurisubharmonic extension to $\mathbf{C}^n$ (see e.g. the proof of Lemma **??** below) so that $\hat{K} \subset \overline{\Omega} \times \mathbf{C}$ easily follows from the second definition we gave above. We can also extend u as a psh function in $\mathbf{C}^n$ and use the function $\psi(z,w) := u(z) + \log|w| \in PSH(\mathbf{C}^{n+1})$ to check that $\hat{K} \subset F$.

We now prove the reverse inclusion. By Lemma **??** below, it suffices to consider psh functions of the type $\varphi(z,w) = \log|w| + v(z)$, $v \in PSH(\mathbf{C}^n)$ to compute $\hat{K}$. Fix $(z_0,w_0) \in \overline{\Omega} \times \mathbf{C} \setminus \hat{K}$ and $v \in PSH(\mathbf{C}^n)$ such that

$$\log|w_0| + v(z_0) > 0 = \sup_{(z,w)\in K} \left[\log|w| + v(z)\right].$$

We need to show that $(z_0,w_0) \notin F$, i.e. $|w_0| > \exp(-u(z_0))$. Observe that $v \leq \Phi$ on $\partial\Omega$, as follows from the condition $\sup_{(z,w)\in K}\left[\log|w| + v(z)\right] = 0$. Therefore

$$|w_0| > \exp(-v(z_0)) \geq \exp(-u(z_0)),$$

as desired. □

Lemma 3.3. *Let $K \subset \mathbf{C}^n \times \mathbf{C}$ be a compact subset that is invariant by rotation in the last coordinate, $(z, e^{i\theta}w) \in K$ whenever $(z,w) \in K$ and $\theta \in \mathbf{R}$. Then*

$$\hat{K} = \left\{ (z,w) \in \mathbf{C}^n \times \mathbf{C} \,/\, |A(z,w)| \leq \sup_K |A|, \text{ for all polynomials } A(z,w) = P(z)w^j \right\}.$$

Proof. Let $\check{K}$ denote the hull on the right hand side, i.e. the polynomial hull restricted to special polynomials of the form $A(z,w) = P(z)w^j$. By definition $\hat{K} \subset \check{K}$.

Assume $(z_0,w_0) \in \check{K}$. Fix $0 < t < 1$. We are going to show that $(z_0, tw_0) \in \hat{K}$. Since $\hat{K}$ is closed, we thus infer that $(z_0,w_0) \in \hat{K}$ by letting t increase to 1. It is an exercise to show that for all $c \geq 1$,

$$\hat{K} = \hat{K}_c := \left\{ (z,w) \in \mathbf{C}^n \times \mathbf{C} \,/\, |P(z,w)| \leq c \sup_K |P|, \text{ for all polynomials } P \right\}.$$

It is therefore sufficient to show that $(z_0, tw_0) \in \hat{K}_c$ where $c = 1/(1-t)$.

Let $P(z,w) = \sum_j P_j(z) w^j$ be the decomposition of a polynomial $P(z,w)$. Note that $\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta}w) e^{-ij\theta}\, d\theta = P_j(z)w^j$, thus the invariance property of K yields

$$\sup_K |P_j(z)w^j| \leq \sup_K |P|$$

Since $(z_0,w_0) \in \check{K}$ we infer

$$|P(z_0, tw_0)| \leq \sum_j t^j |P_j(z_0) w_0^j| \leq \frac{1}{1-t} \sup_K |P|.$$

Therefore $(z_0, tw_0) \in \hat{K}$, hence $(z_0,w_0) \in \hat{K}$. □

It is a particular feature of this construction that $\hat{K} \setminus K$ is filled in with holomorphic disks. This is not the case in general, as we now explain.

3.2. **Stolzenberg and Wermer examples.** Let $K \subset \mathbf{C}^n$ be a compact subset. It is an immediate consequence of the maximum principle that if a complex submanifold V of the unit ball $\mathbf{B}$ has boundary in K, $\partial V \subset K \cap \partial \mathbf{B}$, then $V \subset \hat{K}$. This suggests that $\hat{K} \setminus K$ may be filled in with complex subvarieties whose boundary lies in K.

