ALGORITHMIC NETWORKS: CENTRAL TIME TO TRIGGER EXPECTED EMERGENT OPEN-ENDEDNESS

FELIPE S. ABRAHÃO

KLAUS WEHMUTH

ARTUR ZIVIANI

National Laboratory for Scientific Computing (LNCC)
25651-075 – Petropolis, RJ – Brazil

ABSTRACT. This article investigates emergence and complexity in complex systems that can share information on a network. To this end, we use a theoretical approach from information theory, computability theory, and complex networks. One key studied question is how much emergent complexity arises when a population of computable systems is networked compared with when this population is isolated. First, we define a general model for networked theoretical machines, which we call *algorithmic networks*. Then, we narrow our scope to investigate algorithmic networks that optimize the average fitnesses of nodes in a scenario in which each node imitates the fittest neighbor and the randomly generated population is networked by a time-varying graph. We show that there are graph-topological conditions that make these algorithmic networks have the property of expected emergent open-endedness for large enough populations. In other words, the expected emergent algorithmic complexity of a node tends to infinity as the population size tends to infinity. Given a dynamic network, we show that these conditions imply the existence of a central time to trigger expected emergent open-endedness. Moreover, we show that networks with small diameter compared to the network size meet these conditions. We also discuss future research based on how our results are related to some problems in network science, information theory, computability theory, distributed computing, game theory, evolutionary biology, and synergy in complex systems.

emergence; complexity; open-endedness; networked Turing machines; information; time centrality; small diameter; complex systems

[2010]68Q30; 68Q05; 05C82

1. Introduction

The general context of our research is to mathematically investigate emergence and complexity when a population of complex systems is networked. That is, we study emergence and complexity in a system composed of interacting complex systems. As supported by [9], the pursuit of a universal framework for the problem of complexity in complex networks is paramount for a wide range of topics in complex systems. The present work relies on an intersection between information theory, computability theory, game theory, distributed computing, multiagent systems, communication complexity, adaptive complex systems, and biology. In fact, we discuss in further detail in Section 10 that the investigated problem is connected to questions ranging, for instance, from the problem of symbiosis [53], cooperation [7, 8, 44], and integration [34, 59] to biological [80], economic [7, 70, 71], and social [55] networks.

In particular, we study the problem of how a group of randomly generated computable systems give rise to emergent phenomena through the exchange of information and how it would affect the overall performance, fitness, or payoff. We create a theoretical toy model from which one can investigate how much emergent complexity the whole group has on average when they are networked compared to when they are isolated. As we will formally show later in this paper, the randomly generated algorithmic network $\mathfrak{N}_{BB}$ that plays the Busy Beaver imitation game (BBIG) — see Section 3 — is a mathematical object useful to prove fruitful theorems using well-known results from statistics, computability theory, information theory, and graph theory.

Thus, how does one define a measure of average emergent information/complexity for a randomly generated population of Turing machines? Within the context of this abstract toy model, we answer this question by generalizing the results for open-ended evolutionary systems under algorithmic information theory (AIT), as presented in [1–4, 22–24, 45], in order to present a new mathematical phenomenon of *open-endedness*. We will show that there are graph-topological conditions that trigger *expected emergent open-endedness* (EEOE), i.e., that trigger an unlimited increase of expected emergent complexity as the population size goes to infinity (see Section 4). It is an akin — but different — phenomenon to evolutionary open-endedness [13, 25, 52, 69, 76]: Instead of achieving an unbounded quantity of complexity over time (or successive mutations), an unbounded quantity of emergent complexity is achieved as the population size increases when this population is networked (see Section 4.2).

We start by defining a general encompassing mathematical model that we call algorithmic networks in Section 2. This definition relies on a population of arbitrarily chosen theoretical machines and relies on a multiaspect graph [81], making the nodes in the respective multiaspect graph to correspond to this population of systems/programs (see Definition 2.2.3) such that edges are communication channels that nodes/systems can use to send and receive information. Thus, as it was

our intention described in the first paragraph, an algorithmic network is a network composed of algorithms as nodes, with each node representing a computable system.

Then, we introduce in Section 3 a particular model of synchronous dynamic algorithmic networks $\mathfrak{N}_{BB}$ that is based on simple imitation of the fittest neighbor: a type of algorithmic network that plays the BBIG. A network Busy Beaver game is a game in which each player is trying to calculate the biggest integer[1] it can using the information shared by its neighbors. The BBIG is a variation of the Busy Beaver game in which every node can only propagate the biggest integer, taking into account the one produced by itself and the ones from their neighbors. It configures a simple imitation-of-the-fittest procedure. Thus, these algorithmic networks $\mathfrak{N}_{BB}$ can be seen as playing an optimization procedure where the whole pursues the increase of the average fitness/payoff through diffusing on the network the best randomly generated solution (see discussion on Section 3.1).

We present our main Theorem 8.1 proving that there is a lower bound on the expected emergent algorithmic complexity for algorithmic networks $\mathfrak{N}_{BB}$. Additionally, we prove Corollary 8.1.1 showing that this lower bound can be calculated from a diffusion measure like cover time [29]. Further, from this corollary, we also prove in Theorem 8.2 that there are asymptotic conditions on the increasing power of the cover time as a function of the population size such that they ensure that there is a central time to trigger EEOE. Then, we introduce in Section 9 a small modification on the family of multiaspect graphs of $\mathfrak{N}_{BB}$ with the purpose of investigating what would happen if the time-varying networks present a small diameter compared to the network size, i.e. $D(G_t, t) = \mathbf{O}(\lg(N))$. Indeed, in this case, we show in Corollary 9.1 that a small diameter is sufficient to ensure the existence of a central time[2] to trigger EEOE for algorithmic networks $\mathfrak{N}_{BB}$ — even in a computably bigger number of cycles compared to the temporal diffusion diameter.

Additionally, we discuss in Section 10 future research from how our results are related to problems in network science, statistical (or probabilistic) information theory, computability theory, distributed computing, game theory, evolutionary biology, and synergy in complex systems. We also choose to add discussions in the Sections in order to improve the readability and explanation of the new definitions and new models that we will introduce in the present article. Finally, Section 11 concludes the paper. Complementarily, the appendix shows extended versions of the proofs of Lemmas 8.1, 8.2, 8.3, 8.4, 8.5, and 8.6, as well as of Theorems 8.1 and 8.2.

2. Algorithmic Networks

2.1. **Discussion on algorithmic networks.** In this section we will define a general mathematical model for the study of networked machines which can share information with each other across their respective network while performing their computations. We want to define it in a general sense in order to allow future variations, to add specificities and to extend the model presented in Section 3, while still being able to formally grasp a mathematical analysis of systemic features like the emergence of information and complexity along with its related phenomena:

---

[1] As established as our measure of fitness or payoff. See Section 3.
[2] Or just to trigger EEOE, if the graph is static. See 9.1.

for example, the average emergent open-endedness in our case (see discussion 3.1 and Section 4).

Since, following this general approach, one can see these mathematical models as a merger of algorithmic (and statistical) information theory and complex networks theoretically combining both distributed computing (or multiagent systems) and game theory, we refer to it as '**algorithmic networks**'. The main idea is that a population of formal theoretical machines can use communication channels over the graph's edges. Thus, the graph topology makes this population be networked. Once the elements of the population start to exchange information, it forms a overarching model for a system composed of interacting subsystems. So, note that algorithmic networks will be networks of algorithms — which is the reason of its chosen name —, should each theoretical machine represent a computable system. Indeed, in the present article we will consider each node as a program of a universal Turing machine (see definitions 3.5 and 3.13), which justifies calling either the nodes or the elements of the population of an algorithmic network as **nodes/programs** hereafter.

The term 'algorithmic network' is also employed to visually represent systems or processes with circuits of algorithms [47, 49, 54] in order to tackle the problem of computationally modeling these processes. Differently, we employ the expression with the purpose of mathematically representing a model for networked populations of computable systems from which one can investigate systemic properties and prove theorems. Thus, a possible disambiguation may be using the expression '**circuit-modeling** algorithmic network' for the former case and '**population-systemic** algorithmic network' for our present approach — on which, for the sake of simplifying our notation, we will use only the expression 'algorithmic network' in this article. Although the usage in the present article shares the general goal of mathematically representing models for systems through networked algorithms, it may be seen as a generalization[3] of the circuit-modeling approach. Note that, as we will discuss in Subsection 2.4, there may be several different aspects of the graph corresponding to different properties of the population such that a "circuit of algorithms" would not grasp. For example, there may be different nodes being the same Turing machine (that is, the population of programs may contain repeated elements), there may be no need of a 'delay operator' for multi-step loops, and the graph may not be static. In fact, as we will present in Section 3, these and other features will occur in $\mathfrak{N}_{BB}$.

2.2. **Discussion on graphs and complex networks.** One can have populations with very different properties and several different graphs linking them. Thus, first a general mathematical representation for graphs is paramount — see Definition 2.1 . Second, we need a way to make abstract aspects of these graphs correspond to properties of the population of theoretical machines, which will be formalized in Definition 2.2.

Aiming a wider range of different network configurations, as mentioned in Sections 1 and 2.1, we ground our formalism about graph representations on Multi-Aspects graphs (MAG) as presented in [81]. Therefore, one can mathematically represent abstract aspects that could appear in complex networks. For example, dynamical (or time-varying) networks, multicolored nodes and multilayer networks.

---

[3] If one accepts a formalization of algorithms using theoretical machines (e.g. Turing machines).

Moreover, it facilitates network analysis by showing that their aspects can be isomorphically mapped into a classical directed graph. Thus, MAG abstraction has proven to be crucial to establish connections between the characteristics of the network and the properties of the population composed of theoretical machines (see Definition 2.2.3).

2.3. **Definition.**

Definition 2.1. As defined in [29, 81], let $G = (\mathscr{A}, \mathscr{E})$ be a graph, where $\mathscr{E}$ is the set of edges of the graph and $\mathscr{A}$ is a class of sets, each of which is an aspect.

Note 2.1.1. Note that $\mathscr{E}$ determines the (dynamic or not) topology of G.

Note 2.1.2. Each aspect in $\mathscr{A}$ determines which variant of a graph G will be (and how the set $\mathscr{E}$ will be defined). As in [29], we will deal only with time-varying graphs G_t hereafter, so that there will be only two aspects ($|\mathscr{A}| = 2$): the set of nodes (or vertices) $\mathrm{V}(G_t)$ and the set of time instants $\mathrm{T}(G_t)$. An element in $V(G_t) \times T(G_t)$ is a **composite vertex** (or composite node). The graphs G_t will be better explained in Section 3 .

2.4. **Discussion on the general model.** One can think of an algorithmic network in a broad sense as a theoretical distributed computing representation model in which each node (or vertex) computes using network's shared information, returning a final output[4] or not[5] . The computation of each node may be seen in a combined point of view or taken as individuals. Respectively, nodes/programs may be computing using network's shared information to solve a common purpose — as the classical approach in distributed computing — or, for example, nodes may be competing[6] with each other. For the present purposes, we are interested in the average fitness (or payoff), and its related emergent complexity that may arise from a process that increases the average fitness.

An algorithmic network may have several different configurations, so that the following Definition 2.2 will not specify how a particular algorithmic network works. Instead, our formalism enables one to represent every (or most) variation of algorithmic networks with the purpose of modeling a particular problem that may arise from a networked complex system. For example, the networked population may be synchronous 3.8 or asynchronous, have a set of information sharing protocols 3.9 or none, a randomly generated population 3.6 or a fixed one, with communication costs or without them[7] etc. In addition, the network topology that determines the communication channels may be dynamical 3.1 , with weighted edges, multilayer etc. In fact, in accordance with the goal of this article, we will prove theorems on a particular model of algorithmic networks which will be defined in Section 3 . However, in this section we limit ourselves to present a general definition of algorithmic networks. Also the reader is invited to note that a circuit-modeling

---

[4] Which is our current case. If the maximum number of cycles is finite, then every node/program must return a final output.

[5] An algorithmic network may have a non limited number of cycles, so that each node may remain returning an endless number of partial outputs. This is also the case for circuit-modeling algorithmic networks. However, we do not tackle this problem in the present article.

[6] This game theoretical approach will be discussed further in Section 10.

[7] The current model of this article does not consider any cost of communication or of using the communication channels to send information.

algorithmic network $[47, 49, 54]$ is one of these possible different configurations of population-systemic algorithmic network (see Subsection 2.1).

2.5. **Definitions.**

Definition 2.2. We define an **algorithmic network** $\mathfrak{N} = (G, \mathfrak{P}, b)$ upon a population of theoretical machines $\mathfrak{P}$, a graph $G = (\mathscr{A}, \mathscr{E})$ and a function b that makes aspects of G correspond[8] to properties of $\mathfrak{P}$, so that a node in $V(G)$ corresponds one-to-one to an element of $\mathfrak{P}$. The graph G was previously defined in 2.1, and we will define $\mathfrak{P}$ and b in definitions 2.2.1 and 2.2.3 , respectively.

