Stability of electron wave spectra in weakly magnetized plasmas: local spectra and correlations

M. Kono¹ and H. L. Pécseli²

¹*Chuo University, Faculty of Policy Studies, Hachioji, Tokyo 192-0393, Japan*

²*University of Oslo, Department of Physics, Box 1048 Blindern, N-0316 Oslo, Norway*

(Dated: October 24, 2017)

Analytical models for weakly nonlinear electron plasma waves in weakly magnetized plasmas are considered in order to obtain dynamic equations for the space-time evolution and the stability of their power spectra. The analysis is based on ideas originally proposed for studying wave mechanics, but they were later applied also to describe two dimensional wave-trains on water surfaces.

I. INTRODUCTION

The nonlinear evolution of high frequency electron plasma waves have been studied in great detail in part because these waves are often found in nature [1–5]. They are easily excited also in a laboratory. Most of the studies addressed homogeneous and isotropic condition which can serve as a reference case for illustrating some general nonlinear wave phenomena. For conditions in nature we usually find that the effects of even weak magnetic field will be important, and we here take these effects into account. The basic equations are then changed significantly.

The standard model for weakly nonlinear electron plasma waves assumes the dominant nonlinearity to be modification of the local plasma density (i.e. the index of refraction) [6, 7]. Kinetic effects as linear Landau damping and nonlinear particle trapping are ignored from the outset, arguing that these effects will become negligible when the characteristic wavelengths λ are sufficiently long. (For laboratory conditions it might become difficult to fulfill this condition due to the finite size of the apparatus.) Nonlinearities associated with, for instance, the adiabatic electron motion will be of the order of $(\lambda_D/\lambda)^2$ and can be ignored [8], assuming again that the Debye length λ_D is much smaller than any relevant wavelength. The origin of the dominating nonlinear term is here the space-time varying density in the electron continuity equation.

The present study considers the space-time evolution of high frequency electrostatic wave spectra in weakly magnetized plasmas [9–14]. The assumption of weak magnetic fields will be essential for the dispersion of the waves, where it is known that when $\omega_{ce} > \omega_{pe}/\sqrt{3}$ kinetic effects give rise to a change in the curvature of the dispersion relation at small wavenumbers [15, 16]. Strong magnetic fields thus necessitate a kinetic rather than a fluid model for the wave dynamics, although some simplifications can be obtained also for the kinetic model [16]. The possibility of modulational instability of plane weakly nonlinear electron waves in weakly magnetized plasmas was studied elsewhere [17, 18], the nonlinear frequency shifts in particular.

The following analysis will be based on ideas originally proposed for studying wave mechanics [19], but they were later applied also to two dimensional wave-trains on water surfaces [20] and signal analysis [21]. We use first a fluid model thereby ignoring nonlinear Landau damping [8, 22] as well as nonlinear ion cyclotron damping [23].

II. BASIC EQUATIONS FOR THE HIGH FREQUENCY WAVE-FIELD

As our basic equation for weakly nonlinear electron plasma waves in weakly magnetized plasmas with $\omega_{ce} \ll \omega_{pe}$ we take [11, 24]

$$\begin{aligned} \nabla^2 \left(\frac{\partial^2}{\partial t^2} - v^2 \nabla^2 + \omega_{pe}^2 \right) \phi + \omega_{ce}^2 \nabla_{\perp}^2 \phi = \\ - \nabla \cdot \left(\omega_{pe}^2 \frac{\bar{n}}{n_0} \nabla \phi \right), \end{aligned} \quad (1)$$

where $\phi = \phi(\mathbf{r}, t)$. Here and in the following, the electron plasma frequency is denoted ω_{pe} , the electron-cyclotron frequency is ω_{ce} , the electron Debye length by λ_D , a characteristic particle velocity is defined as $v \equiv \sqrt{3T_e/m}$, while $\nabla_{\perp}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, with the magnetic field being along the z -axis. The physical potential is related to the amplitude (or envelope) potential by $\frac{1}{2}\phi(\mathbf{r}, t)e^{-i\omega t} + \text{c.c.}$ with $\omega \approx \omega_{pe}$. It is assumed that the potential amplitude varies over a timescale much longer than the plasma period $2\pi/\omega_{pe}$. The analysis of the high frequency wave is thus based on a fluid model, which imposes some limitations even for weak magnetic fields, i.e. some wave decay possibilities will be lost [25].

The linear dispersion relation associated with (1) is readily found as

$$\omega^2 = \omega_{pe}^2 + v^2 k^2 + \omega_{ce}^2 \frac{k_{\perp}^2}{k^2}. \quad (2)$$

The upper-hybrid frequency is $\omega_{uh} \equiv \sqrt{\omega_{pe}^2 + \omega_{ce}^2}$. The waves are assumed to be electrostatic also in the weakly nonlinear limit considered here: for a linearized analysis this restriction is immaterial for wave propagation along magnetic field lines, but for off-angle propagation at some θ we have the restriction $(\omega/\omega_{ce})(\omega_{pe}/kc)^2 \sin \theta \ll 1$. For $\theta \neq 0$ the waves will have a fluctuating magnetic component for small wavenumbers [17].

FIG. 1: Illustration by surface plot and contour curves of the approximate normalized linear electron wave dispersion relation in the form $\Omega = K^2 + K_{\perp}^2/K^2$ where $K^2 \equiv K_{\perp}^2 + K_{\parallel}^2$.

We now remove the rapidly varying, high frequency part of the signal by the standard substitution $\phi \rightarrow \phi(\mathbf{r}, t) \exp(-i\omega_0 t)$, with ω_0 being the carrier wave frequency. When $\partial\phi/\partial t \ll \omega_0\phi$ we obtain

$$\left(i \frac{\partial}{\partial t} + \frac{1}{2} \frac{v^2}{\omega_{pe}} \nabla^2 \right) \nabla^2 \phi - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} \nabla_{\perp}^2 \phi = \frac{1}{2} \omega_{pe} \nabla \cdot \left(\frac{\bar{n}}{n_0} \nabla \phi \right). \quad (3)$$

We explicitly used $\omega_0 \approx \omega_{uh} \approx \omega_{pe}$ as appropriate for weakly magnetized plasmas [11], with $\omega_{ce} \ll \omega_{pe}$. The approximation to the normalized linear electron wave dispersion relation obtained from (2) is shown in Fig. 1.

A. Equations for correlation functions

In order to obtain an expression for the correlation function $\langle \phi(t_1, \mathbf{r}_1) \phi^*(t_2, \mathbf{r}_2) \rangle$ of the electrostatic waves, we rewrite (3) for the two sets of variables $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$ [20, 26–30]. After complex conjugating one of them we multiply the two equations by $\phi^*(t_2, \mathbf{r}_2)$ and $\phi(t_1, \mathbf{r}_1)$, respectively, and subtracting the two resulting equations to find

$$\begin{aligned} & \left[i \left(\frac{\partial}{\partial t_1} \nabla_1^2 + \frac{\partial}{\partial t_2} \nabla_2^2 \right) + \frac{1}{2} \frac{v^2}{\omega_{pe}} (\nabla_1^4 - \nabla_2^4) \right] \rho \\ & - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} (\nabla_{1\perp}^2 - \nabla_{2\perp}^2) \rho = \\ & \frac{1}{2} \omega_{pe} \left(\nabla_1 \cdot \frac{\bar{n}_1}{n_0} \nabla_1 - \nabla_2 \cdot \frac{\bar{n}_2}{n_0} \nabla_2 \right) \rho, \end{aligned} \quad (4)$$

with $\rho = \rho(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) \equiv \phi(\mathbf{r}_1, t_1) \phi^*(\mathbf{r}_2, t_2)$ generally complex. We have $\rho(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \rho^*(\mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$. The basic idea is clear already now: the ensemble average of ρ gives the space-time varying correlation function of the electrostatic wave field. We first derive a dynamic equation for ρ .