This is actually far from being true in general. Stolzenberg has produced in [**?**] an example of a compact set $K \subset \partial \mathbf{B}$ such that $\hat{K} \setminus K$ is non empty and does not contain any germ of holomorphic disk.

It is a general principle that the hulls of subsets of $\partial \mathbf{D} \times \mathbf{D}$ are better behaved than those of general compact sets. Thus, one may guess that for compact subsets $K \subset \partial \mathbf{D} \times \mathbf{D}$, these complex subvarieties exist. This is equally wrong and Wermer has produced [**?**] an example of a compact set $K \subset \partial \mathbf{D} \times \mathbf{D}$ with non trivial polynomial hull and such that $\hat{K} \setminus K$ does not contain any germ of holomorphic disk. More generally, we define a *Wermer example* as a closed horizontal subset of the unit bidisk, which contains no holomorphic disk, and which is the polynomial hull of $\overline{X} \cap (\partial \mathbf{D} \times \mathbf{D})$. Recall that a subset of $\mathbf{D}^2$ is said to be horizontal if it is contained in $\mathbf{D} \times D(0, 1 - \varepsilon)$ for some $\varepsilon > 0$.

We now present a construction of Wermer examples, following a slight modification of the original Wermer construction due to Duval and Sibony [**?**].

Let (a_n) be a dense sequence in $\mathbf{D}$, and (r_n) a sequence of positive real numbers, decreasing to zero. We construct (P_n) and (δ_n) by induction as follows. Set $P_0 = w$ and $\delta_0 = 1/2$ so that X_0 is the cylinder $\mathbf{D} \times \{|w| < 1/2\}$.

Define P_1 by $P_1(z, w) = w^2 - \varepsilon_1(z - a_1)$. Choose ε_1 small enough so that $\{P_1 = 0\} \subset X_0$. If δ_1 is small enough, $X_1 := \{|P_1| < \delta_1\}$ is contained in X_0 and neither contain any continuous section (relative to the first coordinate) over $D(a_1, r_1)$, nor any vertical disk of size r_1.

Repeat the same process by induction, by setting

$$P_{n+1}(z, w) = P_n^2(z, w) - \varepsilon_{n+1}(z - a_{n+1})^2,$$

and choosing the constants δ_{n+1} and ε_{n+1} so small that

- $X_{n+1} \subset X_n$ where we define X_n as $X_n = \{|P_n| < \delta_n\}$;
- X_{n+1} neither contains any graph over $D(a_{n+1}, r_{n+1})$, nor any vertical disk of size r_{n+1}

It is now clear that $X = \bigcap X_n$ contains no germ of holomorphic disk. Furthemore it is easily shown that $X = \hat{K} \setminus K$, where $K = \overline{X} \cap (\partial \mathbf{D} \times \mathbf{D})$. Thus X is a Wermer example.

It is obvious that X supports positive closed currents: consider indeed any cluster value T of the sequence $\frac{1}{2^n}[P_n = 0]$. This mere observation provides a second proof of the fact that there exist extremal closed positive currents which are not supported on analytic varieties: this is the case for the currents appearing in the Choquet decomposition of this current T.

There is no reason for these currents to have well-defined self intersection –especially if the δ_n are too small, in which case X tends to become pluripolar. In §**??** below we address this problem and find examples of Wermer examples supporting regular maximal currents T.

4. Maximality with no holomorphic disk

4.1. **The Sibony construction.** Using Stolzenberg-like examples, Sibony exhibited examples of positive closed $(1,1)$ currents T in the unit ball $\mathbf{B}$ of $\mathbf{C}^2$ with $\mathcal{C}^{1,1}$ potential, and which are not uniformly laminar (see [**?**, **?**]). Let us explain this construction.