Definition 2.2.1. Let the **population** $\mathfrak{P}$ be a subset of L, where L is the language on which the chosen theoretical machine U are running.[9] Each member of this population may receive inputs and return outputs through communication channels.

Note 2.2.1.1. The choice of L and U determines the class of nodes/systems. For example, one may allow only time-bounded Turing machines in the population. In the present work, **L** will be a self-delimiting universal programming language for a simple extended universal Turing machine **U'** (see Definition 3.5) — i.e., an oracle Turing machine — that returns zero whenever a non-halting computation occur.

Definition 2.2.1.1. Let $\mathfrak{C}$ be a set of the maximum number of cycles that any node/program o in the population $\mathfrak{P}$ can perform in order to return a final output. A **node cycle** in an algorithmic network $\mathfrak{N}$ is defined as a node/program[10] returning a **partial output** (which, depending on the language and the theoretical machine the nodes are running on, is equivalent to a node completing a halting computation)[11] and sharing (or not) this partial output with its neighbors (accordingly to a specific information sharing protocol or not — see Definition 3.11).

Note 2.2.1.1.1. So, if the algorithmic network is asynchronous, a cycle can be seen as an individual communication round that doesn't depend on whether its neighbors are still running or not, while if the network is synchronous a cycle can be seen as the usual communication round in synchronous distributed computing. Also note that one may also refer to a **network cycle**, which denotes when all nodes of the algorithmic network have returned their **final outputs** (if it is the case). Thus, a network cycle must not be confused with a node cycle.

Definition 2.2.2. A **communication channel** between a pair of elements from $\mathfrak{P}$ is defined in $\mathscr{E}$ by an edge (whether directed or not) linking this pair of nodes/programs.

Note 2.2.2.1. A directed edge (or arrow) determines which node/program sends output to another node/program that takes this information as input. An undirected edge (or line) may be interpreted as two opposing arrows.

Definition 2.2.3. Let

$$b \colon Y \subseteq \mathscr{A}(G) \to X \subseteq Pr(\mathfrak{P})$$
$$\overline{\mathbf{a}} \quad \mapsto \quad b(\overline{\mathbf{a}}) = \overline{\mathbf{p_r}}$$

---

[8] See Definition 2.2.3 .

[9] A population of machines $\mathfrak{P}$ is a set of programs $p \in$ L (i.e., more formaly, Turing machines or other variants on language L).

[10] Once there is a mapping of the set of nodes into the population of programs, the expression "node/program" becomes well-defined within the theory of algorithmic networks.

[11] In the present article for example — see Definition 3.5 .

be a function that **maps** a subspace of aspects Y in $\mathscr{A}$ into a subspace of properties X in the set of properties $Pr(\mathfrak{P})$ of the respective population in the graph $G = (\mathscr{A}, \mathscr{E})$ such that there is an bijective function $f_{V\mathfrak{P}}$ such that, for every $(v, \overline{\mathbf{x}}) \in Y \subseteq \mathscr{A}(G)$ where $b(v, \overline{\mathbf{x}}) = (o, b_{|Y|-1}(\overline{\mathbf{x}})) \in X$, v is a vertex, o is a element of $\mathfrak{P}$ and

$$f_{V\mathfrak{P}} \colon V(G) \to \quad \mathfrak{P}$$
$$v \quad \mapsto f_{V\mathfrak{P}}(v) = o$$

Definition 2.3. We say a element $o_i \in \mathfrak{P}$ is **networked** when there is $\mathfrak{N} = (G, \mathfrak{P}, b)$ such that o_i is running on it.

Definition 2.3.1. We say o_i is **isolated** otherwise. That is, it is only functioning as a element of $\mathfrak{P}$.

Definition 2.4. We say that an input $w \in \mathbf{L}$ is a **network input** *iff* it is the only external source of information every node/program receives and it is given to every node/program before the algorithmic network begins any computation.

3. The Busy Beaver Imitation Game (BBIG)

3.1. **Discussion on the Busy Beaver imitation model, emergence, complexity, optimization, and synchronicity.** In this section, we will formalize a toy model for comparing the emergence of algorithmic complexity from one of the simplest forms of information sharing that percolates through the network: imitation of the fittest. As a toy model, our theoretical simple object of investigation must be general, abstract and highly tractable, while enabling further variations and extensions with the purpose of studying more properties of networked complex systems for example (see Section 9). In the present section we choose to give a full discussion on how one characterizes the algorithmic network $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ (formerly denoted only as $\mathfrak{N}_{BB}$) before listing any formal definition. This way, the reader might get a full picture of underlying ideas and motivations around this model of algorithmic network that plays the Busy Beaver imitation game.

Take a randomly generated set of programs. They are linked, constituting a network which is represented by a graph. Each node/program is trying to return the "best solution" it can. And eventually one of these nodes/programs end up being generated carrying beforehand a "best solution" for the problem in question. This "best solution" is spread through the network by a diffusion process in which each node imitates the fittest neighbor if, and only if, its shared information is "better" than what the very node can produce. The question is: how much more complexity can this diffusion process generates on the average compared to the best nodes/programs could do if isolated?

A comparison between the complexity of what a node/program can do when networked and the complexity of the best a node/program can do when isolated will give the **emergent algorithmic complexity** of the algorithmic network. In the present case, the networked "side of the equation" relies only on the simple imitation of the fittest neighbor. Since this kind of imitating procedure is one of the simplest or "worst" ways to use neighbors' partial outputs to get closer to a

best solution[12] , we are interested in obtaining the emergent algorithmic complexity that arises from a "worst" networked case compared to the best isolated case.

Indeed, a possible interpretation of the diffusion described to the above is **average optimization through diffusion** in a random sampling. While optimization through selection in a random sampling may refer to evolutionary computation or genetic algorithms for example [36], in which the best solution eventually appears and remains sufficiently stable over time, in our model optimization is obtained in a manner that a best solution also eventually appears, but is diffused over time in order to make every individual as averagely closer to the best solution as they can. Therefore, the underlying goal of this process would be to **optimize the average fitness** of the population using the least amount of diffusion time — we will come back to this in Section 10. This type of optimization would particularly be better suited for the cases when adding new nodes/programs (or subparts) is cheaper than adding new cycles (or computational resources).

But how does one measure complexity of what a node/program computes? We ground our complexity analysis on algorithmic information theory (AIT). As a well-established mathematical field in theoretical computer science and information science it has proven to be a powerful tool to achieve analytical results, proving sound lemmas and theorems, in order to investigate and build theories on how complexity changes over time. See [1, 3, 4, 22–24, 45, 46]. These works present mathematical results on evolutionary **open-endedness** for computable complex systems. That is, a process in which systems that could be fully simulated on a theoretical Turing machine gain a unlimited amount of complexity over time as random mutations and natural selection[13] apply on them.

Therefore, as mentioned before, a different form of open-endedness plays the central role in the fundamental characteristics and consequences of the results we will present here (see Section 4): the **average emergent open-endedness**. Instead of asking about how complex systems become over time, as in evolutionary open-endedness, we are focusing another akin question: how complex systems (in fact, systems composed of interacting systems [71] [7]) become when the number of its subparts increases? Or, more specifically in our case, how much more emergent complexity arises on the average when the number of networked systems increases? In other words, we are interested in how synergy [14, 40, 41, 65] (see also discussion 10) among interacting systems may have an impact on the emergence of complexity.

In order to tackle this problem we choose to first work under the framework of classical algorithmic complexity, as in [1–4, 22–24, 45]. Furthermore, it will give a direct way to measure fitness of each node/program as explained below. Despite being a theoretical and abstract model for complex systems, it gives the solid foundation and fruitful framework to develop further quasi-isomorphic extensions from resource-bounded — with a closer application to computer simulations — and hipercomputable versions, as analogously did in [1,3,4]. However, in this article the population of nodes/programs will be composed of arbitrary programs running on

---

[12] Not necessarily given by the randomly generated node/program with the higher fitness, since a node/program could use its neighbor's partial output to calculate a even bigger integer for example. Thus, this is the reason we say non formally that simply imitating the fittest neighbor is one of the "worst". Nevertheless, we leave the actual mathematical investigation of what would be the less effective way to get averagely closer to a better solution for future research.

[13] Which can be mathematically described by a fitness function and a selection procedure.

an arbitrarily chosen universal Turing machine. It makes the definition of algorithmic complexity 3.4 straightforwardly applicable. But, analogously in [22], it forces the need of a simple and restrict hipercomputable procedure to deal with eventual non-halting programs (see definitions 3.5 and 3.7).

The second toy-modeling property that our algorithmic network $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ will have is **synchronicity**, which will be defined as a population property in 3.8 . As in synchronous distributed computing, communication rounds are forced to happen at the same time, so that only at the end of each cycle 2.2.1.1 each node/program is allowed to exchange information by sending partial outputs and receiving partial outputs from its neighbors as input for the next cycle. Indeed, an asynchronous version of our results and how it relates to one presented here are paramount for future research.

3.2. **Discussion on dynamic graphs and measures of diffusion.** As in Section 2, we will start defining topological properties of networks. Following a pursuit of overarching mathematical theorems, we choose to deal with **time-varying** (or dynamical) directed graphs [60] [43] [29]. The static case is covered by a particular case of dynamical networks in which topology does not change over time — that is, each temporal snapshot $\mathcal{G}_t$ is the equal [43]. And the undirected case can be seen as a graph in which each undirected edge (or line) represents two opposing directed edges (or arrows). Moreover, the dynamical case sets proper theoretical foundations for future research, should one be interested in studying systemic emergent properties of algorithmic networks with a cycle-varying population for example (i.e., an algorithmic network that changes its population size as the number of cycles increases).

Since we aim a formalization of a model for optimizing diffusion with the purpose of investigating emergent complexity, we need a diffusion measure that can be applied on dynamical networks. **Cover time** is a diffusion measure that gives the average time intervals in which a fraction τ of the nodes is "infected" [43] [29]. Besides being useful in order to measure diffusion on dynamical networks, it offers some advantages on domain conditions that other measures like the average geodesic distance (or average shortest path length) does not. See notes 3.4.2 and 3.4.2.1. The algorithmic networks that we will define below get their graph topologies from a family of dynamical graphs that has a certain cover time function as a common feature. This function has the domain on population sizes, time instants and fractions τ of nodes (see Definition 3.4).

Next we will define the properties of the population composed of nodes/programs. Remember in Section 2 that both aspects of a graph and properties of a population on a theoretical machine (as well as a function b_j that maps aspects into properties) are necessary in order to properly formalize an algorithmic network. As mentioned in the beginning of the present section, the diffusion process must rely on an imitation of the fittest. So one does not only need to define how fitness[14] is measured but also to mathematically state the exact procedure from which each diffusion step is determined.

3.3. **Discussion on the population playing the BBIG.** As in [1–4, 22–24], we use the **Busy Beaver function** as our **fitness function**. Naming bigger integers relates directly to increasing algorithmic complexity [22], which will allow us to

---

[14] Or payoff in a game-theoretical approach.

establish crucial probabilistic and statistical properties of a randomly generated population in Lemma 8.1. Remember that there is a diffusion of the "best solutions" during the cycles. Now the "best solution" assumes a formal interpretation of fittest final output (or payoff). The choice of the word "solution" for naming bigger integers now strictly means a solution for the Busy Beaver problem. Also note that several uncomputable problems are equivalently reduced to the Busy Beaver, including the halting problem. Thus, these mathematical features supports the Busy Beaver function as a sound and meaningful choice for a fitness function for a toy model [1, 23]. Not only on a resource-boundless case (e.g. Turing machines) in which finding the best solution might be reducible to a first order uncomputable problem (in the Turing hierarchy) — which is the case presented in this article —, but also for more realistic resource-bounded versions of the Busy Beaver, as shown in [1, 3, 4, 32, 86]. Such resource-bounded versions may be useful to model optimization problems in which finding the best solution falls under a higher time complexity class[15] and, as we will mention in Section 10, we leave for future research.

Thus, with a fixed fitness function that works as a universal[16] parameter for every node/program's final (and partial) output it makes sense to have an interpretation of these running algorithmic networks $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ as playing a **network Busy Beaver game**: during the cycles each node is trying to use the information shared by its neighbors to return the biggest integer it can. The bigger the final output integer the better the payoff (or fitness). However, only after we define the language and the theoretical machine in 3.5 the notions of partial output and algorithmic complexity — necessary for a fitness measure — will be well-defined.

Furthermore, a definition of the *imitation* part of the game is still required. Under what circumstances should a node/program imitate a neighbor? How the best partial outputs are diffused through the network? We will give a formal definition of the **information sharing protocols** that each node/program must follow when networked in 3.9. But before we will define the language and universal Turing machine for a population of programs. Doing so, it makes the **algorithmic complexity** — as mentioned in open-endedness — well-defined as a direct consequence and, moreover, its correspondent algorithmic probabilities ground our definition of **randomly generated population**.