We introduce separation and average coordinates $\mathbf{R} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, respectively, and similarly for the temporal variable $T \equiv (t_1 + t_2)/2$ and $\tau \equiv t_1 - t_2$. In terms of these new variables we find trivially $\nabla_1 = \frac{1}{2}\nabla_R + \nabla_r$, $\partial/\partial t_1 = \frac{1}{2}\partial/\partial T + \partial/\partial \tau$, $\nabla_1^2 = \frac{1}{4}\nabla_R^2 + \nabla_r^2 + \nabla_R \cdot \nabla_r$, etc. In particular we have $\nabla_1^2 - \nabla_2^2 = 2\nabla_R \cdot \nabla_r$ and $\nabla_1^4 - \nabla_2^4 = \nabla_r \cdot \nabla_R(\nabla_R^2 + 4\nabla_r^2)$.

Introducing for simplicity $\eta_1 \equiv \bar{n}_1/n_0$ and $\eta_2 \equiv \bar{n}_2/n_0$ we have the operator

$$\begin{aligned} & \nabla_1 \cdot \eta_1 \nabla_1 - \nabla_2 \cdot \eta_2 \nabla_2 = \\ & \frac{1}{4} \nabla_R \cdot (\eta_1 - \eta_2) \nabla_R + \nabla_r \cdot (\eta_1 - \eta_2) \nabla_r \\ & + \frac{1}{2} \nabla_r \cdot (\eta_1 + \eta_2) \nabla_R + \frac{1}{2} \nabla_R \cdot (\eta_1 + \eta_2) \nabla_r, \end{aligned}$$

etc. On the other hand we have also

$$\begin{aligned} \eta_1 &= \eta \left(\mathbf{R} + \frac{1}{2}\mathbf{r}, T + \frac{1}{2}\tau \right) = \\ & \eta(\mathbf{R}, T) + \frac{1}{2} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right) \eta(\mathbf{R}, T) + \dots \\ & + \frac{1}{n!2^2} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^n \eta(\mathbf{R}, T) + \dots, \end{aligned}$$

and similarly for $\eta_2 \equiv \bar{n}_2/n_0 = \eta(\mathbf{R} - \frac{1}{2}\mathbf{r}, T - \frac{1}{2}\tau)$. All in all we find the operator expressions

$$\begin{aligned} \eta_1 - \eta_2 &= \\ 2 \left(\sum_n \frac{1}{(1+2n)!} \frac{1}{2^{1+2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{1+2n} \right) \eta(\mathbf{R}, T), \end{aligned}$$

and

$$\begin{aligned} \eta_1 + \eta_2 &= \\ 2 \left(\sum_n \frac{1}{(2n)!} \frac{1}{2^{2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{2n} \right) \eta(\mathbf{R}, T). \end{aligned}$$

From (4) we then find after some algebra

$$\begin{aligned} & \left[i \left(\left(\frac{1}{4} \nabla_R^2 + \nabla_r^2 \right) \frac{\partial}{\partial T} + 2 \nabla_R \cdot \nabla_r \frac{\partial}{\partial \tau} \right) + 2 \frac{v^2}{\omega_{pe}} \left(\frac{1}{4} \nabla_R^2 + \nabla_r^2 \right) \nabla_R \cdot \nabla_r - \frac{\omega_{ce}^2}{\omega_{pe}} \nabla_{R\perp} \cdot \nabla_{r\perp} \right] \rho(\mathbf{R}, T, \mathbf{r}, \tau) = \\ & \frac{\omega_{pe}}{2} \left(\frac{1}{2} \nabla_R \cdot \sum_n \frac{1}{(1+2n)!} \frac{1}{2^{1+2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{1+2n} \eta(\mathbf{R}, T) \nabla_R \rho(\mathbf{R}, T, \mathbf{r}, \tau) \right. \\ & + \nabla_r \cdot \sum_n \frac{1}{(1+2n)!} \frac{1}{2^{1+2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{1+2n} \eta(\mathbf{R}, T) \nabla_r \rho(\mathbf{R}, T, \mathbf{r}, \tau) \\ & + 2 \nabla_r \cdot \sum_n \frac{1}{(2n)!} \frac{1}{2^{2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{2n} \eta(\mathbf{R}, T) \nabla_R \rho(\mathbf{R}, T, \mathbf{r}, \tau) \\ & \left. + \nabla_R \cdot \sum_n \frac{1}{(2n)!} \frac{1}{2^{2n}} \left(\mathbf{r} \cdot \nabla_R + \tau \frac{\partial}{\partial T} \right)^{2n} \eta(\mathbf{R}, T) \nabla_r \rho(\mathbf{R}, T, \mathbf{r}, \tau) \right). \end{aligned} \quad (5)$$

Care should be taken here to interpret the differential operations correctly. The differential operators in the summations originate from the sums and differences, $\eta_1 + \eta_2$ and $\eta_1 - \eta_2$, respectively, and therefore operate only on $\eta(\mathbf{R}, T)$, while the “outer” operators ∇_r and ∇_R act on all that follows.

We now Fourier transform $\rho(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \rho(\mathbf{R}, T, \mathbf{r}, \tau)$ with respect to both spatial and temporal separation variables $\mathbf{r}$ and τ and denote this Fourier transform $F(\mathbf{R}, T, \mathbf{k}, \Omega)$. We have $\rho(\mathbf{R}, T, \mathbf{r}, \tau) = \rho^*(\mathbf{R}, T, -\mathbf{r}, -\tau)$ which imposes some symmetries on $F(\mathbf{R}, T, \mathbf{k}, \Omega)$. For the particularly simple case of a plane wave where $\phi(\mathbf{r}_j, t_j) = a \exp(i(\mathbf{k}_0 \cdot \mathbf{r}_j - \omega_0 t_j))$ we have $\rho(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = a^2 \exp(i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 \tau))$ with a real Fourier transform $F(\Omega, \mathbf{k}) = a^2 \delta(\Omega - \omega_0) \delta(\mathbf{k} - \mathbf{k}_0)$. Note that there are here no symmetries for $\mathbf{k} \rightarrow -\mathbf{k}$.

For time stationary and spatially homogeneous processes, we hereby obtain the power spectrum for the electrostatic potential as function of frequency Ω and wave vector $\mathbf{k}$, using in particular that $\mathcal{F}t[xf(x)] = i\nabla_k f(\mathbf{k})$, where $\hat{f}(\mathbf{k}) \equiv \mathcal{F}t[f(x)]$ denotes the Fourier transform and $\nabla_k \equiv (\partial/\partial k_x)\hat{\mathbf{x}} + (\partial/\partial k_y)\hat{\mathbf{y}} + (\partial/\partial k_z)\hat{\mathbf{z}}$. In the present case we can interpret the ensuing results as a set of coupled equations for the space-time variation of the local power spectrum [29] of the electrostatic potential here written as $F(\mathbf{R}, T, \Omega, \mathbf{k})$ being a function of frequency and wave-vector $(\Omega, \mathbf{k})$ at a position and time $(\mathbf{R}, T)$. We also use the series expansions $\cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$ and $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ when expressing the series expansion of the differential operators.