Let $X \subset \partial\mathbf{B}$ be a Stolzenberg example, i.e. a compact subset of $\partial\mathbf{B}$ such that the polynomial hull $\hat{X}$ is non trivial but does not contain any holomorphic disk $\mathbf{D}$. Fix $\Phi \in \mathcal{C}^\infty(\partial\mathbf{B})$ a non-negative function such that $X = \{\Phi = 0\}$. By Bedford-Taylor's result [**?**], there exists a unique $u \in PSH(\mathbf{B}) \cap Lip(\overline{\mathbf{B}}) \cap \mathcal{C}^{1,1}(\mathbf{B})$ such that

$$(dd^c u)^2 = 0 \text{ in } \mathbf{B} \quad \text{ and } \quad u_{|\partial\mathbf{B}} = \Phi.$$

This is actually the Perron-Bremermann envelope,

$$u(z) := \sup\{v(z) \,/\, v \in PSH(\Omega) \text{ with } \limsup v \leq \Phi \text{ on } \partial\Omega\}.$$

The main step is the following.

Proposition 4.1. *The current $T = dd^c u$ is not uniformly laminar.*

Proof. Let us assume for the moment that $\hat{X} = \{u = 0\}$. Let $p \in \hat{X} \cap \mathbf{B}$, and let us show that T is not uniformly laminar near p. Otherwise there would exist a holomorphic disk Δ through p such that $u_{|\Delta}$ is harmonic. Observe that $u \geq 0$ in $\mathbf{B}$ since $\Phi \geq 0$. Therefore since u vanishes at p, it has to be identically zero on $\mathbf{D}$ by the maximum principle so that $\mathbf{D} \subset \hat{X}$, a contradiction. □

It remains to establish the following result:

Lemma 4.2. $\hat{X} = \{u = 0\}$.

Proof. Observe that u admits a plurisubharmonic extension to $\mathbf{C}^2$. Indeed consider first Φ_1 a smooth extension of Φ with compact support in $\mathbf{C}^2$ and add a large multiple of $\log[1 + ||z||^2] - \log 2$ to obtain an extension Φ_2 of Φ which is moreover plurisubharmonic. By its upper envelope nature, the function u dominates Φ_2 in $\mathbf{B}$ and coincides with it on $\partial\mathbf{B}$, thus we can extend u by setting $u(z) = \Phi_2(z)$ in $\mathbf{C}^2 \setminus \mathbf{B}$.

It readily follows from the definition of $\hat{X}$ via psh functions that $\hat{X} \subset \{u = 0\}$.

For the reverse inclusion, note that $u > 0$ in $\mathbf{C}^2 \setminus (\mathbf{B} \cup X)$ so that $\hat{X} \subset \mathbf{B} \cup X$. What remains to be proved is thus that $u(p) > 0$ whenever $p \in \mathbf{B} \setminus \hat{X}$. Let $p \in \mathbf{B} \setminus \hat{X}$. There exists a psh function on $\mathbf{C}^2$ such that $v < 0$ on X and $v(p) > 0$. Fix δ such that $v < 0$ on $\Phi \leq \delta$, and M such that $v \leq M$ on $\partial\mathbf{B}$. If $0 < \varepsilon < \delta/M$, we infer that $\varepsilon v \leq \Phi$ on $\partial\mathbf{B}$. From the definition of u as an upper envelope, we deduce that $u \geq \varepsilon v$ on $\mathbf{B}$. In particular $u(p) > 0$ □

This interesting example shows that the answer to Question **??** is "no" in the $\mathcal{C}^{1,1}$ setting. Regularity $\mathcal{C}^{1,1}$ is important in pluripotential theory for it is the regularity of solutions to the homogeneous Monge-Ampère equation with smooth boundary data. On the other hand it can be shown (see [**?**, Prop. 4.1]) that for these examples, the Stolzenberg example $\hat{X}$ has zero trace measure, that is, we are proving non-laminarity on a *negligible* set for T. This raises the following natural question:

Question 4.3. *Is the above current T laminar ?*

We don't know the answer to this question. What we explain in the next paragraph is that, at the expense of a small loss of regularity, we can ensure non-laminarity *everywhere* on the support of T.