We use the term ***protocol*** as an abstraction of its usage in distributed computing and telecommunications. A protocol is understood as a set of rules or algorithmic procedures that nodes/program must follow at the end of each cycle when communicating. For example, it can be seen as the "rules for the communications" under a game-theoretical perspective. So there must be a computable procedure determining what each node/program do when receiving inputs and sending outputs to its neighbors. In fact, as our model of algorithmic network works under a simple imitation of the fittest neighbor, the protocol which must be followed by every node/program determines the exact procedure in doing this imitation. Hence, we call this global information sharing protocol as **imitation-of-the-fittest protocol** (IFP). We will define these simple algorithmic procedures in 3.9 . The main

---

[15] Note that time complexity refers to computational complexity in which computation is limited by a certain amount of time.

[16] As a universal parameter it may not work as a measure of how well adapted a system is in respect to its respective environment or in respect to generate more offsprings. It is not in our present scope to discuss the problem of measuring fitness and adaptation in adaptive complex systems.

idea is that each node/program o_i compares its neighbors' partial output (that is, the integer they have calculated in the respective cycle) and runs the program of the neighbor that have output the biggest integer if, and only if, this integer is bigger than the one that the node/program o_i has output. Since $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ is playing the Busy Beaver game on a network while limited to simple imitation performed by a randomly generated population of programs, we say it is playing a **Busy Beaver imitation game**.

Finally, we end this section by using these previous definitions and the definition of the function b_j, which "binds together" network and programs, in order to define $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ in 3.13 . It is an algorithmic network populated by N nodes/programs (that constitutes population $\mathfrak{P}_{BB}(N)$) such that, after the first (or c_0 cycles) cycle, it starts a diffusion[17] process of the biggest partial output (given at the end of the first cycle) determined by a time-varying graph G_t that belongs to a family of graphs $\mathbb{G}(f, t, \tau)$. Then, at the last time instant diffusion stops and one cycle (or more) is spent in order to make each node return a final output.

3.4. Definitions.

Definition 3.1. As defined in [29], let $G_t = (\mathrm{V}, \mathscr{E}, \mathrm{T})$ be a **time-varying graph** (TVG), where V is the set of nodes, T is the set of time instants, and $\mathscr{E} \subseteq \mathrm{V} \times \mathrm{T} \times \mathrm{V} \times \mathrm{T}$ is the set of edges.

Notation 3.1.1. Let $\mathrm{V}(G_t)$ denote the set of nodes (or vertices) of G_t.

Notation 3.1.2. Let $|\mathrm{V}(G_t)|$ be the size of the set of nodes in G_t.

Notation 3.1.3. Let $\mathrm{T}(G_t)$ denote the set of time instants in $G_t = (\mathrm{V}, \mathscr{E}, \mathrm{T})$.

Notation 3.1.4. Let $G_t(t)$ denote the graph G_t at time instant $t \in T(G_t)$

Definition 3.1.1. We define the set of time instants of the graph G_t as $T(G_t) = \{t_0, t_1, \ldots, t_{|T(G_t)|-1}\}$.

Note 3.1.1.1. For the sake of simplifying our notations in the theorems below one can take a natural ordering for $T(G_t)$ such that

$$\forall i \in \mathbb{N} \ (0 \leq i \leq |T(G_t)| - 1 \implies t_i = i + 1)$$

.

Definition 3.2. Let $d_t(G_t, t_i, u, \tau)$ be the number of time instants (steps, time intervals [60] or, in our case, cycles) for a diffusion starting on node u at time instant t_i to reach a fraction τ of nodes in the graph G_t.

Definition 3.3. We define the **cover time** for time-varying graphs as

$$CT(G_t, t_i, \tau) = \begin{cases} \frac{1}{|\mathrm{V}(G_t)|} \displaystyle\sum_{u \in \mathrm{V}(G_t)} d_t(G_t, t_i, u, \tau) & \text{if } \tau \text{ reached;} \\ \infty & \text{otherwise;} \end{cases}$$

---

[17] There are other diffusions too, since it is possible that two randomly generated neighbors are not close to the node/program with the highest fitness and have different integers as first partial outputs. However, only the one from the biggest partial output is independent of neighbor's partial outputs, so that it disseminates in any situation.

Notation 3.3.1. Let $D(G_t, t)$ denote the **temporal diffusion diameter** of the graph G_t taking time instant t as the starting time instant of the diffusion process. That is,

$$D(G_t, t) = \begin{cases} max\{x \mid x = d_t(G_t, t, u, 1) \land u \in V(G_t)\}; \\ \infty \quad if \ \forall x \in \mathbb{N} \forall u \in V(G_t)\big(x \neq d_t(G_t, t, u, 1)\big); \end{cases}$$

Definition 3.4. Let

$$\mathbb{G}(f, t, \tau) = \{G_t \mid i = |V(G_t)| \land f(i, t, \tau) = CT(G_t, t, \tau) \land$$
$$\land \ \forall i \in \mathbb{N}^* \exists! G_t \in \mathbb{G}(f, t, \tau)(|V(G_t)| = i)\}$$

where

$$f \colon \mathbb{N}^* \times X \subseteq T(G_t) \times Y \subseteq]0, 1] \to \mathbb{N}$$
$$(x, t, \tau) \qquad \mapsto y$$

be a **family** of unique time-varying graphs which shares $f(i, t, \tau) = CT(G_t, t, \tau)$, where i is the number of nodes, as a common property. The number of nodes of each graph in this family may vary from 1 to ∞.

Note 3.4.1. The results in this article can be equally achieved with a weaker assumption within this family: one can instead define a family of graphs upon an arbitrary function f such that $f(i, t, \tau) = CT(G_t, t, \tau)$ **only** when $N \to \infty$.

Note 3.4.2. Note that a well-defined family $\mathbb{G}(f, t, \tau)$ for every $t \in T(G_t)$ and $\tau \in]0, 1]$ implies that $CT(G_t, t, \tau) \neq \infty$. Hence, in this case, the graphs $G_t \in \mathbb{G}(f, t, \tau)$ must be strongly temporal-connected for every time instant (and, thus, also cyclic) — See [81] —, so that every composite node can reach any other composite node given a sufficient amount of time intervals.

Note 3.4.2.1. For our main results to hold one may weaken conditions on τ and t, because our theorems assume arbitrary values as long as they belong to their respective intervals or set and the cover time has a function f well-defined for that respective domain. In fact, this is one of the advantages of using the cover time as a diffusion metric for dynamic graphs.

Notation 3.1. Let $\lg(x)$ denote the binary logarithm $\log_2(x)$.

Notation 3.2. Let $\mathbf{U}(x)$ denote the output of a universal Turing machine $\mathbf{U}$ when x is given as input in its tape. So, $\mathbf{U}(x)$ denote a partial recursive function $\varphi_{\mathbf{U}}(x)$ that is a universal partial function [68]. If x is a non-halting program on $\mathbf{U}$, then this function $\mathbf{U}(x)$ is undefined for x.

Notation 3.3. Let $\mathbf{L_U}$ be a binary self-delimiting universal programming language for a universal Turing machine $\mathbf{U}$ such that there is a concatenation of strings $w_1, ..., w_k$ in the language $\mathbf{L_U}$, which preserves[18] the self-delimiting (prefix-free) property of the resulting string, denoted by

$$w_1 \circ ... \circ w_k \in \mathbf{L_U}$$

Notation 3.4. Let $\mathbf{L_U}$ be a binary self-delimiting universal programming language for a universal Turing machine $\mathbf{U}$. The (prefix) **algorithmic complexity** (Kolmogorov complexity, program-size complexity or Solomonoff-Komogorov-Chaitin

---

[18] For example, by adding a prefix to the entire concatenated string $w_1 w_2 ... w_k$ that encodes the number of concatenations. Note that each string was already self-delimiting.

complexity) of a string $w \in \mathbf{L_U}$, denoted by $A(w)$, is the size of the smallest program $p^* \in \mathbf{L_U}$ such that $\mathbf{U}(p^*) = w$.

Note 3.4.1. The reader may also find in the literature the prefix algorithmic complexity denoted by $H(w)$ or — more frequently used — $K(w)$. As introduced in Section 1 and presented in Section 10, there might be several intersections with other fields in future work. Thus, we choose a self-explaining approach on notation in order to avoid ambiguity and notation conflicts in future work, such that we would denote the (prefix) algorithmic complexity/information[19] by $I_A(w)$. However, for the sake of simplifying our notation, we choose to denote it only by $A(w)$ in the present article.

Definition 3.5. Given a binary self-delimiting universal programming language $\mathbf{L_U}$ for a universal Turing machine $\mathbf{U}$, where there is a constant $\epsilon \in \mathbb{R}$, with $0 < \epsilon \leq 1$, and a constant $0 \leq C_L \in \mathbb{N}$ such that, for every $N \in \mathbb{N}$,

$$A(N) \leq \lg(N) + (1 + \epsilon)\lg(\lg(N)) + C_L$$

we then define an oracle[20] Turing machine $\mathbf{U'}$ such that for every $w \in \mathbf{L_U}$:

$$\mathbf{U'}(w) = \begin{cases} \mathbf{U}(w) + 1 & \text{if } \mathbf{U} \text{ halts on } w \\ 0 & \text{if } \mathbf{U} \text{ does not halt on } w \end{cases}$$

Note 3.5.1. The oracle Turing machine is basically (except for a trivial bijection) the same as the chosen universal Turing machine. The oracle is only triggered to know whether the program halts or not in first place. Also note that $\mathbf{U}'(w)$ is a **total** function, and not a partial function as $\mathbf{U}(w)$ — see Definition 3.2.

Note 3.5.2. Note that from algorithmic information theory (AIT) we know that the algorithmic complexity (see next Definition 3.4) $A(\mathbf{U'}(w))$ only differs from $A(\mathbf{U}(w))$ by a constant if $\mathbf{U}$ halts on w. This constant is always limited by the size of the smallest program that always adds 1 to any other halting computation. This is the reason why the algorithmic complexity of the final outputs of node/programs in $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ only differ by a constant, should nodes be halting programs of $\mathbf{U}$ instead of $\mathbf{U'}$. And it also makes Lemma 8.2 sound.

Definition 3.6. We say a a population $\mathfrak{P} \subseteq \mathbf{L_U}$ is **randomly generated** *iff* $\mathfrak{P}$ is a sample generated by $|\mathfrak{P}|$ trials accordingly to an algorithmic probability where, for a constant C,

$$p \in \mathfrak{P}$$

$$iff$$

$$\mathbf{P}[p] = C\frac{1}{2^{|p|}}$$

Note 3.6.1. The constant C is important for us because it allows us to characterise population $\mathfrak{P}_{BB}(N)$ in Definition 3.11 as randomly generated, taking into account that there are global information sharing protocols.

---

[19] That is, the algorithmic information contained in a object about itself [50].

[20] Or any hypercomputer with a respective Turing degree higher than or equal to 1.

Definition 3.7. We say a population $\mathfrak{P}$ is **sensitive to oracles** *iff* whenever an oracle is triggered during any cycle in order to return a partial output the final output of the respective node/program is also 0.[21] Or more formally:

Let $p_{net_{\mathbf{U}}}$ be a program such that $\mathbf{U}(p_{net_{\mathbf{U}}} \circ o_i \circ c)$ computes on machine $\mathbf{U}$ cycle-by-cycle what a node/program $o_i \in \mathfrak{P}$ does on machine $\mathbf{U}$ until cycle c when networked. Let $p_{o_i,c}$ be the **partial output** sent by node/program o_i at the end of cycle c to every neighbor (at the end of the arrow or directed edge) of o_i. Also, $p_{o_i,max\{c|c\in\mathfrak{C}\}}$ denotes the final output of the node/program o_i. Then, for every $o_i \in \mathfrak{P}$,

$$\nexists p_{net_{\mathbf{U}}} \implies p_{o_i,max\{c|c\in\mathfrak{C}\}} = 0$$

Definition 3.8. In a **synchronous** population of an algorithmic network each node/program is **only** allowed to receive inputs from its incoming neighbors and to send information to its outgoing neighbors at the end of each cycle (or communication round[22]), except for the last cycle. Each cycle always begins and ends at the same time even if the computation time of the nodes/programs is arbitrarily different. Or more formally:

Let G_t be a time-varying graph such that each time interval corresponds to nodes sending information to their neighbors at the same time. Then, there is an injective function

$$f \colon T(G_t) \to \quad \mathfrak{C}$$
$$t \quad \mapsto f(t) = t$$

Note 3.8.1. If the population is isolated, then only each partial output counts in respect to the respective element of the population.

Definition 3.8.1. In the **last cycle**, every node only returns its **final output**.