$$\begin{aligned} & \left[\left(\frac{1}{4} \nabla_R^2 - k^2 \right) \frac{\partial}{\partial T} + 2 \left(\Omega + \frac{v^2}{\omega_{pe}} \left(\frac{1}{4} \nabla_R^2 - k^2 \right) \right) \mathbf{k} \cdot \nabla_R - \frac{\omega_{ce}^2}{\omega_{pe}} \mathbf{k}_{\perp} \cdot \nabla_{R\perp} \right] F(\mathbf{R}, T, \mathbf{k}, \Omega) = \\ & \frac{\omega_{pe}}{2} \left(\frac{1}{2} \nabla_R \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \nabla_R F(\mathbf{R}, T, \mathbf{k}, \Omega) \right. \\ & \quad - 2\mathbf{k} \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \mathbf{k} F(\mathbf{R}, T, \mathbf{k}, \Omega) \\ & \quad + \mathbf{k} \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \nabla_R F(\mathbf{R}, T, \mathbf{k}, \Omega) \\ & \quad \left. + \nabla_R \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \mathbf{k} \eta(\xi, T') F(\mathbf{R}, T, \mathbf{k}, \Omega) \right). \end{aligned} \quad (6)$$

In order to make it clear which variables the differential operators refer to, we introduced the operators

$$\hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \equiv \int \int_{-\infty}^{\infty} d^3\xi dT' \delta(\mathbf{R} - \xi) \delta(T - T') \sin \left(\frac{1}{2} \nabla_k \cdot \nabla_{\xi} - \frac{1}{2} \frac{\partial}{\partial \Omega} \frac{\partial}{\partial T'} \right),$$

and

$$\hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \equiv \int \int_{-\infty}^{\infty} d^3\xi dT' \delta(\mathbf{R} - \xi) \delta(T - T') \cos \left(\frac{1}{2} \nabla_k \cdot \nabla_{\xi} - \frac{1}{2} \frac{\partial}{\partial \Omega} \frac{\partial}{\partial T'} \right),$$

where the integrals run over all ξ -space, and also all times T' .

We obtained two equations by writing (3) as functions of two sets of variables labeled by 1 and 2. By the difference of these two equations we obtained (6) after some manipulations as outlined before. We can also take the *sum* of the two equations and after the very similar operations as before (in principle) we obtain another equation in the form

$$\left(i \left(\frac{\partial}{\partial t_1} \nabla_1^2 - \frac{\partial}{\partial t_2} \nabla_2^2 \right) + \frac{1}{2} \frac{v^2}{\omega_{pe}} (\nabla_1^4 + \nabla_2^4) - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} (\nabla_{1\perp}^2 + \nabla_{2\perp}^2) \right) \rho = \frac{1}{2} \omega_{pe} (\nabla_1 \eta_1 \nabla_1 + \nabla_2 \eta_2 \nabla_2) \rho. \quad (7)$$

After some algebra, following the previous procedure, we obtain

$$\begin{aligned} & \left[\left(\frac{1}{2} \nabla_R^2 - 2k^2 \right) \Omega - \mathbf{k} \cdot \nabla_R \frac{\partial}{\partial T} + \frac{1}{2} \frac{v^2}{\omega_{pe}} \left(\frac{1}{8} \nabla_R^4 + 2k^4 - k^2 \nabla_R^2 - 2(\mathbf{k} \cdot \nabla_R)^2 \right) \right. \\ & \quad \left. - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} \left(\frac{1}{2} \nabla_{R\perp}^2 - 2k_{\perp}^2 \right) \right] F(\mathbf{R}, T, \mathbf{k}, \Omega) = \\ & \frac{\omega_{pe}}{2} \left(\frac{1}{2} \nabla_R \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \nabla_R F(\mathbf{R}, T, \mathbf{k}, \Omega) \right. \\ & \quad - 2\mathbf{k} \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \mathbf{k} F(\mathbf{R}, T, \mathbf{k}, \Omega) \\ & \quad - \mathbf{k} \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \eta(\xi, T') \nabla_R F(\mathbf{R}, T, \mathbf{k}, \Omega) \\ & \quad \left. - \nabla_R \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \xi, T') \mathbf{k} \eta(\xi, T') F(\mathbf{R}, T, \mathbf{k}, \Omega) \right). \end{aligned} \quad (8)$$

The relations (6) and (8) presume that $\eta(\mathbf{k}, t)$ is given otherwise: it could in principle be representing some a priori given density variations, a dc-density gradient for instance [31] where η is time-stationary so the expressions become somewhat simpler. For the present study we are mostly interested in a self-consistent nonlinear model relating $F(\mathbf{R}, T, \mathbf{k}, \Omega)$ and $\eta(\mathbf{R}, T)$.

Most studies restrict the analysis to the a difference equation as (4), but it should be emphasized that the starting point is two complex equations, in the present case (3) for two sets of variables $(t_1, \mathbf{r}_1)$ and $(t_2, \mathbf{r}_2)$. Reducing this set to only one implies loss of information, which should be accounted for by some means. Here we retain both the sum and difference equations (4) and (7), which contain the same information as the two original equations, albeit in a different form, that is at first sight not as accessible. A detailed analysis of the relations (6) and (8) will, however, make the physical content of these equations more evident.

1. Homogeneous spectra

One particular spectral solution can be found readily by inspection. We assume that $F(\mathbf{R}, T, \mathbf{k}, \Omega) = F(\mathbf{k}, \Omega)$, i.e. independent of $\mathbf{R}$ and T . In this limit we have $\hat{S} \rightarrow 0$ and $\hat{C} \rightarrow 1$. For this particular solution we find that (6) is trivially fulfilled, while (8) gives the condition

$$\left[-2k^2\Omega + \frac{v^2}{\omega_{pe}}k^4 + \frac{\omega_{ce}^2}{\omega_{pe}}k_{\perp}^2 \right] F(\mathbf{k}, \Omega) = -\omega_{pe}k^2\eta(0)F(\mathbf{k}, \Omega), \quad (9)$$

where we introduced $\eta(0)$ as independent of $\mathbf{R}$ and T .

Recalling that the equation $xF(x) = 0$ has a solution $F(x) = G\delta(x)$ with arbitrary G , we find the result

$$F(\mathbf{k}, \Omega) = G_0(\mathbf{k})\delta\left(\Omega - \frac{v^2}{2\omega_{pe}}k^2 - \frac{\omega_{ce}^2}{2\omega_{pe}}\frac{k_{\perp}^2}{k^2} - \frac{1}{2}\omega_{pe}\eta(0)\right), \quad (10)$$

where G_0 is an arbitrary (but positive) wave energy distribution. The last term is a nonlinear frequency shift due to a shift in plasma frequency $\frac{1}{2}\omega_{pe}\eta(0)$ as induced by a yet unspecified nonlinear coupling between the waves and the plasma density. The remaining part of the argument in the δ -function is the long wavelength limit of the linear electron wave dispersion relation, also to be discussed in more detail later.

B. Equations for the low-frequency density variations

We now relate the variations of the high-frequency potential variations to the slow variations in plasma density. In general we have the normalized density variation $\eta = \eta(\mathbf{R}, T, \mathbf{r}, \tau)$, but as we have learned by (8) it suffices to find an expression terms of the reduced expression $\eta = \eta(\mathbf{R}, T)$. The simplest relation relevant here assumes that ponderomotive forces associated with the spatial variation of the high frequency wave-field sets the electrons into slow bulk motions, and the resulting charge imbalance gives rise to slowly varying electric fields that sets also the ion component into motion, thereby inducing the space-time variations of the normalized quantity η .