4.2. **No holomorphic disk at all.** We finish these notes by explaining the main ideas of the proof of the following recent result of the first author [**?**]:

Theorem 4.4. *There exists a plurisubharmonic function u in the unit polydisk $\mathbf{D}^2$ such that*
(1) u is of class $\mathcal{C}^{1,\alpha}$ for all $0 < \alpha < 1$;
(2) u is maximal (i.e. $(dd^c u)^2 = 0$);
(3) the support of $dd^c u$ does not contain any holomorphic disk.

The basic idea of the proof is to reconsider the Wermer construction of [**?**], as presented in §**??** and make explicit estimates of the quantities δ_n, ε_n in terms of r_n. Then we take a cluster value T of the sequence $T_n := \frac{1}{2^n}[P_n = 0]$ (which will actually be convergent), and arrange the parameters so that T has continuous potential u. It turns out that such T are always maximal. An unfortunate fact is that in this construction the potential is never more regular than merely continuous (and even never Hölder continuous!). So to achieve $\mathcal{C}^{1,\alpha}$ regularity we will need to find a way of "thickening" the Wermer example.

Let us be more specific. We work in $\mathbf{D}^2$. Let (a_n) be a dense sequence in $\mathbf{D}$ and (r_n) a sequence of real numbers decreasing to zero. We define by induction a sequence of polynomials by the formula $P_{n+1} = P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$ and a sequence of horizontal subsets $X_n = \{|P_n| < \delta_n\}$ so that
(i) $X_{n+1} \subset X_n$;
(ii) X_n does not contain any holomorphic graph over $\mathbf{D}(a_n, r_n)$.
This imposes some explicit conditions on the parameters, respectively
(i) $\delta_{n+1} + \varepsilon_{n+1} < \delta_n^2$;
(ii) $\delta_n < \varepsilon_n r_n$.
An essentially optimal choice for this is (our goal is to get estimates from below for δ_n):

$$\varepsilon_{n+1} = \frac{\delta_n^2}{2} \text{ and } \delta_{n+1} = \frac{\delta_n^2 r_{n+1}}{2}$$

(say that $r_n \leq 1/10$). Note that, declaring that $\delta_0 = 1/2$ as above, (δ_n) depends only on (r_n).

Now introduce the potential $u_n = \frac{1}{2^n}\max(\log|P_n|, \log\delta_n)$ of T_n. Using the definition of P_{n+1} in terms of P_n it may be shown that $|u_{n+1} - u_n| = O\big(\frac{1}{2^n}|\log r_n|\big)$. In particular if the series $\sum \frac{1}{2^n}|\log r_n|$ converges, we infer that u_n converges uniformly to some continuous function u. In particular the sequence T_n converges to a current T with continuous potential, supported on X.

It is an easy fact that $(dd^c u_n)^2 = 0$ so by uniform convergence we infer that $(dd^c u)^2 = 0$.

With our definition of P_n, we are actually only certain that X does not contain any holomorphic graph, but we cannot rule out the possibility of vertical disks. For this, we slightly modify the inductive step by introducing an oblique projection: we assume that (a_{2n}) and (a_{2n+1}) are dense and define $P_{n+1} = P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$ if n is even, and $P_n^2 - \varepsilon_{n+1}(z + \frac{w}{100} - a_{n+1})$ if n is odd.

At this point we have constructed a maximal current with continuous potential, whose support does not contain any holomorphic disk. We want to understand more precisely the regularity of its potential.