Definition 3.9. We say a population $\mathfrak{P}$ follows an **imitation-of-the-fittest protocol** (IFP) *iff* each node/program always obeys protocols defined in 3.9.1 , 3.9.2 and 3.9.3 **when networked**. Or more formally:

Let $\mathbf{X}_{neighbors}(o_j,c)$ be the set of incoming neighbors of node/program o_j that have sent partial outputs to it at the end of the cycle c. Let $\{p_{o_i,c} \mid o_i \in \mathbf{X}_{neighbors}(o_j,c) \wedge i \in Y \subset \mathbb{N} \wedge c \in \mathfrak{C}\}$, where Y is finite, be the set of partial outputs relative to $\mathbf{X}_{neighbors}(o_j,c)$. Let w be the network input as defined in 2.4. Let $\circ$ denote a recursively determined concatenation of finite strings. Then, for every $o_j, o_i \in \mathfrak{P}$ and $c, c-1 \in \mathfrak{C}$,

(1) if $max\{c \mid c \in \mathfrak{C}\} = 1$, then

$$p_{o_j,c} = \mathbf{U}'(o_j \circ w)$$

(2) if $c = 1$ and $c \neq max\{c \mid c \in \mathfrak{C}\}$, then

$$p_{o_j,c} = w \circ o_j \circ \mathbf{U}'(o_j \circ w)$$

(3) if $c \neq 1$ and $c \neq max\{c \mid c \in \mathfrak{C}\}$, then
$p_{o_j,c} = w \circ o_i \circ max\{x \mid p_{o_j,c-1} = w \circ o_i \circ x \ \vee \ w \circ o_i \circ x \in \{p_{o_i,c-1} \mid o_i \in \mathbf{X}_{neighbors}(o_j,c-1) \wedge i \in Y \subset \mathbb{N} \wedge c-1 \in \mathfrak{C}\}\}$

(4) if $c = max\{c \mid c \in \mathfrak{C}\}$ and $p_{o_j,c-1} = w \circ o_i \circ x$, then

$$p_{o_j,c} = x$$

---

[21] Since we are assuming 0 as the assigned non-halting output for $\mathbf{U}'$ in relation to the machine $\mathbf{U}$.

[22] As in distributed computing.

Note 3.9.1. Since we will be working with synchronous algorithmic networks, these global sharing protocols applies at the end of each cycle (or communication round) — see Definition 2.2.1.1 . So, after the first cycle the diffusion of the biggest partial output works like a **spreading** in time-varying networks [43] [29]. And the last cycle (or more cycles — see Definition 3.13) is spent in order to make each node/program return a number — from which we measure the complexity of the respective node/program as discussed in Subsection 3.1 .

Note 3.9.2. In order to simplify our notation we let $w \circ \mathbf{U}'(x)$ denote the prefix preserving concatenation $\circ$ (see Notation 3.3) of the string $w \in \mathbf{L_U}$ with the string $y \in \mathbf{L_U}$ such that y represents the number $\mathbf{U}'(x)$ in the language $\mathbf{L_U}$.

Definition 3.9.1. We call a **Busy Beaver contagion protocol** as a global information sharing protocol in which every node/program runs the node/program of the neighbor that have output — a partial output — the biggest integer instead of its own program *iff* the partial output of this neighbor is bigger than the receiver's own partial output.

Note 3.9.1.1. Note that a node/program only needs to take into account the biggest partial output that any of its neighbors have sent. If more than one sends the biggest integer as partial output, the receiver node/program choose one of these respective neighbors accordingly to an arbitrary rule. Then, this partial output is the one that will be compared to the partial output from the receiver node/program.

Definition 3.9.2. In a **maximally cooperative protocol** every node/program shares its own program and latest partial output with all its neighbors at the end of each cycle and before the next cycle begins.

Note 3.9.2.1. In the model defined in 3.11 sharing only the last partial output turns out to be equivalent[23] to the definition of maximal cooperation.

Note 3.9.2.2. Remember that $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ only lets its nodes/programs perform computation in the first cycle (see Definition 3.9.3). This is the reason why only the network input w matters in its respective maximally cooperative protocol.

Note 3.9.2.3. As in [29] this diffusion process may be interpreted as following a Breadth-First Search (BFS), in which each node starts a diffusion by sending the specified information in Definition 3.9 to all of its adjacent nodes. Then, these adjacent nodes relay information for their own adjacent nodes in the next time instant, and so on.

Note 3.9.2.1. Typically, one can set a collection of time instants in which the diffusion of information is limited to 1 step. However, the presented final results 8.1 , 8.1.1 and 8.2 also hold if more than one step per time instant of diffusion (in our case, the application of the global sharing protocols) is allowed at the end of each cycle. In fact, it can only make $\tau_{\mathbf{E}(max)}(N, f, t_i, \tau)|_t^{t'}$ bigger — see Lemma 8.6 .

Definition 3.9.3. We call a **contagion-only protocol** as a global information sharing protocol in which every node/program only plays the Busy Beaver contagion and does not perform any other computation after the first cycle **when networked** in some $\mathfrak{N} = (G, \mathfrak{P}, b)$.

---

[23] However, at the expense of using more computation time.

Note 3.9.3.1. This is the condition that allows us to investigate the "worst" (see discussion 3.1) case in which no node/program spends computational resources other than playing its global sharing protocols.[24] In other words, it forces the algorithmic network to rely on a diffusion process only.

Note 3.9.3.2. In the main model presented in this article the first time instant occurs after the first cycle — see 3.13 .

Definition 3.10. Let $\mathbf{L}_{BB} \subset \mathbf{L_U}$ be a language of programs of the form $P_{prot} \circ p$ where $p \in \mathbf{L_U}$ and the prefix P_{prot} is any program that always makes $P_{prot} \circ p$ obey when running on $\mathbf{U'}$ the **imitation-of-the-fittest protocol when it is networked**. Otherwise, if it is **isolated**, then $\mathbf{U'}(P_{prot} \circ p) = \mathbf{U'}(p)$ and every subsequent cycle works like a reiteration of partial outputs as immediate next input for the same node/program.[25]

Definition 3.11. Let $\mathfrak{P}_{BB}(N) \subseteq \mathbf{L}_{BB}$ be a **randomly generated** population of N elements that is **synchronous**[26], **sensitive to oracles** and with randomly generated[27] suffixes $p \in \mathbf{L}_{\mathfrak{P}_{BB}(N)}$ such that[28]

$$p \in \mathbf{L}_{\mathfrak{P}_{BB}(N)}$$
$$iff$$
$$P_{prot} \circ p \in \mathfrak{P}_{BB}(N) \subseteq \mathbf{L}_{BB}$$

Note 3.11.1. Note that all conditions and protocols in 3.11 define the set of properties $Pr(\mathfrak{P}_{BB}(N))$ of the population $\mathfrak{P}_{BB}(N)$ as in 2.2.3 .

Note 3.11.2. There is a misplaced usage of the operator $\subseteq$ in $\mathfrak{P}_{BB}(N) \subseteq \mathbf{L}_{BB}$ here. Since $\mathfrak{P}_{BB}(N)$ is a population, it may contain repeated elements of $\mathbf{L}_{BB}$. However, for the sake of simplicity, we say a population $\mathfrak{P}$ is contained in a language $\mathbf{L}$ *iff*

$$\forall p_i, 1 \le i \le |\mathfrak{P}| \left(p_i \in \mathfrak{P} \implies p_i \in \mathbf{L} \right)$$

Definition 3.12. For the sake of simplifying our notation, we denote the set of the size-ordered[29] N smallest

$$p \in \lim_{N \to \infty} \mathbf{L}_{\mathfrak{P}_{BB}(N)}$$

as

$$\mathbf{L_U}(N)$$

---

[24] In our case, generating expected emergent algorithmic complexity of a node.

[25] Thus, the isolated case may be represented (and is equivalent to) by same algorithmic network built on a population in language $\mathbf{L_U}$ that does not follow any information sharing protocol and the topology of the multiaspect graph is composed by one-step loops on each node/program only.

[26] The procedure responsible for performing the synchronization may be abstract-hypothetical or defined on an underlying oracle machine that makes each individual cycle start at the same time (or after every node/program returns its partial output in the respective cycle).

[27] Note that we are dealing with self-delimiting languages, so that one can always define algorithmic probabilities.

[28] Hence, $\mathbf{L}_{\mathfrak{P}_{BB}(N)}$ is defined as the set of suffix programs that were randomly generated in order to constitute $\mathfrak{P}_{BB}(N)$.

[29] That is, an ordering from the smallest to the biggest size. Also note that ordering members with the same size may follow a chosen arbitrary rule.

Definition 3.13. Let

$$\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$$

be an algorithmic network where f is an arbitrary well-defined function such that

$$f \colon \mathbb{N}^* \times X \subseteq T(G_t) \times Y \subseteq]0,1] \to \mathbb{N}$$
$$(x, t, \tau) \qquad\qquad \mapsto y$$

and $G_t \in \mathbb{G}(f, t, \tau)$, $|V(G_t)| = N$, $|T(G_t)| > 0$ and there are arbitrarily chosen[30] $c_0, n \in \mathbb{N}$ where $c_0 + |T(G_t)| + 1 \le n \in \mathbb{N}$ such that b_j is a injective function

$$b_j \colon V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n$$
$$(v, t_{c-1}) \quad \mapsto b_j(v, t_{c-1}) = (o, c_0 + c)$$

such that[31] , since one has fixed the values of c_0 and n,

$$\left| \left\{ b_j \middle| \begin{array}{l} b_j \colon V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n \\ (v, t_{c-1}) \quad \mapsto b_j(v, t_{c-1}) = (o, c_0 + c) \end{array} \right\} \right| \le N^N$$

Note 3.13.1. Thus, $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ is an algorithmic network populated by N nodes/programs from $\mathfrak{P}_{BB}(N)$ such that, after the first (or c_0 cycles) cycle, it starts a diffusion[32] process of the biggest partial output (given at the end of the first cycle) determined by a time-varying graph G_t that belongs to a family of graphs $\mathbb{G}(f, t, \tau)$. Then, at the last time instant diffusion stops and one cycle (or more) is spent[33] in order to make each node return a final output. Each node returns as final output its previous partial output determined at the last time instant — see Definition 3.9.3 .

Note 3.13.2. Note that in this model the aspects of the graphs in the family $\mathbb{G}$ that are mapped into the properties of the population $\mathfrak{P}_{BB}(N)$ by functions b_j are nodes and time instants.

Note 3.13.3. The reader is invited to note that the main results presented in this paper also hold for only one function b_j per graph in the family $\mathbb{G}$ (see 8.1.3).

Definition 3.13.1. We denote $\mathbb{N}|_1^n$ as $\mathfrak{C}_{\mathfrak{BB}}$.

4. Average Emergent Open-Endedness (AEOE)

4.1. **Discussion on emergence and complexity.** In this section we will introduce a formal definition of average emergent open-endedness (AEOE). Some of the current discussion is already presented in 3.1 [34] . Thus, we will abstain from addressing the problem of how AEOE relates to the BBIG. Instead, we will discuss general aspects and the idea that the former mathematically grasps.

As discussed in Section 2 , on a general approach one can understand the theory of algorithmic networks as a mathematical framework for studying interacting complex systems. However, we are limiting ourselves to deal with computable complex

---

[30] Since they are arbitrarily chosen, one can take them as minimum as possible in order to minimize the number of cycles for example. That is, $c_0 = 0$ and $n = |T(G_t)| + 1$ for example.

[31] See Definition 2.2.3 .

[32] There other diffusions too. However, only the one from the biggest partial output is independent of neighbor's partial outputs.

[33] This condition is necessary to make this algorithmic network defined even when $|T(G_t)| = 1$.

[34] And we recommend its reading before the current subsection.

systems that sends and receives information from neighbors (during cycles) in order to return final outputs. This way, as mentioned in discussion 3.1 , we can apply the tools from theoretical computer science and information theory — specially algorithmic information theory — on the study of how a systemic feature like complexity "behaves" as this system of interacting subsystems runs (see discussion 3.1). So, our central inquiry falls under investigating how much complexity emerges when subsystems are interacting compared to when they are running completely isolated from each other.

We follow a consensual notion of **emergence** $[28, 33, 37, 45, 58, 63]$ as a systemic feature or property that appears only when the system is analyzed (theoretically or through computer simulations) as a "whole". Thus, the algorithmic complexity (i.e., an irreducible number of bits of information) of a node/program's final output when networked[35] minus the algorithmic complexity of a node/program's final output when isolated formally defines an irreducible quantity[36] of information that **emerges** in respect to a node/program[37] that belongs to an algorithmic network. We call it as **emergent algorithmic complexity (EAC)** of a node/program. Note that if a system is analyzed as a separated — the subparts do not need to be necessarily apart from each other, but each part in this case would be taken as an object of investigation where no information enters or exits anyway — collection of "subparts", the EAC of a node/program will be always 0.

An important distinction is crucial: the EAC of a node/program must be not confused with EAC of the **entire** algorithmic network. Measuring the emergent algorithmic complexity of the algorithmic network taking into account every node/program "at the same time" is — as our intuition demands to be — mathematically different from looking at each individual final output's algorithmic complexity. For example, one may consider the algorithmic information of each node/program combined (in a non-trivial way) with the algorithmic information of the network's topology $[18, 56, 85]$. Within the framework of algorithmic networks, this "whole" emergent algorithmic complexity can be formally captured by the **joint** algorithmic complexity of each node/program's final output when networked minus the **joint** algorithmic complexity of each node/program's final output when isolated. However, analyzing this systemic property is not of the scope of the present article and it will be a necessary future research. Specially in investigating different emergent complexity phase transitions as discussed in Subsection 4.2.