Averaging the electron momentum over the fast time scale ω_{pe}^{-1} we obtain an expression for the balance between the electrostatic force, the electron pressure force and the Miller force (or radiation pressure from the Langmuir waves). We thus have a simplified linear expression for the quasi-equilibrium of electrons at temperature T_e in the electrostatic and Miller force potentials as

$$\nabla \left(en_0\bar{\phi} - T_e\bar{n} - \frac{1}{2}\varepsilon_0|\nabla\phi|^2 \right) = 0. \quad (11)$$

The ion equations of motion give a relation between $\bar{\phi}$ and the quasi neutral plasma density $\bar{n}$ which finally gives a relation between η and $|\nabla\phi|^2$ to enter for instance (6) and (8).

Irrespective of the low frequency model equation to be applied, the essential element in the closure of the present set of equations is to express $|\nabla\phi|^2$ in terms of $F(\mathbf{R}, T, \mathbf{k}, \Omega)$, where we have $|\phi(\mathbf{R}, T)|^2 = \int \int_{-\infty}^{\infty} F(\mathbf{R}, T, \mathbf{k}', \Omega') d\mathbf{k}'^3 d\Omega'$. This expression can be derived in several ways. The temporal variables are immaterial here and are omitted for simplicity. Similarly we can outline the derivation in terms of one spatial coordinate without loss of generality. In terms of average and separation coordinates $x_1 \equiv X + \frac{1}{2}r$ and $x_2 \equiv X - \frac{1}{2}r$ we have

$$\begin{aligned} (\phi(x_1) - \phi(x_2))(\phi^*(x_1) - \phi^*(x_2)) &= |\phi(x_1)|^2 + |\phi(x_2)|^2 - \phi(x_1)\phi^*(x_2) - \phi^*(x_1)\phi(x_2) \\ &= \left| \phi\left(X + \frac{1}{2}r\right) \right|^2 + \left| \phi\left(X - \frac{1}{2}r\right) \right|^2 - \phi\left(X + \frac{1}{2}r\right)\phi^*\left(X - \frac{1}{2}r\right) - \phi^*\left(X + \frac{1}{2}r\right)\phi\left(X - \frac{1}{2}r\right) \\ &\equiv \rho\left(X + \frac{1}{2}r, 0\right) + \rho\left(X - \frac{1}{2}r, 0\right) - \rho(X, r) - \rho^*(X, r) \\ &\equiv \rho\left(X + \frac{1}{2}r, 0\right) + \rho\left(X - \frac{1}{2}r, 0\right) - \rho(X, r) - \rho(X, -r) \\ &\equiv \rho\left(X + \frac{1}{2}r, 0\right) + \rho\left(X - \frac{1}{2}r, 0\right) - 2\rho(X, 0) - [\rho(X, r) + \rho(X, -r) - 2\rho(X, 0)]. \end{aligned} \quad (12)$$

For small r we find that the left side of this expression approximates $r^2|d\phi/dX|^2$ while the right hand side can be approximated by terms containing two second derivatives to give

$$r^2 \left(\left| \frac{d\phi(X, 0)}{dX} \right|^2 - \frac{1}{4} \frac{d^2\rho(X, 0)}{dX^2} + \frac{d^2\rho(X, r)}{dr^2} \right) \Big|_{r=0} \approx 0$$

for small r . For the equality to be satisfied for all small r we need the parenthesis to be vanishing. We thus have generally

$$|\nabla\phi(\mathbf{R}, T)|^2 = \frac{1}{4}\nabla_R^2 \int \int_{-\infty}^{\infty} F(\mathbf{R}, T, \mathbf{k}', \Omega') dk'^3 d\Omega' + \int \int_{-\infty}^{\infty} k'^2 F(\mathbf{R}, T, \mathbf{k}', \Omega') dk'^3 d\Omega', \quad (13)$$

where the integrations run over all Ω and all of wavenumber space. The two terms in (13) have a self-evident interpretation.

1. Quasi stationary conditions

It is common practice to assume the simple quasi static limit for slow time variations

$$-C_s^2 \eta(\mathbf{R}, T) \approx \frac{\varepsilon_0}{4Mn_0} |\nabla\phi|^2, \quad (14)$$

where $C_s \equiv \sqrt{T_e/M}$, the sound speed. However, this limit implicitly imposes some restrictions on the initial conditions, and these need not always apply. The model for the nonlinear shift in plasma frequency $\frac{1}{2}\omega_{pe}\eta(0)$ becomes particularly simple in this quasi static limit by relating $\eta(0)$ to F by using (13) for $|\nabla\phi|^2$, giving

$$\eta(0) = -\frac{\varepsilon_0}{4Mn_0C_s^2} \int \int_{-\infty}^{\infty} k^2 F(\mathbf{k}', \Omega') dk'^3 d\Omega', \quad (15)$$

independent of $(\mathbf{R}, T)$.

2. Ion fluid models

For consistency we require the ion dynamics to be described by magnetized models as well. First we use a simple cold ion fluid model to identify the basic wave modes entering the problem. Since we consider weak magnetic field, we assume $\Omega_{ci} < \Omega_{pi}$, in terms of ion cyclotron and ion plasma frequency, respectively. It is well known that the dispersion relation for such a model can be written as

$$\omega^2(\omega^2 - \Omega_{ci}^2) - C_s^2 \left(k_{\perp}^2 \omega^2 + k_{\parallel}^2 (\omega^2 - \Omega_{ci}^2) \right) = 0. \quad (16)$$

illustrated in Fig. 2 and with contour curves also in Fig. 3. We identify two dispersion relations: a high frequency part $\omega > \Omega_{ci}$ and a low frequency part $\omega < \Omega_{ci}$. As limiting cases we recognize ion sound waves $\omega = C_s k_{\parallel}$ along magnetic field lines and ion cyclotron waves $\omega = \sqrt{\Omega_{ci}^2 + C_s^2 k_{\perp}^2}$ propagating in the direction perpendicular to the magnetic field lines.

The two branches shown in Fig. 2 have particular interest for decay of the high frequency electron waves by opening for a one more ion wave decay product in addition to the standard ion sound waves. For $k_{\perp} = 0$ also have the ion cyclotron resonance at $\omega = \Omega_{ci}$.

Linearizing the basic equations, with the assumption of a simple equilibrium state, with constant plasma density n_0 and homogeneous magnetic fields $\mathbf{B}_0$, the set of equations are then reduced to the ion continuity equation

$$\frac{\partial}{\partial t} \bar{n} = -n_0 \nabla \cdot \bar{\mathbf{u}} = -n_0 \nabla_{\perp} \cdot \bar{\mathbf{u}} - n_0 \frac{\partial \bar{u}_{\parallel}}{\partial z}, \quad (17)$$

and the ion momentum equation

$$n_0 M \frac{\partial}{\partial t} \bar{\mathbf{u}} = -\gamma T_i \nabla \bar{n} - e n_0 (\nabla \bar{\phi} - \bar{\mathbf{u}} \times \mathbf{B}_0), \quad (18)$$

for singly charged ions. We used the ideal gas law for the ion equation of state, $p_i = T_i n$, introducing $\gamma \equiv C_P/C_V \approx 5/3$ as the adiabatic exponent (at time called the Poisson constant), where we usually take the value for ideal gases. An overline on the potential $\bar{\phi}$ distinguishes this potential from the amplitude of the rapidly varying electron wave potential. With the sound speed defined here as $C_s^2 = (\gamma T_i + T_e)/M$, we find [17] the equation relating the low frequency density variation

$$\frac{\partial^4 \eta}{\partial t^4} - C_s^2 \nabla^2 \frac{\partial^2 \eta}{\partial t^2} - C_s^2 \Omega_{ci}^2 \frac{\partial^2 \eta}{\partial z^2} + \Omega_{ci}^2 \frac{\partial^2 \eta}{\partial t^2} = \frac{1}{4} \frac{\varepsilon_0}{n_0 M} \left(\nabla^2 \frac{\partial^2 |\nabla\phi|^2}{\partial t^2} + \Omega_{ci}^2 \frac{\partial^2 |\nabla\phi|^2}{\partial z^2} \right), \quad (19)$$

FIG. 2: Ion wave dispersion relation for weakly magnetized plasmas with $\Omega_{ci} < \Omega_{pi}$ in terms of normalized variables ω/Ω_{ci} and $\mathbf{k}a_i$ with $a_i \equiv \sqrt{(\gamma T_i + T_e)/M} / \Omega_{ci}$.