For this we analyze the vertical slice measures of T. Keeping the same construction, we enlarge the bidisk and look at the curves $[P_n = 0]$ and the sets $X_n = \{|P_n| < \delta_n\}$ in $3\mathbf{D} \times \mathbf{C}$. Over the annulus $3\mathbf{D} \setminus 2\mathbf{D}$ the curves $\{P_n = s\}$, $|s| < \delta_n$ have no ramifications (relative to the vertical projection), so the intersection of each of these curves with a vertical fiber consists of exactly 2^n points. Thus in those vertical fibers[2], X_n is the union of 2^n disjoint topological disks, and each component of X_n contains two components of X_{n+1}. These disks are actually essentially "round" (in the sense of distortion theory of conformal mappings), and we can estimate their size. Up to exponential terms, the size is of the order of magnitude of $\prod_1^n r_k$ (which is superexponentially small). To say it differently, each component of X_{n+1} has relative size $\approx r_{n+1}$ times the size of the component of X_n in which it sits. In particular the fibers of X have zero Hausdorff dimension.

On the other hand, the slice measure of T (which is the same as the Laplacian of u restricted to the vertical fiber) is the "balanced" measure on the Cantor set X, so that each component of X_n has mass 2^{-n}. This prevents u from being Hölder continuous. Indeed, the Laplacian of a Hölder continuous plane subharmonic function gives mass $O(r^\alpha)$ to a disk of radius r (α is the Hölder exponent).

Another way to say this is that a measure with Hölder continuous potential cannot charge sets of Hausdorff dimension 0.

Therefore, to upgrade the regularity of our examples, we need to modify the construction in order to "thicken" the set X. To understand this modification, let us first imagine a model situation. Suppose that we have a process which at time n replaces a disk D of radius r in the plane by two smaller disks $D_i \subset D$, of *relative* size r_n and with mutual distance $r/2$. Start with the unit disk, and apply this process repeatedly. Then at time n we have 2^n disjoint disks of size $\prod_1^n r_k$. Taking their union we get a nested sequence of subsets, and the result is a Cantor set of Hausdorff dimension zero.

To increase dimension we do as follows. We now consider a process in two steps. Given a disk D of radius r, we first replace it by $\approx N^2$ evenly spaced small disks of radius r/N, and then we apply the previous doubling process to each of the small disks. Then if at each step, N is sufficiently large with respect to r_n, the limiting Cantor set will have dimension 2.

To implement this strategy for our Wermer examples, before the ramification process $P_n \mapsto P_n^2 - \varepsilon_{n+1}(z - a_{n+1})$, we replace $\{|P_n| < \delta_n\}$ by a large number of smaller subsets $\left\{|P_n - s| < \frac{\delta_n}{N}\right\}$, filling out most of $\{|P_n| < \delta_n\}$ (s ranges over a finite set $\mathcal{S}_n$). Then we apply the ramification process to each $P_n - s$ and get a family of polynomials $P_{n+1,s}$. Let then $X_n := \bigcup_s \{|P_{n+1,s}| < \delta_{n+1}\}$ for well chosen δ_{n+1} and $X = \bigcap X_n$. It can be shown that the vertical slices of X_n over $3\mathbf{D} \setminus 2\mathbf{D}$ indeed "look like" the model situation described above, in the sense of plane conformal geometry. In particular we can ensure Hausdorff dimension 2 for the vertical fibers of X.

Let further u_n be defined by

$$u_n = \frac{1}{2^n \# \mathcal{S}_n} \sum_{s \in \mathcal{S}_n} \log \max\left(|P_{n+1,s}|, \delta_{n+1}\right).$$

As before, (u_n) converges to a maximal continuous psh function u, such that $\operatorname{supp}(dd^c u)$ is contained in X, thus contains no holomorphic disk.

By standard estimates in potential theory, if at each step N is chosen to be sufficiently large, u will be of class $\mathcal{C}^{1,\alpha}$ for all $\alpha < 1$ in $(3\mathbf{D} \setminus 2\mathbf{D}) \times \mathbf{C}$. Then by regularity theory for solutions of Monge-Ampère equations, this regularity propagates to $3\mathbf{D} \times \mathbf{C}$. The reader is referred to [**?**] for details.

[2] We use the same notation for X_n (resp. X) and its vertical fibers

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