Therefore, instead of investigating the joint EAC of an algorihtmic network, one may look for a mean value of EAC for all nodes/programs. The **average** emergent algorithmic complexity (AEAC) is defined by the mean on all nodes/programs' (and possible network's topologies) EAC (see Definition 4.3). It gives the average complexity of the nodes/programs' respective fitnesses (or, in a game-theoretical interpretation, payoff) in a networked population, once there is a fitness function that evaluates final outputs. For further discussions, see Subsection 3.1 .

As the main object of mathematical investigation of this article is an algorithmic network $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ in which the population is randomly generated from a stochastic process of independent and identically distributed random variables

---

[35] That is, interacting with other parts of the system.

[36] Note that this quantity of bits may be 0 or negative.

[37] More precisely, in respect to a node's final output, i.e. the complexity of its fitness/payoff as discussed in Subsection 3.3.

under an algorithmic probability distribution, we refer to the average EAC in $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ as **expected** emergent algorithmic complexity (EEAC), denoted here 4.4 by

$$\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c) \right)$$

4.2. **Discussion on open-endedness.** Definition 4.3 will tell us how much emergent algorithmic complexity emerges on the average in an algorithmic network $\mathfrak{N}$. The question is: which algorithmic networks produces more AEAC? And an immediate question would be: how does AEAC increase as the population grows in size? More AEAC means that a node/program needs more irreducible information on the average than it already contains, should it try to compute isolated what it does networked. A system with a bigger AEAC "informs" or "adds" more information to its parts on the average. Thus, the investigation of these questions are directly related to important topics in complex systems, like integration [59], synergy (see Subsection 10), and emergence [28, 33, 37, 45, 58, 63].

Another important concept mentioned in discussion 3.1 is open-endedness. It is commonly defined in evolutionary computation and evolutionary biology as the inherent potential of a evolutionary process to trigger an endless increase in complexity [1, 5, 13, 25, 45, 52, 69, 76]. That means that in the long run eventually will appear an organism that is as complex as one may want.[38] Given a certain complexity value as target, one would just need to wait a while in order to appear an organism with a bigger complexity than or equal to the target value, no matter how big this value is. In fact, within the framework of algorithmic information theory as shown in [1, 22–24], a cumulative[39] evolution reaches N bits of complexity after — realistic fast — $\mathbf{O}(N^2(\log(N))^2)$ successive mutations on one organism at the time — whether your organisms are computable, sub-computable or hyper-computable (see [1, 3, 4]).

What we have found is that within the theory of algorithmic networks open-endedness is a mathematical phenomenon closely related to the core questions in the first paragraph of this subsection. However, it emerges as an akin — but different — phenomenon to evolutionary open-endedness: instead of achieving an unbounded quantity of algorithmic complexity over time (or successive mutations), an unbounded quantity of emergent algorithmic complexity is achieved as the population size increases. Since it is a property that emerges depending on the amount of parts of a system only when these parts are interacting somehow (e.g. exchanging information), it arises as an emergent property. So, we refer to it as **emergent open-endedness**.

The main results of this paper (see 8.1, 8.1.1, 8.2 and 9.1) prove that there are conditions of the network and protocols of the population that trigger a type of emergent open-endedness as the randomly generated populations grows. They make the AEAC increases as one may want, should the population size increases sufficiently. Since it is an increase in average complexity (see discussion 4.1), we refer to it as **average emergent open-endedness (AEOE)** (see Definition 4.5). Moreover, since the population is randomly genarated so that AEAC turns to EEAC in the case of algorithmic networks $\mathfrak{N}_{BB}(N, f, t, \tau, j)$, one may refer to AEOE as

---

[38] As a consequence, an infinite number of different organisms tends to appear in the evolutionary line after an infinite time, or after an infinite amount of successive mutations.

[39] Which allows organisms to recall its predecessors

expected emergent open-endedness (EEOE) (see Definition 4.5.1). For example, even if the AEAC is 0 or negative for some algorithmic networks with finite size populations, for bigger enough population sizes the probability that these algorithmic networks have a bigger AEAC tends to 1 — see note 8.1.1. In other words, we will show that there are conditions for algorithmic networks such that a phase transition in AEAC eventually occurs, should the AEAC be non-positive for small populations. Thus, one may call this transition as **average emergent complexity phase transition**.

4.3. **Definitions.**

Definition 4.1. The **emergent algorithmic complexity (EAC)** of a node/program o_i in c cycles is given in an algorithmic network that always produces partial and final outputs by

$$\overset{net(b)}{\underset{iso}{\Delta}} A(o_i, c) = A(\mathbf{U}(p_{net}^b(o_i, c))) - A(\mathbf{U}(p_{iso}(o_i, c)))$$

where:

(1) $o_i \in \mathbf{L}$;
(2) p_{net}^b is the program that computes cycle-per-cycle the partial outputs of o_i when networked assuming the position v, where $b(v, \bar{\mathbf{x}}) = (o_i, b(\bar{\mathbf{x}}))$, in the graph G in the specified number of cycles c with network input w;
(3) p_{iso} is the program that computes cycle-per-cycle the partial outputs of o_i when isolated in the specified number of cycles c with network input w;

Note 4.1.1. Note that:

(1) $A(\mathbf{U}(p_{net}^b(o_i, c)))$ is the algorithmic complexity of what the node/program o_i does **when networked**;
(2) $A(\mathbf{U}(p_{iso}(o_i, c)))$ is the algorithmic complexity of what the node/program o_i does **when isolated**;

Note 4.1.2. While program p_{iso} may be very simple, since it is basically a program that reiterates partial outputs of o_i as inputs to itself at the beginning of the next cycle (up to c times), program p_{net}^b may also comprise giving the sent partial outputs from o_i's incoming neighbors at the end of each cycle as inputs to o_i at the beginning of the respective next cycle, so that it may be only described by a much more complex procedure.

Note 4.1.3. Note that the algorithmic complexity of p_{iso} or p_{net}^b may be not directly linked to $A(\mathbf{U}(p_{iso}(o_i, c)))$ or $A(\mathbf{U}(p_{net}^b(o_i, c)))$ respectively, since the $A(\mathbf{U}(p_{iso}(o_i, c)))$ and $A(\mathbf{U}(p_{net}^b(o_i, c)))$ are related to the final outputs (if any) of each node's computation.

Note 4.1.4. Remember definitions 3.7 and 3.5 which states that even when a node/program does not halt in some cycle, machine $\mathbf{U}'$ was defined in order to assure that there is always a partial output for every node/program for every cycle. If population $\mathfrak{P}$ is defined in a way that eventually a partial or final output is not obtained when running on the respective theoretical machine (see Definition 2.2.1), Definition 4.1 would be inconsistent. This is the reason we stated in its formulation that there always is partial and final outputs. However, it is not necessary in the case of $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ (see Definition 3.11 and 4.1.1).

Note 4.1.5. If one defines the **emergent creativity** of a node/program as

$$A(\mathbf{U}(p_{net}^b(o_i,c))|\mathbf{U}(p_{iso}(o_i,c))$$

our results also hold for replacing the expected emergent algorithmic complexity (EEAC) with expected emergent algorithmic creativity (EEACr). Since we are estimating lower bounds, note that from AIT we have that

$$A(\mathbf{U}(p_{net}^b(o_i,c))|\mathbf{U}(p_{iso}(o_i,c)) \geq A(\mathbf{U}(p_{net}^b(o_i,c))) - A(\mathbf{U}(p_{iso}(o_i,c))) + \mathbf{O}(1)$$

Definition 4.1.1. More specifically, one can denote the **emergent algorithmic complexity** of a node/program o_i **in an algorithmic network** $\mathfrak{N}_{BB}(N,f,t,\tau,j)$ (**EAC$_{\mathbf{BB}}$**) during c cycles as

$$\overset{net(b_j)}{\underset{iso}{\Delta}} A(o_i,c) = A(\mathbf{U}(p_{net}^{b_j}(o_i,c))) - A(\mathbf{U}(p_{iso}(p_i,c)))$$

where:

(1) $o_i = P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N) \subseteq \mathbf{L}_{BB}$;
(2) $p_{net}^{b_j}$ is the program that computes cycle-per-cycle what a program o_i does **when networked** assuming the position v, where $b_j(v) = (o_i)$, in the graph G_t in c cycles with network input w;
(3) p_{iso} is the program that computes cycle-per-cycle what a program p_i does **when isolated**[40] in c cycles with network input w;

Note 4.1.1.1. In order to check if Definition 4.1.1 is well-defined, remember the definitions 3.5 and 3.10 .

Definition 4.2 (Notation)**.** For the sake of simplifying our notation, let $\{b_j\}$ denote

$$\left\{ b_j \left| \begin{array}{l} b_j \colon V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n \\ \quad (v,t_{c-1}) \quad \mapsto b_j(v,t_{c-1}) = (o,x+c) \end{array} \right. \right\}$$

and b_j in the sum $\sum\limits_{b_j}$ denote

$$\begin{array}{l} b_j \colon V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n \\ \quad (v,t_{c-1}) \quad \mapsto b_j(v,t_{c-1}) = (o,x+c) \end{array}$$

Definition 4.3. We denote the **average emergent algorithmic complexity of a node/program (AEAC)** for an algorithmic network $\mathfrak{N} = (G,\mathfrak{P},b)$, as

$$\mathbf{E}_{\mathfrak{N}}\left(\overset{net}{\underset{iso}{\Delta}} A(o_i,c) \right) =$$

$$= \sum_b \frac{\frac{\sum\limits_{o_i \in \mathfrak{P}} \overset{net(b)}{\underset{iso}{\Delta}} A_{(o_i,c)}}{N}}{|\{b\}|}$$

Note 4.3.1. As in note 3.13.3 , if only one function b exists per population, then there is only one possible network's topology linking each node/program in the

---

[40] Which is a redundancy since we are refering to p_i instead of o_i here.

population. So, in this case,

$$\mathbf{E}_{\mathfrak{N}}\left(\underset{iso}{\overset{net}{\Delta}}A(o_i,c)\right) = \frac{\sum\limits_{o_i\in\mathfrak{P}}\overset{net(b)}{\underset{iso}{\Delta}}A(o_i,c)}{N}$$

Definition 4.4. We denote the **expected emergent algorithmic complexity of a node/program** for algorithmic networks $\mathfrak{N}_{BB}(N,f,t,\tau,j)$ (**EEAC$_{\mathbf{BB}}$**) with network input w, where $0 < j \leq |\{b_j\}|$ [41], as

$$\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)}\left(\underset{iso}{\overset{net}{\Delta}}A(o_i,c)\right) =$$

$$= \sum_{b_j}\frac{\dfrac{\sum\limits_{o_i\in\mathfrak{P}_{BB}(N)}\overset{net(b_j)}{\underset{iso}{\Delta}}A_{(o_i,c)}}{N}}{|\{b_j\}|} =$$

$$= \frac{\sum\limits_{b_j}\dfrac{\sum\limits_{o_i\in\mathfrak{P}_{BB}(N)}A(\mathbf{U}(p_{net}^{b_j}(o_i,c))) - A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|}$$

Note 4.4.1. As in note 3.13.3 and 4.3.1 , if only one function b_j exists per population, then only one possible network's topology linking each node/program in the population. So, in this case,

$$\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)}\left(\underset{iso}{\overset{net}{\Delta}}A(o_i,c)\right) = \frac{\sum\limits_{o_i\in\mathfrak{P}_{BB}(N)}\overset{net(b_j)}{\underset{iso}{\Delta}}A(o_i,c)}{N}$$

Definition 4.5. We say an algorithmic network $\mathfrak{N}$ with a population of N nodes has the property of **average emergent open-endedness (AEOE)** for a given network input w in c cycles *iff*

$$\lim_{N\to\infty}\mathbf{E}_{\mathfrak{N}}\left(\underset{iso}{\overset{net}{\Delta}}A(o_i,c)\right) = \infty$$

Definition 4.5.1. We say an algorithmic network $\mathfrak{N}_{BB}(N,f,t,\tau)$ has the property of **expected emergent open-endedness (EEOE)** for a given network input w in c cycles *iff*

$$\lim_{N\to\infty}\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)}\left(\underset{iso}{\overset{net}{\Delta}}A(o_i,c)\right) = \infty$$

---

[41] For the sake of simplifying our notation, let $\{b_j\}$ denote

$$\left\{b_j \,\middle|\, \begin{array}{l} b_j: V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n \\ \quad (v,t_{c-1}) \quad \mapsto b_j(v,t_{c-1}) = (o,x+c) \end{array}\right\}$$ and b_j in the sum denote

$$b_j: V(G_t) \times T(G_t) \to \quad \mathfrak{P}_{BB}(N) \times \mathbb{N}|_1^n \quad .$$
$$\quad (v,t_{c-1}) \quad \mapsto b_j(v,t_{c-1}) = (o,x+c)$$

5. Cycle-bounded conditional halting probability

5.1. **Discussion on halting probabilities.** As presented in [11, 19–21, 50], the number Ω is the probability of randomly picking a halting program given an algorithmic probability distribution for a self-delimiting language. So, a conditional probability $\Omega(w)$ is the probability of randomly picking a halting program with w as its input. A variation of halting probabilities on self-delimiting universal programming languages is the resource-bounded halting probability: the probability of randomly picking a halting program on universal machine with limited computation time or memory [1, 6, 50]. However, for algorithmic networks composed of populations of arbitrary self-delimiting programs, another variation is needed. The "scarce" resource is not time or memory, but cycles. Since nodes/programs are computing during cycles (see 2.2.1.1) which determine the maximum number of times a node/program may reuse (through successive halting computations) its partial output before returning a final output, the **cycle-bounded conditional halting probability** $\Omega(w, c)$ is the probability of randomly picking a program that halts for every cycle until c with w as its initial input.