FIG. 3: The low ($\omega \leq \Omega_{ci}$) and high frequency ($\omega \geq \Omega_{ci}$) branches of the ion wave dispersion shown in Fig. 2, here shown with contour curves.

which reproduces the linear dispersion relation (16) if we set $|\nabla\phi|^2 = 0$.

We note that finite ion temperatures enter only through the sound speed C_s in (19). Without any significant loss of generality we may thus set $T_i = 0$ in the present fluid model, which gives a slight advantage for the ensuing normalizations.

The set of equations (3) and (19) form the generalization of the so called Zakharov equations [6], here applicable for weakly magnetized plasmas.

3. Kinetic models for the ion dynamics

We have the general relation for the ion contribution to the relative dielectric function of the plasma in the form

$$\varepsilon_{ir}(\Omega, \mathbf{K}) = 1 - \frac{e\bar{n}_i(\Omega, \mathbf{K})}{K^2\varepsilon_0\bar{\phi}(\Omega, \mathbf{K})}, \quad (20)$$

where $\bar{n}_i$ is the fluctuating ion density. In the literature we find [32] the analytical expression

$$\varepsilon_{ir}(\Omega, \mathbf{K}) = 1 + \frac{2\pi}{K^2} \Omega_{pi}^2 \int_0^\infty u_\perp du_\perp \sum_{j=-\infty}^\infty J_j \left(\frac{K_\perp u_\perp}{\Omega_{ci}} \right) \int_{-\infty}^\infty \frac{du_\parallel \left(\frac{j\Omega_{ci}}{u_\perp} \frac{\partial}{\partial u_\perp} + K_\parallel \frac{\partial}{\partial u_\parallel} \right) f_0(u_\parallel, u_\perp)}{\Omega - j\Omega_{ci} - K_\parallel u_\parallel}. \quad (21)$$

With (20) and (21) together with (11) and the quasi neutral approximation, $\bar{n}_e \approx \bar{n}_i \equiv \bar{n}$, we find a kinetic model for the relation between η and $|\nabla\phi|^2$ which now includes ion Landau and ion cyclotron damping [23]. We can simplify (21) for the case where the unperturbed ion velocity distribution function $f_0(u_\parallel, u_\perp)$ is a Maxwellian, possibly with different temperatures $T_\parallel$ and $T_\perp$ in the directions parallel and perpendicular to the magnetic field $\mathbf{B}_0$. We have for this case in terms of the plasma Z -function

$$\varepsilon_{ir}(\Omega, \mathbf{K}) = 1 + \frac{\Omega_{pi}^2}{K^2} \frac{M}{T_\parallel} \sum_{j=-\infty}^\infty \exp(-p^2) I_j(p^2) [1 + (z_j + jh\tau)Z(z_n)], \quad (22)$$

where $h \equiv \Omega_{ci}/(K_\parallel a_\parallel)$, $\tau \equiv T_\parallel/T_\perp$, $a_\parallel \equiv \sqrt{2T_\parallel/M}$ and $z_j \equiv (\Omega - j\Omega_{ci})/(K_\parallel a_\parallel)$. In terms of a Larmor radius $\rho_\perp \equiv (2T_\perp/M)^{1/2}/\Omega_{ci}$ we have $p \equiv K_\perp \rho_\perp/\sqrt{2}$. We introduced I_j as the Bessel function of order j . The equations containing the form (22) are amenable for a numerical analysis.

It is readily seen that for wave propagation exactly perpendicular to an externally applied homogeneous magnetic field, i.e. $k_\parallel = 0$, there will be no kinetic damping of the waves. There is an interesting paradox associated with this observation: if we let $B_0 \rightarrow 0$ we should recover the unmagnetized ion Landau damping for $\mathbf{K} \perp \mathbf{B}_0$, but as said, this effect vanishes identically for any B_0 no matter how small. The resolution of this paradox is found in noting that phase mixing of the many infinitesimally separated modes exactly reproduces the ion Landau damping [33]. This effect has its equivalent also in plasma models using multi-water bag velocity distributions [34, 35].

C. Simple exact solutions

Ignoring for the moment a small correction from η_0 in (10) we have the solution

$$F_0(\mathbf{k}, \Omega) = G_0(\mathbf{k}) \delta \left(\Omega - \frac{1}{2} \frac{v^2}{\omega_{pe}} k^2 - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} \left(\frac{k_\perp}{k} \right)^2 \right), \quad (23)$$

where $G_0(\mathbf{k}) \geq 0$ is a wavenumber power spectrum for the potential fluctuations. Since the model does not include internal frequency-wavenumber couplings in the spectrum, we have in principle $G_0(\mathbf{k})$ to be an arbitrary function. In the argument of the δ -function we recognize the approximation to the linear dispersion relation

$$\Omega + \omega_0 = \sqrt{v^2 k^2 + \omega_{pe}^2 + \omega_{ce}^2 (k_\perp/k)^2} \approx \omega_{pe} \left(1 + \frac{1}{2} (k\lambda_D)^2 + \frac{1}{2} \left(\frac{\omega_{ce}}{\omega_{pe}} \right)^2 \left(\frac{k_\perp}{k} \right)^2 \right)$$

as obtained from (1), and recalling our approximation $\omega_0 \approx \omega_{pe}$, see also Fig. 1.

Within the present approximation, we have for later reference the components of the group velocity to be

$$v_{g\parallel} \equiv \frac{\partial \Omega}{\partial k_\parallel} \approx k_\parallel \lambda_D^2 - \frac{k_\parallel k_\perp^2}{k^4} \left(\frac{\omega_{ce}}{\omega_{pe}} \right)^2, \quad (24)$$

and

$$v_{g\perp} \equiv \frac{\partial \Omega}{\partial k_\perp} \approx k_\perp \lambda_D^2 + \frac{k_\perp k_\parallel^2}{k^4} \left(\frac{\omega_{ce}}{\omega_{pe}} \right)^2, \quad (25)$$

where $\perp$ and $\parallel$ refers to the magnetic field direction also here.

D. Limit of long wavelengths and slow variations

The expressions given by (6) and (8) has a simpler limit when the variations with R and T are slow [36]. Retaining everywhere the lowest order terms in gradients of R and derivatives of T , we find from (6) the approximation

$$\begin{aligned} & \left(-\frac{\partial}{\partial T} + 2 \left(\frac{\Omega}{k^2} - \frac{v^2}{\omega_{pe}} \right) \mathbf{k} \cdot \nabla_R - \frac{\omega_{ce}^2}{\omega_{pe} k^2} \mathbf{k} \cdot \nabla_{R\perp} \right) F(\mathbf{R}, T, \mathbf{k}, \Omega) = \\ & -\frac{\omega_{pe}}{2k^2} \mathbf{k} \cdot [\nabla_R \eta(\mathbf{R}, T) \cdot \nabla_k \mathbf{k} F(\mathbf{R}, T, \mathbf{k}, \Omega)] \\ & + \frac{\omega_{pe}}{2} \left(\frac{\partial}{\partial T} \eta(\mathbf{R}, T) \right) \left(\frac{\partial}{\partial \Omega} F(\mathbf{R}, T, \mathbf{k}, \Omega) \right) \\ & + \frac{\omega_{pe}}{2k^2} (\eta(\mathbf{R}, T) \mathbf{k} \cdot \nabla_R F(\mathbf{R}, T, \mathbf{k}, \Omega) + \mathbf{k} \cdot \nabla_R (\eta(\mathbf{R}, T) F(\mathbf{R}, T, \mathbf{k}, \Omega))), \end{aligned} \quad (26)$$

where we recall that the first term in the expansion of the $\hat{C}$ -operator gives unity.

If we now assume that $F(\mathbf{R}, T, \mathbf{k}, \Omega) = F(\mathbf{R}, T, \mathbf{k}) \delta \left(\Omega - \frac{1}{2} \frac{v^2}{\omega_{pe}} k^2 - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} \left(\frac{k_\perp}{k} \right)^2 \right)$, i.e. it follows the same separation of variables as F_0 , then we can integrate (26) with respect to Ω and find with a little algebra that

$$\begin{aligned} & \left(\frac{\partial}{\partial T} + \mathbf{v}_g \cdot \nabla_R \right) F(\mathbf{R}, T, \mathbf{k}) = \frac{\omega_{pe}}{2k^2} \mathbf{k} \cdot [\nabla_R \eta(\mathbf{R}, T) \cdot \nabla_k \mathbf{k} F(\mathbf{R}, T, \mathbf{k})] \\ & - \frac{\omega_{pe}}{2k^2} \mathbf{k} \cdot [\eta(\mathbf{R}, T) \nabla_R F(\mathbf{R}, T, \mathbf{k}) + \nabla_R (\eta(\mathbf{R}, T) F(\mathbf{R}, T, \mathbf{k}))], \end{aligned} \quad (27)$$

in terms of the group velocity vector $\mathbf{v}_g$ from (24) and (25). The first term on the right hand side can be simplified somewhat, but there is little purpose in doing so here and now. The expression (27) has the generic features of a wave kinetic equation of the form used also in related studies for unmagnetized plasmas [37–39]

III. ENSEMBLE AVERAGES

As they stand, the set of basic equations (6), (8) and e.g. (14) is deterministic, and contain basically the same information as the original two equations obtained from (3) in terms of the two sets of coordinates $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$. Ensemble averaging of this original set of equations does not, however, give useful results for the present analysis. The ensemble average $\langle \rho(\mathbf{R}, T, \mathbf{r}, \tau) \rangle$ is on the other hand interpreted as the space-time varying local potential auto-correlation function and the corresponding average $\langle F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$ is the space-time varying local power spectrum of the potential fluctuations. In case the system constitutes a homogeneous time stationary random process these expressions will be independent of $\mathbf{R}$ and T , and we find the usual expression for the correlation function $\langle \rho(\mathbf{r}, \tau) \rangle$ and power-spectrum $\langle F(\mathbf{k}, \Omega) \rangle$. This ensemble averaging is straight forward for most of the terms, since they are linear in ρ and F . The only problems arise with the product $\langle \eta(\mathbf{r}, T) F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$ appearing on the right hand sides of (6) and (8). To discuss the averaging of these terms we note that the one dimensional sound wave equation can be solved analytically with the result [28]

$$\eta(Z, T) = -\frac{\varepsilon_0}{4Mn_0} \int_0^T dT' \int_{Z-C_s(T-T')}^{Z+C_s(T-T')} dZ' \frac{\partial^2}{\partial Z'^2} |E(Z', T')|^2, \quad (28)$$

where the electric field E is here assumed to be a function of the spatial coordinate Z only. The assumed initial condition is $\eta(Z, T=0) = 0$, which is natural if we imagine the high frequency wave-field to be generated initially in an otherwise uniform plasma background density. This initial condition would not be consistent with the approximation (14). More general conditions at $T=0$ can be introduced by adding the appropriate condition to (28). Partial integration is not making (28) any simpler because the integral limits have to be retained.

To get some insight into the structure of the seemingly complicated solution (28) we assume merely as an illustration, that the integrand has an analytical structure propagating with some characteristic velocity U , which could for instance represent a group velocity of a wave-packet, i.e. $\partial^2 |E|^2 / \partial Z'^2 \sim H(Z' - UT')$, where H is derived from the shape of the wave-packet. For this simple model case we readily find

$$\eta(Z, T) = -\frac{\varepsilon_0}{8Mn_0} \left(2 \frac{H(Z - UT)}{C_s^2 - U^2} - \frac{H(Z - C_s T)}{C_s - U} - \frac{H(Z + C_s T)}{C_s + U} \right), \quad (29)$$

that satisfies also the given initial condition. The first term in the parenthesis of (29) is the response in η to the forcing by H , while the two other terms represent left and right traveling sound pulses that are needed in order to fulfill the assumed initial condition. From this simple model we can conclude that for an infinitely extended initially modulated wave field we might for late times at any spatial position expect sound pulses to arrive from widely separated spatial regions. If the separation of these regions is sufficiently large, we expect that the corresponding contributions can be taken to be statistically independent. If the initial condition on $|\nabla\phi|^2$ is spatially compact, we can for large times assume that the sound pulses corresponding to the two last terms in (29) have disappeared, so that the precise initial condition is of minor importance. If the spatial support of the initial condition is $\mathcal{L}$, the time for this assumption to be applicable is $t > \mathcal{L}/C_s$. In the present study we will be mostly interested in spatially distributed systems, and will consider the limit where $\mathcal{L}/C_s \rightarrow \infty$.

The quantity η is statistically distributed over the ensemble of realizations and its statistical distribution is derived from that of $|E|^2$. We have

$$\langle \eta(Z, T) |E(Z, T)|^2 \rangle = -\frac{\varepsilon_0}{4Mn_0} \int_0^T dT' \int_{Z-C_s(T-T')}^{Z+C_s(T-T')} dZ' \frac{\partial^2}{\partial Z'^2} \langle |E(Z', T')|^2 |E(Z, T)|^2 \rangle, \quad (30)$$

where in general $|E(\mathbf{r}, t)|^2$ is related to $\langle F(\boldsymbol{\xi}, T, \mathbf{k}, \Omega) \rangle$ as already discussed. The ensemble average in the integrand of (30) invites the assumption of quasi neutrality to be imposed. In a formal notation, this assumption can be expressed symbolically as $\langle E_1 E_2 E_3 E_4 \rangle \approx \langle E_1 E_2 \rangle \langle E_3 E_4 \rangle + \langle E_2 E_3 \rangle \langle E_4 E_1 \rangle + \langle E_2 E_4 \rangle \langle E_1 E_3 \rangle$. An equivalent of this approximation will be applied here. In (6) and (8) we encounter ensemble averages as $\langle \eta(\boldsymbol{\xi}, T') F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$ where $\eta(\boldsymbol{\xi}, T')$ is given via (28), so in effect we need averages like $\langle F(\boldsymbol{\xi}, T', \mathbf{k}', \Omega') F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle$. In the present work we will advocate the approximation

$$\langle F(\boldsymbol{\xi}, T', \mathbf{k}', \Omega') F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle \approx \langle F(\boldsymbol{\xi}, T', \mathbf{k}', \Omega') \rangle \langle F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle. \quad (31)$$

Unfortunately we have no a priori reason to assume that we are dealing with a Gaussian process. On the other hand we can safely assume that the integral in (28) for large times follows a Gaussian statistic [40] and this is after all what is relevant here. The effect of the “free sound” contributions illustrated in (29) has been discussed elsewhere [28, 41]. We assume that making the assumption (31) will not imply any significant differences in the statistical properties of the integral term in (28).