In Subsection 3.1 we have discussed that we want to compare the average algorithmic complexity of an algorithmic network playing the BBIG with its population when nodes/programs are completely isolated from each other. Thus, in order to prove the central Theorem 8.1 one will need to calculate an upper bound on the expected algorithmic complexity of what a node/program does when isolated during the cycles. The cycle-bounded conditional halting probability $\Omega(w, c)$ is used with the purpose of normalizing the probabilities of halting nodes/programs in Lemmas 8.3 and 8.5 . Remember that there may be programs that halt on every cycle and programs that do not. Since we need to calculate probability in each case and the population is randomly generated, $\Omega(w, c)$ necessarily arises as an important constant in Theorem 8.1.

5.2. **Definitions.**

Definition 5.1. Let L be an arbitrary population. Let $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ be an algorithmic network. We denote the set of $p_i \in L$, where $1 \leq i \leq N$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$, such that p_i always halts on network input w in every cycle until c when $P_{prot} \circ p_i$ is **isolated** as

$$Halt_{iso}(L, w, c)$$

Definition 5.1.1. Analogously, we denote the set of $p_i \in L$, where $1 \leq i \leq N$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$ with $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$, such that p_i does **not** halt on network input w in at least one cycle until c when $P_{prot} \circ p_i$ is isolated as

$$\overline{Halt}_{iso}(L, w, c)$$

Definition 5.2. We denote the **cycle-bounded conditional halting probability**[42] of a program in a language $\mathbf{L_U}$ that always halts for an initial input w in c

---

[42] That is, a conditional Chaitin's Omega number for isolated programs in a population of an algorithmic network

cycles as

$$\Omega(w,c) = \sum_{p_i \in Halt_{iso}(\mathbf{L_U},w,c)} \frac{1}{2^{|p_i|}} = \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} \frac{1}{2^{|p_i|}} =$$

$$= \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{1}{2^{|p_i|}}$$

Note 5.2.1. Since $\mathbf{L_U}$ is self-delimiting, the algorithmic probability of each program is well-defined. Hence, one can define the halting probability Ω [43] for $\mathbf{L_U}$. Further, the same holds for conditional halting probability $\Omega(w)$, i.e. the probability that a program halts when w is given as input. Then, the set of programs that always halt on initial input w in c cycles is a proper subset of the set of programs that halt, so that

$$\Omega(w,c) \leq \Omega(w) = \Omega(w,1) < 1$$

In fact, one can prove that

$$\Omega(w,c') \leq \Omega(w,c)$$

when $c' \geq c > 0$. On the other hand, one can also build programs that always halts for every input and for every number of cycles. So, for every $c > 0$

$$0 < \Omega(w,c) < 1$$

6. AVERAGE DIFFUSION DENSITY

6.1. **Discussion on diffusion in the BBIG.** On the "networked side of the equation" in Theorem 8.1, as discussed in Subsection 3.1, we are looking for a lower bound on the expected emergent algorithmic complexity of what a node/program does when networked playing the Busy Beaver imitation game (BBIG). This is the value from which will be subtracted the expected algorithmic complexity of what a node/program does when isolated.

Remember that in the BBIG once nodes/programs start sharing their partial outputs the only relevant feature that matters from then on is who has the biggest integer. In other words, the neighbor with a bigger partial output will "infect" a node/program with a smaller partial output. Note that it may be analogous to a multi-"disease" spreading, since there are several different graph arrangements of randomly generated nodes/programs in which there might be separated clusters with different respective "diseases" (in our case, different integers) spreading at the same time for a while. However, the BBIG rules make one type of partial output spreads over any other partial output: the biggest partial output of the randomly generated population. As we will stress in proof of Theorem 8.1, this is one of the mathematical shortcuts to reach a desired proof of a lower bound on the expected emergent algorithmic complexity of what a node/program does when networked playing the BBIG. It is given using the average density of nodes/programs infected by a single source in Definition 6.2.1.1 — in the case, the source is the randomly generated node/program that returns the biggest integer. Thus, we call it as **average singleton diffusion density**. Note that as there is a probability of two or more nodes/programs being generated which return the biggest integer, the average singleton diffusion density is actually a lower bound on the **average diffusion**

---

[43] Note that the Greek letter Ω here does not stand for an asymptotic notation opposed to the big O notation.

density (which might be multisource). Although Lemma 8.6 is proved using the average singleton diffusion density, it also holds for the average diffusion density, since it works like a lower bound. The average diffusion density is related to prevalence in [61,62], although there is no "curing" process in our case. And it might be also related to flooding time in [27] — see also Section 10.

Indeed, this analogy with diffusion or spreading in complex networks is not only a coincidence but is also responsible for bringing the aspects and proprieties studied in complex networks to our current results in algorithmic networks. The EEAC in the BBIG is lower bounded by a value that depends on how powerful the diffusion is on a network. We address more specifically some of theses relations with complex networks in Section 10.

6.2. Definitions.

Definition 6.1. Let $A_{max}(\mathfrak{N}, c)$ denote the algorithmic complexity of the biggest final output returned by a member of the population $\mathfrak{P}$ in a maximum number number of cycles c, where $\mathfrak{N} = (G, \mathfrak{P}, b)$.

Definition 6.1.1. In the case of $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ and $c = 1$, for the sake of simplifying our notation, we will just denote it as A_{max}.

Definition 6.2. Let $t_i, t, t' \in T(G_t)$ with $t_i \leq t \leq t'$. We denote the fraction to which the node/program with the "best" partial output on time instant t_i in the algorithmic network $\mathfrak{N} = (G, \mathfrak{P}, b)$ diffuses its partial output during time instants t until t', given that this diffusion started at time instant t_i, as

$$\tau_{max}(\mathfrak{N}, t_i)|_t^{t'} = \frac{\left|\mathbf{X}_{\tau_{max}(\mathfrak{N}, t_i)|_t^{t'}}\right|}{|V(G)|}$$

or

$$\tau_{max}(G, \mathfrak{P}, b, t_i)|_t^{t'} = \frac{\left|\mathbf{X}_{\tau_{max}(\mathfrak{N}, t_i)|_t^{t'}}\right|}{|V(G)|}$$

Note 6.2.1. The notion of what is the "best" partial output may vary on how the algorithmic network $\mathfrak{N}$ is defined — in some algorithmic networks the notion of "best" partial output may even be not defined. We consider the "best" partial output as being the one that always affects the neighbors to which it is shared by making them return a final result that is at least as "good" as the one that the node with the "best" partial output — that is, the one that started this diffusion — initially had. How good is a final result also depends on how $\mathfrak{N}$ is defined and on how one defines what makes a result better than another (e.g. a fitness function). While this general definition is not formally stated, in the particular case of the present paper, these matters become formal and precise in Definition 6.2.1 .

Note 6.2.2. Note that a node/program does only belong to $\mathbf{X}_{\tau_{max}(\mathfrak{N}, t_i)|_t^{t'}}$ *iff* its partial outputs between time instants t and t' historically depends on the information diffused by the node/program with the "best" partial output on time instant t_i.

Note 6.2.3.
$$0 \leq \tau_{max}(\mathfrak{N}, t_i)|_t^{t'} \leq 1$$

Definition 6.2.1. We denote the fraction to which the node/program with the biggest partial output at the first cycle in the algorithmic network $\mathfrak{N}_{BB}(N, f, t_i, \tau, j)$ diffuses this partial output during time instants t until t', given that this diffusion started at time instant t_i, as

$$\tau_{max}(N, f, t_i, \tau, j)|_t^{t'} = \frac{\left| \mathbf{X}_{\tau_{max}(N,f,t_i,\tau,j)|_t^{t'}} \right|}{N}$$

Definition 6.2.1.1. In the average case for all possible respective node mappings into the population, we denote the **average (or expected) singleton diffusion density** as

$$\tau_{\mathbf{E}(max)}(N, f, t_i, \tau)|_t^{t'} = \sum_{b_j} \frac{\tau_{max}(N, f, t_i, \tau, j)|_t^{t'}}{|\{b_j\}|}$$

7. Time centralities

7.1. **Discussion on how to trigger EEOE.** We have previously mentioned that our model of algorithmic networks playing the BBIG was built on time-varying graphs (see Section 3), so that the static case is also covered as an instance of time-varying graphs which every time-instant snapshot $\mathcal{G}_t$ (see [43]) is equal — see also note 9.1. In fact, a preliminar result for static graphs was presented in [2]. The reader is also invited to note that a model in which the population size varies during the cycles is a particular case of $\mathfrak{N}_{BB}(N, f, t_i, \tau, j)$ where $N = \omega$ [44] and each new edge is added in G_t in the exact cycle (or time instant) that an edge links the older network's population to a newborn node/program [45] — and we leave for future research proving the present results for graphs with $|V(G_t)| = \omega$. Within the framework of algorithmic dynamic networks in which one is interested in investigating conditions that trigger AEOE, a natural question is when is the best time instant (or cycle) to allow communication between nodes/programs with the purpose of generating more and more AEAC. Note that there may be cases where there is a cost in using a communication channel (i.e., an edge) for example. Since the information sharing relies only on diffusion processes in the BBIG played by algorithmic networks $\mathfrak{N}_{BB}(N, f, t_i, \tau, j)$, we tackle this problem following the approach and metrics developed in [29, 43]. The more powerful the diffusion, the more $\mathfrak{N}_{BB}(N, f, t_i, \tau, j)$ can be optimized (see discussion 3.1). So, in the present case, finding the central time to start a diffusion process is directly related to finding the central time to trigger expected emergent open-endedness.

We will present two related types of time centralities on $\mathfrak{N}_{BB}(N, f, t_i, \tau, j)$ (see 7.2.1). Both relies on cover time 3.3 3.4 as our diffusion measure as discussed in Subsection 3.2 and both depends on how time instants correspond to cycles (see function $c(x)$ in 7.1 and 7.2) . First, we define the central time to trigger expected emergent open-endedness in 7.1. Second, we define the central time to trigger maximum expected emergent algorithmic complexity in 7.2 . One can think of these time centralities as the minimum value t for which the function $f(N, t, \tau)$ minimizes the number of cycles, that depends on $t + f(N, t, \tau)$, in order to generate

---

[44] Note that ω is the first ordinal number and it an ordered set with the same cardinality of the natural numbers $\mathbb{N}$.

[45] And analogously a similar procedure applies when nodes are being "killed" during the cycles.

increasing expected emergent algorithmic complexity of a node when N tends to ∞, given a fraction τ of the nodes/programs as a diffusion "yardstick".

7.2. Definitions.

Definition 7.1. Let $w \in \mathbf{L_U}$ be a network input. Let $0 < N \in \mathbb{N}$. Let $c(x)$ be a non-decreasing total computable function where

$$c\colon \mathbb{N} \to \quad \mathbb{N}^*$$
$$x \mapsto c(x) = y$$

Let $\mathfrak{N}_{BB}(N, f, t_z, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$, where $0 \leq z \leq |T(G_t)| - 1$, be well-defined, where there is $t_{z_0} \in T(G_t)$ such that[46]

$$\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_{z_0}, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(z_0 + f(N, t_{z_0}, \tau) + 2)) \right) = \infty$$

We define the central time t_{cen_1} in generating **unlimited** expected emergent algorithmic complexity of a node on a diffusion process over a fraction τ as the population grows that minimizes the number of cycles[47] as a function[48] of the topological diffusion measure $f(N, t, \tau)$ as

$$t_{cen_1}(c) = \min \left\{ t_j \,\middle|\, j + f(N, t_j, \tau) = \min \left\{ t_i + f(N, t_i, \tau) \,\middle|\, t_i + f(N, t_i, \tau) \leq \right.\right.$$

$$\left.\left. \leq z_0 + f(N, t_{z_0}, \tau) \wedge \lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_i, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(i + f(N, t_i, \tau) + 2)) \right) = \infty \right\} \right\}$$

Definition 7.2. Let $w \in \mathbf{L_U}$ be a network input. Let $0 < N \in \mathbb{N}$. Let $c(x)$ be a non-decreasing total computable function where

$$c\colon \mathbb{N} \to \quad \mathbb{N}^*$$
$$x \mapsto c(x) = y$$

Let $\mathfrak{N}_{BB}(N, f, t_z, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$, where $0 \leq z \leq |T(G_t)| - 1$, be well-defined, where there is $t_{z_0} \in T(G_t)$ such that[49]

$$\forall x \in \mathbb{N}|_0^{|T(G_t)|-1} \left(\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_x, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(x + f(N, t_x, \tau) + 2)) \right) \leq \right.$$

$$\left. \leq \lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_{z_0}, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(z_0 + f(N, t_{z_0}, \tau) + 2)) \right) \right)$$

We define the central time t_{cen_2} in generating the **maximum** expected emergent algorithmic complexity of a node on a diffusion process over a fraction τ as the population grows that minimizes the number of cycles as a function of the topological diffusion measure $f(N, t, \tau)$ as

---

[46] This condition directly assures that this definition of time centrality is well-defined.
[47] And communication rounds.
[48] A computable function.
[49] This condition directly assures that this definition of time centrality is well-defined.

$$t_{cen_1}(c) = \min\left\{ t_j \,\middle|\, j + f(N, t_j, \tau) = \min\left\{ t_i + f(N, t_i, \tau) \,\middle|\, i + f(N, t_i, \tau) \leq \right.\right.$$

$$\leq z_0 + f(N, t_{z_0}, \tau) \wedge \forall x \left(\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_x, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(x + f(N, t_x, \tau) + 2)) \right) \right) \leq$$

$$\leq \left. \left. \lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_i, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(i + f(N, t_i, \tau) + 2)) \right) \right) \right\}\right\}$$

Note 7.2.1. Note that, by definition, since $|T(G_t)| \leq \infty$, if t_{cen_1} is well-defined, then $t_{cen_2} = t_{cen_1}$.