The formalism summarized here can be seen as a supplement to or an extension of classical weak turbulence theories [38, 42] and many others. The aim of the present study is restricted by considering only the propagation of electrostatic electron waves in weakly magnetized plasmas. On the other hand we take care to account also for the wave dispersion, offering dynamic equations for the full space-time evolution of the local wavenumber-frequency spectrum of these waves.

IV. PERTURBATION ANALYSIS

In the following we consider ensemble averaged quantities and carry out a stability analysis of $\langle F_0(\mathbf{k}, \Omega) \rangle$ by a perturbation analysis of $\langle F(\mathbf{R}, T, \mathbf{k}, \Omega) \rangle = F_0(\mathbf{k}, \Omega) + F_1(\mathbf{R}, T, \mathbf{k}, \Omega)$ with $F_1 \ll F_0$ and $\langle \eta \rangle = \eta_0 + \eta_1$ again with $\eta_1 \ll \eta_0$. Ignoring second order terms we find that several terms disappear in the right hand sides of (6) and (8). We obtain

$$\begin{aligned} & \left[\left(\frac{1}{4} \nabla_R^2 - k^2 \right) \frac{\partial}{\partial T} + 2 \left(\Omega + \frac{v^2}{\omega_{pe}} \left(\frac{1}{4} \nabla_R^2 - k^2 \right) \right) \mathbf{k} \cdot \nabla_R - \frac{\omega_{ce}^2}{\omega_{pe}} \mathbf{k} \cdot \nabla_{R\perp} \right] F_1(\mathbf{R}, T, \mathbf{k}, \Omega) = \\ & - \frac{\omega_{pe}}{2} \left(2\mathbf{k} \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \boldsymbol{\xi}, T') \eta_1(\boldsymbol{\xi}, T') \mathbf{k} F_0(\mathbf{k}, \Omega) \right. \\ & \left. - \nabla_R \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \boldsymbol{\xi}, T') \mathbf{k} \eta_1(\boldsymbol{\xi}, T') F_0(\mathbf{k}, \Omega) \right), \end{aligned} \quad (32)$$

and

$$\begin{aligned} & \left[\left(\frac{1}{2} \nabla_R^2 - 2k^2 \right) \Omega - \mathbf{k} \cdot \nabla_R \frac{\partial}{\partial T} + \frac{1}{2} \frac{v^2}{\omega_{pe}} \left(\frac{1}{8} \nabla_R^4 + 2k^4 - k^2 \nabla_R^2 - 2(\mathbf{k} \cdot \nabla_R)^2 \right) \right. \\ & \left. - \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}} \left(\frac{1}{2} \nabla_{R\perp}^2 - 2k_\perp^2 \right) \right] F_1(\mathbf{R}, T, \mathbf{k}, \Omega) = \\ & - \frac{\omega_{pe}}{2} \left(2\mathbf{k} \cdot \hat{C}(\mathbf{R}, \mathbf{k}, T, \Omega | \boldsymbol{\xi}, T') \eta_1(\boldsymbol{\xi}, T') \mathbf{k} F_0(\mathbf{k}, \Omega) \right. \\ & \left. + \nabla_R \cdot \hat{S}(\mathbf{R}, \mathbf{k}, T, \Omega | \boldsymbol{\xi}, T') \mathbf{k} \eta_1(\boldsymbol{\xi}, T') F_0(\mathbf{k}, \Omega) \right). \end{aligned} \quad (33)$$

We now Fourier transform with respect to T , so that $\partial/\partial T \rightarrow -iW$ and spatial positions so that $\nabla_R \rightarrow i\mathbf{K}$. The operators $\hat{C}$ and $\hat{S}$ can be expressed in terms of series containing W and $\mathbf{K}$, best seen by (5).

We have [20] the relation

$$\sinh\left(\frac{1}{2}\mathbf{K} \cdot \nabla_k - \frac{1}{2}W \frac{\partial}{\partial \Omega}\right) F_0(\mathbf{k}, \Omega) = \frac{1}{2} \left(F_0\left(\mathbf{k} + \frac{1}{2}\mathbf{K}, \Omega - \frac{1}{2}W\right) - F_0\left(\mathbf{k} - \frac{1}{2}\mathbf{K}, \Omega + \frac{1}{2}W\right) \right)$$

and

$$\cosh\left(\frac{1}{2}\mathbf{K} \cdot \nabla_k - \frac{1}{2}W \frac{\partial}{\partial \Omega}\right) F_0(\mathbf{k}, \Omega) = \frac{1}{2} \left(F_0\left(\mathbf{k} + \frac{1}{2}\mathbf{K}, \Omega - \frac{1}{2}W\right) + F_0\left(\mathbf{k} - \frac{1}{2}\mathbf{K}, \Omega + \frac{1}{2}W\right) \right),$$

as readily understood by identifying the series expansions of cosh and sinh with the appropriate Taylor expansions of $F_0(\mathbf{k} \pm \frac{1}{2}\mathbf{K}, \Omega \pm \frac{1}{2}W)$. Using (23) for F_0 we can integrate with respect to Ω , but we will not here pursue solutions of these basic equations here.

We have now obtained a set of equations that can account for the space-time evolution and in particular also the stability of a given turbulent spectrum of electron waves in a weakly magnetized plasma. It is interesting to note how significant the differences (as compared to the unmagnetized case [41]) are in the present study: even weak magnetic fields give rise to significant changes in the basic model. These changes are reflected also in the corresponding stability analysis.

V. CONCLUSION

The foregoing analysis can have interest as a general formalism, but its short time goal, a stability analysis, can be derived differently. As it turns out, the perturbation analysis can be formulated in a much simpler way, with the principle being illustrated for instance in the appendix of [43]. This analysis will be reported separately.

Acknowledgments

One of the authors (H.L.P.) acknowledges valuable discussions with K. B. Dysthe, L. Stenflo and J. K. Trulsen.

-
- [1] J. Bonnell, P. Kintner, J.-E. Wahlund, and J. A. Holtet. Modulated Langmuir waves: Observations from Freja and SCIFER. *J. Geophys. Res.: Space Phys.*, 102:17233–17240, 1997. doi:10.1029/97JA01499.
 - [2] T. M. Burinskaya, J. L. Rauch, and M. M. Mogilevskii. Spectra of Langmuir waves in a magnetized plasma with low-frequency turbulence. *Plasma Phys. Reports*, 30:756–760, 2004. doi:10.1134/1.1800220.
 - [3] J. LaBelle, I. H. Cairns, and C. A. Kletzing. Electric field statistics and modulation characteristics of bursty Langmuir waves observed in the cusp. *J. Geophys. Res.*, 115:A10317, 2010. doi:10.1029/2010JA015277.
 - [4] D. Píša, G. B. Hospodarsky, W. S. Kurth, O. Santolík, J. Souček, D. A. Gurnett, A. Masters, and M. E. Hill. Statistics of Langmuir wave amplitudes observed inside Saturn’s foreshock by the Cassini spacecraft. *J. Geophys. Res.: Space Phys.*, 120:2531–2542, 2015. doi:10.1002/2014JA020560.
 - [5] C. Briand. Langmuir waves across the heliosphere. *J. Plasma Phys.*, 81:325810204, 2015. doi:10.1017/S0022377815000112.
 - [6] V. E. Zakharov. Collapse of Langmuir waves. *Zh. Eksp. Teor. Fiz.*, 62:1745–1759, 1972. See also (1972) Sov. Phys.-JETP, **35**, 908–914.
 - [7] M. Kono and M. M. Škorić. *Nonlinear Physics of Plasmas*. Number 62 in Springer Series on Atomic, Optical, and Plasma Physics. Springer, Heidelberg, Germany, 2010.
 - [8] K. B. Dysthe and H. L. Pécseli. Non-linear Langmuir wave modulation in collisionless plasmas. *Plasma Phys.*, 19:931–943, 1977. doi:10.1088/0032-1028/19/10/004.
 - [9] V. I. Petviashvili. Three-dimensional solitons of extraordinary and plasma waves. *Fiz. Plazmy*, 1:128, 1975. Russian original Sov. J. Plasma Phys. **1**, 15 (1975).
 - [10] M. Porkolab and M. V. Goldman. Upper-hybrid solitons and oscillating-two-stream instabilities. *Phys. Fluids*, 19:872–881, 1976. doi:10.1063/1.861553.