8. An EEOE Phenomenon in the BBIG

In this section we will prove lemmas, theorems and corollaries with the purpose of showing that $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ is an algorithmic network capable of EEOE (see Definition 4.5.1) under certain topological conditions of the graph G_t (see Corollary 8.1.1). We will show that there is a trade-off between the average diffusion density and the number of cycles (see Theorem 8.1). Moreover, once these topological properties are met, the concept of central time to trigger EEAC within the minimum number of cycles becomes well-defined (see Theorem 8.2).

We split the proof of the main Theorem 8.1 in six Lemmas 8.1, 8.2, 8.3, 8.4, 8.5 and 8.6. Corollary 8.1.1 does not only enables one to apply Theorem 8.1 to a network diffusion measure like cover time 3.3 but also bridges EEAC to time centrality by linking Theorems 8.1 and 8.2 . The main idea behind the construction of the proof of the main Theorem 8.1 comes from combining an estimation of a lower bound for the average algorithmic complexity of a **networked** node/program and a estimation of the expected algorithmic complexity of a **isolated** node/program. While the estimation of the former comes from the very BBIG dynamics, the estimation of the latter comes from the strong law of large numbers, Gibb's inequality and algorithmic information theory applied on the randomly generated population $\mathfrak{P}_{BB}(N)$. Thus, calculating the former estimation minus the latter gives directly the EEAC (see defnition 4.4). Corollary 8.1.1 follows directly from Theorem 8.1. The proof of Theorem 8.2 is a mathematical adjustment of definitions 7.1 and 7.2 into Corollary 8.1.1.

In appendix 12 we present extended versions of these proofs, except for the Corollary 8.1.1 and 9.1, which will be presented in the next section.

Lemma 8.1 (12.1.1). Let $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be an algorithmic network. Let N be the size of a randomly generated population $\mathfrak{P}_{BB}(N)$. Then, on the average as N grows, we will have that there is a constant C_4 such that[50]

$$A_{max} \geq \lg(N) - C_4$$

Note 8.1.1. Let $\mathbf{P}[X = a]$ be the usual notation for the probability of a random variable X assuming value a. Or $\mathbf{P}[statement\ S]$ denote the probability of a

---

[50] This result — on random samples following algorithmic probabilities — is inspired by a similar argument found in [22] with the purpose of building a true open-ended evolutionary model for software — see also [1, 24].

statement S be true. Thus, the conclusion of this theorem can be also stated using the strong law of large numbers [15] as: there is a constant C_4 such that

$$\mathbf{P}\left[\lim_{N\to\infty} A_{max} - \lg(N) + C_4 \geq 0\right] = 1$$

In fact, the main theorems and Corollaries 8.1, 8.1.1, 8.2 and 9.1 presented in this article can be translated into such probabilistic form by putting their last statements (or conclusions) into the square brackets like we just did to the above. Note that they all originate from Lemma 8.1. For the sake of simplifying the notation and shortening the formulas, we have chosen to state our results without using this probabilistic form.

Short proof. Let $\phi_N(p)$ be the frequency that p occurs in a random sample of size N. From the **strong law of large numbers** and **algorithmic information theory (AIT)** we have that

(1) $$\mathbf{P}\left[\lim_{N\to\infty} \phi_N(p) = \frac{1}{2^{|p|}}\right] = 1$$

Let $BB(k)$ be the Busy Beaver value for an arbitrary $k \in \mathbb{N}$ on machine $\mathbf{U}$. From AIT, we know there are constants $C_\Omega, C_{BB} \geq 0$ and a program $p_{BB(k)}$ such that for every $w \in \mathbf{L_U}$

$$\mathbf{U}(p_{BB(k)} \circ w) = \mathbf{U}(p_{BB(k)}) = BB(k)$$

and

$$k - C_\Omega \leq A(BB(k)) \leq |p_{BB(k)}| \leq k + C_{BB}$$

and

$$\forall x > BB(k) \left(A(x) + C_{BB} \geq A(BB(k)) \right)$$

Since k was arbitrary, let $k = \lg(N) - C_{BB}$. From step 1 we will have that, when N is big enough, one should expect that $p_{BB(\lg(N)-C_{BB})}$ occurs

$$\frac{N}{2^{|p_{BB(\lg(N)-C_{BB})}|}} \geq \frac{N}{2^{\lg(N)}} = 1$$

times within N random tries. Let $C_4 = C_\Omega + 2\,C_{BB}$. Thus, from definitions 3.10 and 6.1.1, since any node/program count as isolated from the network when $c = 1$, we will have that, when $N \to \infty$,

$$A_{max} \geq A(\mathbf{U}(P_{prot} \circ p_{BB(\lg(N)-C_{BB})})) - C_{BB} = A(\mathbf{U}(p_{BB(\lg(N)-C_{BB})})) - C_{BB} =$$
$$= A(BB(\lg(N) - C_{BB})) - C_{BB} \geq \lg(N) - C_{BB} - C_\Omega - C_{BB} = \lg(N) - C_4$$

$\square$

Lemma 8.2 (12.1.2). Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)}, w, c)$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$ and $N \in \mathbb{N}^*$ is arbitrary, there is a constant C_1 such that

$$A(\mathbf{U}(p_{iso}(p_i, c))) \leq C_1 + |p_i| + A(w) + A(c)$$

Short proof. We have that there is at least one program p' such that, for every $p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)}, w, c)$,

$$\mathbf{U}(p_{iso}(p_i, c)) = \mathbf{U}(p' \circ p_i \circ w \circ c)$$

Take the smallest such program p' and let $C_1 = |p'| + 5 - |p_{+1}|$, where p_{+1} is the smallest program that adds 1 to any computation (see Definition 3.5). Then, from AIT and Definition 3.5 we will have that

$$A(\mathbf{U}(p_{iso}(p_i, c))) = A(\mathbf{U}(p' \circ p_i \circ w \circ c)) \le C_1 + |p_i| + A(w) + A(c)$$

$\square$

Lemma 8.3 (12.1.3). Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L_U}$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$, we will have that

$$\frac{1}{\Omega(w,c)} \left(\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i|}{2^{|p_i|}} \right) + \lg(\Omega(w,c)) \le \lim_{N \to \infty} \lg(\Omega(w,c)N)$$

Short proof. Since $\mathbf{L_U}$ is self-delimited, from AIT and the Definition 5.2 , we will have that

$$(2) \qquad \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{1}{2^{|p_i| + \lg(\Omega(w,c))}} = 1$$

From step 2 ,

$$(3) \qquad \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i| + \lg(\Omega(w,c))}{2^{|p_i| + \lg(\Omega(w,c))}} =$$

$$= \frac{1}{\Omega(w,c)} \left(\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i|}{2^{|p_i|}} \right) + \lg(\Omega(w,c))$$

Since step 2 holds, from Gibb's (or Jensen's) inequality [30, 51], we will have that

$$(4) \qquad \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i| + \lg(\Omega(w,c))}{2^{|p_i| + \lg(\Omega(w,c))}} \le \lg(|Halt_{iso}(\mathbf{L_U}(N), w, c)|)$$

We also have by Definition 5.2 that

$$(5) \qquad \lim_{N \to \infty} \lg(|Halt_{iso}(\mathbf{L_U}(N), w, c)|) = \lim_{N \to \infty} \lg(\Omega(w,c)N)$$

Therefore, from steps 3 , 4 and 5 we will have that

$$\frac{1}{\Omega(w,c)} \left(\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i|}{2^{|p_i|}} \right) + \lg(\Omega(w,c)) \le \lim_{N \to \infty} \lg(\Omega(w,c)N)$$

$\square$

Lemma 8.4 (12.1.4). Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L_U}$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$, we will have that

$$\lim_{N \to \infty} \frac{\sum_{b_j} \dfrac{\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)}, w, c)} |p_i|}{N}}{|\{b_j\}|} \le \lim_{N \to \infty} \Omega(w,c) \lg(N)$$

Short proof. From the definition of language $\mathbf{L}_{BB}$ in 3.10 , as in 8.1, we will have that

(6)
$$\frac{\sum_{b_j} \frac{\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}}{|\{b_j\}|} = \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)} \frac{|p_i|}{2^{|p_i|}}$$

And from Lemma 8.3 we will have that

(7)
$$\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \leq \lim_{N \to \infty} \Omega(w,c)\lg(N)$$

Then, from steps 6 and 7

$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}}{|\{b_j\}|} =$$

$$= \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \leq \Omega(w,c)\lg(N)$$

$\square$

Lemma 8.5 (12.1.5). Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L}_{\mathbf{U}}$ and $P_{prot} \circ p_i = o_i \in \mathfrak{P}_{BB}(N)$, there is a constant C_0 such that

$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = C_0(1 - \Omega(w,c))$$

Short proof. As in step 6 , $A(\mathbf{U}(p_{iso}(p_i,c)))$ is independent of any topology, so that, from the Definition 5.1.1 as in step 6 , we will have that

(8)
$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = \lim_{N \to \infty} \sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)} \frac{A(\mathbf{U}(p_{iso}(p_i,c)))}{2^{|p_i|}}$$

Let $A(0) = C_0$. Since $p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)$ and $\mathfrak{P}_{BB}(N)$ is sensitive to oracles 3.11 , by the definition of p_{iso} in 4.1.1 we will have that, for every $p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathbf{U}},w,c)$,

(9)
$$A(\mathbf{U}(p_{iso}(p_i,c))) = A(0) = C_0$$

Thus, by the definition of $\Omega(w,c)$ in 5.2 ,

(10)
$$\lim_{N \to \infty} \sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathbf{U}}(N),w,c)} \frac{A(\mathbf{U}(p_{iso}(p_i,c)))}{2^{|p_i|}} = C_0(1 - \Omega(w,c))$$
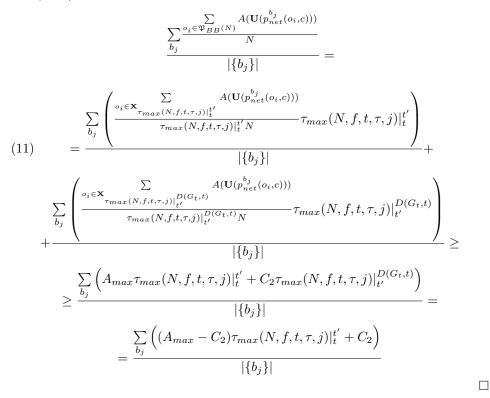
And we conclude from steps 8 , 9 and 10 that

$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = C_0(1 - \Omega(w,c))$$

$\square$

Lemma 8.6 (12.1.6). Let $\mathfrak{P}_{BB}(N)$ be a population in an arbitrary algorithmic network $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ as defined in 3.13 and 3.11 . Let $t_0 \leq t \leq t' \leq t_{|T(G_t)|-1}$. Let $c \in \mathfrak{C}_{\mathfrak{B}\mathfrak{B}}$ be an arbitrary number of cycles where $c_0 + t' + 1 \leq c$. Then, there is a constant C_2 such that

$$\frac{\sum_{b_j} \frac{\sum_{p_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{N}}{|\{b_j\}|} \geq (A_{max} - C_2) \, \tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} + C_2$$

Short proof. Let $C_2 = \min\{A(w) \mid \exists x \in \mathbf{L}_{\mathbf{U}}(\mathbf{U}(x) = w)\}$. From the definitions 6.2.1, 6.2 , 3.10 and 6.1.1 we will have that

$$\frac{\sum_{b_j} \frac{\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{N}}{|\{b_j\}|} =$$

$$(11) \quad = \frac{\sum_{b_j} \left(\frac{\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{\tau_{max}(N,f,t,\tau,j)|_t^{t'} N} \tau_{max}(N, f, t, \tau, j)|_t^{t'} \right)}{|\{b_j\}|} +$$

$$+ \frac{\sum_{b_j} \left(\frac{\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)}}} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)} N} \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} \right)}{|\{b_j\}|} \geq$$

$$\geq \frac{\sum_{b_j} \left(A_{max}\tau_{max}(N, f, t, \tau, j)|_t^{t'} + C_2\tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} \right)}{|\{b_j\}|} =$$

$$= \frac{\sum_{b_j} \left((A_{max} - C_2)\tau_{max}(N, f, t, \tau, j)|_t^{t'} + C_2 \right)}{|\{b_j\}|}$$

$\square$

Theorem 8.1 (12.1.1). $\big($A lower bound on EEAC$_{\text{BB}}$ from an arbitrary number of cycles$\big)$. Let $w \in \mathbf{L}_{\mathbf{U}}$ be a network input. Let $0 < N \in \mathbb{N}$. Let $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined. Let $t_0 \leq t \leq t' \leq t_{|T(G_t)|-1}$. Let

$$c \colon \mathbb{N} \to \quad \mathfrak{C}_{\mathfrak{B}\mathfrak{B}}$$
$$x \mapsto c(x) = y$$

be a total computable function where $c(x) \geq c_0 + t' + 1$. Then, we will have that:

$$\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(x)) \right) \geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) -$$
$$- \Omega(w, c(x)) \lg(x) - \Omega(w, c(x)) \lg(\lg(x)) - A(w) - C_5$$

Note 8.1.1. Thus, note that for bigger enough values of $c(x)$ compared to $\lg(N)$ one can make this lower bound always negative for example. One of the main ideas behind forthcoming results in this paper is to find optimal conditions where this lower bound is not only positive, but also goes to ∞.