- [11] V. V. Krasnosel'skikh and V. I. Sotnikov. Plasma-wave collapse in a magnetized plasma. *Fiz. Plazmy*, 3:872–879, 1977. russian original Sov. J. Plasma Phys., **3**, 491-495 (1977).
- [12] H. P. Freund and K. Papadopoulos. Oscillating two-stream and parametric decay instabilities in a weakly magnetized plasma. *Phys. Fluids*, 23:139–146, 1980. doi:10.1063/1.862854.
- [13] H. L. Pécseli. Solitons and weakly nonlinear waves in plasmas. *IEEE Trans. Plasma Sci.*, PS-13:53–86, 1985. doi:10.1109/TPS.1985.4316366.
- [14] Z. Z. Kasymov, E. Näslund, A. N. Starodub, and L. Stenflo. Upper hybrid turbulence in a plasma with magnetized electrons. *Phys. Scripta*, 31:201–204, 1985. doi:10.1088/0031-8949/31/3/008.
- [15] A. N. Kaufman and L. Stenflo. Upper-hybrid solitons. *Phys. Scripta*, 11:269–270, 1975. doi:10.1088/0031-8949/11/5/005.
- [16] K. B. Dysthe, E. Mjølhus, H. L. Pécseli, and L. Stenflo. Nonlinear electrostatic wave equations for magnetized plasmas. *Plasma Phys. Contr. Fusion*, 26:443–447, 1984. doi:10.1088/0741-3335/26/2/005.
- [17] K. B. Dysthe, E. Mjølhus, H. L. Pécseli, and L. Stenflo. Langmuir solitons in magnetized plasmas. *Plasma Phys.*, 20:1087–1099, 1978. doi:10.1088/0032-1028/20/11/001.
- [18] R. Pozzoli and D. D. Ryutov. Modulational instability produced by Langmuir turbulence in a magnetic field. *Phys. Fluids*, 22:1782–1789, 1979. doi:10.1063/1.862815.
- [19] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, 1932. doi:10.1103/PhysRev.40.749.
- [20] I. E. Alber. Effects of randomness on stability of two-dimensional surface wavetrains. *Proc. Roy. Soc. London*, 363:525–546, 1978. doi:10.1098/rspa.1978.0181.
- [21] K. Rönnmark and J. Larsson. Local spectra and wave distribution functions. *J. Geophys. Res.: Space Phys.*, 93:1809–1815, 1988. doi:10.1029/JA093iA03p01809.
- [22] N. L. Shatashvili and N. L. Tsintsadze. Nonlinear Landau damping phenomenon in a strongly turbulent plasma. *Phys. Scripta*, T2B:511–516, 1982. doi:10.1088/0031-8949/1982/T2B/031.
- [23] K. B. Dysthe and H. L. Pécseli. Non-linear Langmuir wave modulation in weakly magnetized plasmas. *Plasma Phys.*, 20:971–989, 1978. doi:10.1088/0032-1028/20/10/001.
- [24] K. B. Dysthe, E. Mjølhus, H. L. Pécseli, and L. Stenflo. Nonlinear electrostatic wave equations for magnetized plasmas - II. *Plasma Phys. Contr. Fusion*, 27:501–508, 1985. doi:10.1088/0741-3335/27/4/011.
- [25] G. Brodin and L. Stenflo. A new decay channel for upper-hybrid waves. *Phys. Scripta*, 91:104005, 2016. doi:10.1088/0031-8949/91/10/104005.
- [26] N. Marcuvitz. Quasiparticle view of wave-propagation. *Proc. IEEE*, 68:1380–1395, 1980. doi:10.1109/PROC.1980.11880.
- [27] J. C. Bhakta and D. Majumder. Effect of finite spectral width on the modulational instability of Langmuir-waves. *J. Plasma Phys.*, 30:203–209, 1983. doi:10.1017/S0022377800001124.
- [28] K. B. Dysthe, M. S. Espedal, and H. L. Pécseli. Langmuir turbulence - proposal for a closure. *Phys. Scr.*, 33:246–253, 1986. doi:10.1088/0031-8949/33/3/012.
- [29] L. P. J. Kamp. Local frequency spectra for nonlinear wave equations. *J. Phys. A: Mathematical and General*, 20:3279–3291, 1987. doi:10.1088/0305-4470/20/11/031.
- [30] N. Marcuvitz. The quasiparticle view of wave propagation. *Proc. IEEE*, 79:1350–1358, 1991. doi:10.1109/5.104209.
- [31] E. P. Kontar and H. L. Pécseli. Nonlinear development of electron-beam-driven weak turbulence in an inhomogeneous plasma. *Phys. Rev. E*, 65:066408, 2002. doi:10.1103/PhysRevE.65.066408.
- [32] A. V. Timofeev and V. I. Pistunovich. Transport processes in a plasma. In M. A. Leontovich, editor, *Reviews of Plasma Physics*, volume 5, pages 401–445. Consultants Bureau, 1965.
- [33] D. E. Baldwin and G. Rowlands. Plasma oscillations perpendicular to a weak magnetic field. *Phys. Fluids*, 9:2444–2453, 1966. doi:10.1063/1.1761637.
- [34] G. Rowlands. Landau damping and the water-bag model. *Phys. Letters A*, 30:408 – 409, 1969. doi:10.1016/0375-9601(69)90736-1.
- [35] M. Navet and P. Bertrand. Multiple "water-bag" model and Landau damping. *Phys. Letters A*, 34:117 – 118, 1971. doi:10.1016/0375-9601(71)90232-5.
- [36] P. K. Shukla and T. Farid. Nonlinear propagation of broadband upper-hybrid waves in collisional magnetoplasmas. *Phys. Scripta*, T82:60–62, 1999. doi:10.1238/Physica.Topical.082a00060.
- [37] M. Camac, A. R. Kantrowitz, M. M. Litvak, R. M. Patrick, and H. E. Petschek. Shock waves in collision-free plasmas. *Nuclear Fusion, Suppl.*, 2:423–445, 1962.
- [38] A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov. Oscillations and instability of a weakly turbulent plasma. *Plasma Phys.*, 9:719–735, 1967. doi:10.1088/0032-1028/9/6/305.
- [39] A Hasegawa. *Plasma Instabilities and Nonlinear Effects*, volume 8 of *Physics and Chemistry in Space*. Springer-Verlag, Berlin, 1975.
- [40] H. Tennekes and J. L. Lumley. *A First Course in Turbulence*. The MIT press, Cambridge, Massachusetts, 1972.
- [41] H. L. Pécseli. Modulational stability of electron plasma wave spectra. *J. Plasma Phys.*, 80:745–769, 2014. doi:10.1017/S002237781400035X.
- [42] B. B. Kadomtsev. *Plasma Turbulence*. Academic Press, New York, 1965.
- [43] M. Kono and H. L. Pécseli. Parametric decay of wide band Langmuir wave-spectra. *J. Plasma Phys.*, 82:905820606, 2016. doi:10.1017/S0022377816001070.