Note 8.1.2. Note that this lower bound on the expected emergent algorithmic complexity is dependent on the value in the domain of the function c and not on function c itself, even if it grows fast. In fact, one may want to obtain this theorem for fixed values of c in which it is not a function but an arbitrary value. And the same result also hold in this case. The reader is invited to check that a simple substitution of $c(x)$ for c and of x inside the logarithms for c is enough.

Note 8.1.3. The same result also holds if only one possible function b_j is defined for each member of the family $\mathbb{G}(f, t, \tau)$. However, the notion of expected emergent algorithmic complexity (EEAC) gains some redundancy (see note 4.4.1), as only one function b_j will be taken into account within the sum in order to give the mean. Thus, in this case one can replace $\tau_{\mathbf{E}(max)}$ with τ_{max} not only in Theorem 8.1 but also in 8.1.1 and 8.2 . Such variation of these theorems becomes useful when one has algorithmic networks $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ built upon a historical population size growth in which each new node/program is linked (or not) to the previous existing algorithmic network.

Short proof. We have from AIT that there is $C_c \in \mathbb{N}$ for every $x \in \mathbb{N}$ such that

$$(12) \qquad A(c(x)) \leq C_c + A(x)$$

Let $C_5 = C_c + C_L + C_1 + C_4$. From Definition 4.4, we have that the expected emergent algorithmic complexity of a node/program for $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$, where $0 < j \leq |\{b_j\}|$ is given by

(13)

$$\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)}\left(\underset{iso}{\overset{net}{\Delta}} A(o_i, c(x))\right) = \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x)))) - A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}}{|\{b_j\}|}$$

And from definitions 5.1 , 5.1.1 , 5.2 , 6.2.1.1 , 3.5 and Lemmas 8.2 , 8.5 , 8.4 , 8.6 , 8.1 and step (12) we will have that

$$(14) \qquad \lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x)))) - A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}}{|\{b_j\}|} \geq$$

$$\geq \lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x))))}{N}}{|\{b_j\}|} -$$

$$- \left(\frac{\displaystyle\sum_{b_j} \left(\frac{\displaystyle\sum_{p_i \in Halt_{iso}(L_{\mathfrak{P}_{BB}(N)},w,c(x))} C_1 + |p_i| + A(w) + A(c(x))}{N} + C_0(1 - \Omega(w, c(x))) \right)}{|\{b_j\}|} \right) \geq$$

$$\geq \lim_{N \to \infty} (A_{max} - C_2) \, \tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} + C_2 -$$

$$- \left(\Omega(w, c(x)) \lg(N) + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big)\right) \geq$$

$$\geq \lim_{N \to \infty} (\lg(N) - C_4 - C_2) \, \tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} + C_2 -$$

$$- \left(\Omega(w, c(x)) \lg(N) + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big)\right) \geq$$

$$\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} - \Omega(w, c(x))\right) \lg(N) -$$

$$- \Omega(w, c(x)) \lg(x) - \Omega(w, c(x)) \lg(\lg(x)) - C_5 - A(w)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$$

Corollary 8.1.1. (A lower bound on EEAC$_{\mathrm{BB}}$ from a diffusion process as a function of cover time). Let $w \in \mathbf{L_U}$ be a network input. Let $0 < N \in \mathbb{N}$. Let $\mathfrak{N}_{BB}(N, f, t_z, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined. Let $t_z \in T(G_t)$. Let $\tau \in]0, 1]$. Let $c \colon \mathbb{N} \to \mathfrak{C_{BB}}$ be a total computable function where
$$x \mapsto c(x) = y$$
$$c(z + f(N, t_z, \tau) + 2) \geq c_0 + z + f(N, t_z, \tau) + 2$$

Then, we will have that:

$$\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t_z, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(z + f(N, t_z, \tau) + 2))\right) \geq$$

$$\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t_z, \tau)|_{t_z}^{t_z + f(N, t_z, \tau)} - \Omega(w, c(z + f(N, t_z, \tau) + 2))\right) \lg(N) -$$

$$- \Omega(w, c(z + f(N, t_z, \tau) + 2)) \lg(z + f(N, t_z, \tau) + 2) -$$

$$- \Omega(w, c(z + f(N, t_z, \tau) + 2)) \lg(\lg(z + f(N, t_z, \tau) + 2)) - A(w) - C_5$$

Proof. Remember the Definition 3.13 . Note that $z + f(N, t_z, \tau)$ goes as an index of $t_{z+f(N, t_z, \tau)}$, so that one must add 1 on the right side of the condition $c(x) \geq c_0 + t' + 1$ in 8.1 in order to make $c_0 + t_{z+f(N, t_z, \tau)} + 1 = c_0 + z + f(N, t_z, \tau) + 2$. Thus, the proof follows directly from replacing t with t_z, t' with $t_{z+f(N, t_z, \tau)}$ and x with $z + f(N, t_z, \tau) + 2$ in 8.1 . $\qquad\square$

Theorem 8.2 (12.1.2). (Central time in reaching unlimited EEAC from the BBIG played by a randomly generated algorithmic population). Let $w \in \mathbf{L_U}$ be a network input. Let $0 < N \in \mathbb{N}$. Let $\mathfrak{N}_{BB}(N, f, t_z, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined. If there is $0 \leq z_0 \leq |T(G_t)| - 1$ and $\epsilon, \epsilon_2 > 0$ such that

$$z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^C}{\lg(N)}\right)$$

where

$$-1-\epsilon \leq C = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0} + f(N, t_{z_0}, \tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)} \leq \frac{1}{\epsilon_2}$$

Then, for every non-decreasing total computable function $c \colon \mathbb{N} \to \mathfrak{C_{BB}}$ where
$$x \mapsto c(x) = y$$
$t_{z_0} \in T(G_t)$ and $c(z_0 + f(N, t_{z_0}, \tau) + 2) \geq c_0 + z_0 + f(N, t_{z_0}, \tau) + 2$ we will have that there are $t_{cen_2}(c)$ and $t_{cen_1}(c)$ such that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

Short proof. Suppose that there is $t_{z_0} \in T(G_t)$, where $0 \leq z_0 \leq |T(G_t)| - 1$, and a small enough $\epsilon > 0$ such that

$$(15) \qquad z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^C}{\lg(N)}\right)$$

where

$$C = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)}$$

Let

$$C' = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) - \epsilon}{\Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))}$$

Thus, since we are assuming $c(z_0 + f(N, t_{z_0}, \tau) + 2) \geq c_0 + z_0 + f(N, t_{z_0}, \tau) + 2$, for fixed values of $\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)}$ and ϵ, we will have that

$$(16) \quad C' = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) - \epsilon}{\Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))} \geq$$

$$\geq \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)} = C$$

Hence, from steps 15 and 16 we will have that there is ϵ_2 such that

$$(17) \qquad z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^{C'}}{\lg(N)}\right)$$

where

$$-1 - \epsilon \leq C' = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) - \epsilon}{\Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))} \leq \frac{1}{\epsilon_2}$$

And since $z_0 + f(N, t_{z_0}, \tau)$ is now assymptotically dominated by $\frac{N^{C'}}{\lg(N)}$, then by definition we will have from definitions 5.2 and 6.2.1.1 and Corollary 8.1.1 and step 17 that there is a constant C_6 such that

$$(18) \qquad \lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N,f,t_{z_0},\tau)}\left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(z_0 + f(N, t_{z_0}, \tau) + 2))\right) \geq$$

$$\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))\right) \lg(N) -$$

$$- \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) \lg(C_6 \frac{N^{C'}}{\lg(N)}) -$$

$$- \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) \lg(\lg(C_6 \frac{N^{C'}}{\lg(N)})) - A(w) - C_5 \geq$$

$$\geq \lim_{N \to \infty} (\epsilon) \lg(N) - \lg(C_6) - \lg(2) - \lg(\frac{1}{\epsilon_2}) - A(w) - C_5 = \infty$$

Then, directly from the definitions 7.1 and 7.2 and step 18 , since t_{z_0} satisfy these definitions, we will have that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

□

9. EEOE from a small diameter

We have previously established definitions and theorems for an abstract toy model of a simple optimization of average fitness through diffusion performed by a randomly generated population of interacting systems. It is shown in Theorem 8.1 that there are conditions (e.g. in Theorem 8.2) on the average diffusion density 6 and the number of cycles 2.2.1.1 that are sufficient to make the lower bound on the EEAC goes to infinity when the populations size goes to infinity, giving rise to a phenomenon we call expected emergent open-endedness (EEOE).

Following this purpose of theoretical investigation, we present in this section an application of our results using an important property in complex networks that fulfills these conditions mentioned to the above. Characterizing a small-diameter network, in which either the average shortest path length (also called as average geodesic distance) or the diameter (i.e., the maximum shortest path length) is "small" compared to the network size (i.e., when it is dominated by $\mathbf{O}(\lg(N))$), is a relevant topic in the current literature of complex networks [9,10,16,38]. Therefore, in this section we aim to exemplify and reinforce the relevance of the study of models like the BBIG within the theory of algorithmic networks in relation to small-diameter networks (see Sections 1 and 2 for more discussions). In fact, we will show that a small maximum shortest path length (that is, a small **diameter**) gives rise to an emergent phenomenon: it increases the very potential[51] of average (or, in our case, expected) emergent open-endedness (AEOE or EEOE), should the respective network link a population of systems that mathematically behave like $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ (see Definition 3.13). Thus, we not only present a model in which the **small-diameter** phenomenon (SD) plays an central role in achieving the EEOE, but also suggest in a general way that AEOE in algorithmic networks might be related to why the SD would appear in real networks — in which emergent complexity might be somehow advantageous to some degree.

A family $\mathbb{G}_{sm}$ of graphs will be defined in the same way as $\mathbb{G}$ in Definition 3.4. However, with two additional properties: first, there is a time instant on which every node can reach all nodes in a finite number of time intervals; second, taking this time instant as the starting point of diffusion, the graphs have a temporal diffusion diameter (see Definition 3.3.1) small compared to the size of the graph. The main idea is to ensure that a small-diameter phenomenon for dynamic networks, which relies on diffusion processes, eventually appears as the population grows. Then, one can directly define an algorithmic network $\mathfrak{N}_{BB_{sm}}(N, f, t, 1, j)$ that is totally analogous to $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ except for replacing family $\mathbb{G}(f, t, \tau)$ with $\mathbb{G}_{sm}(f, t, 1)$.[52] Thus, we will prove in Corollary 9.1, using in Theorem 8.2 the flexibility of Theorem 8.1 and Corollary 8.1.1, that these conditions on $\mathbb{G}_{sm}(f, t, 1)$ are sufficient for, in a computably bigger number of cycles compared to the temporal diffusion diameter, existing a central time to trigger EEOE.

---

[51] In fact, a lower bound on the EEAC.

[52] Further, as it is straightforward, we choose to omit a formal definition of $\mathfrak{N}_{BB_{sm}}(N, f, t, 1, j)$ in 9.2 .

Definition 9.1. Let

$$\mathbb{G}_{sm}(f,t,1) = \{G_t \mid i = |V(G_t)| \wedge f(i,t,1) = D(G_t,t) = \mathbf{O}\big(\lg(i)\big) \wedge$$

$$\wedge\; \forall i \in \mathbb{N}^* \exists! G_t \in \mathbb{G}(f,t,\tau)\big(|V(G_t)| = i\big) \wedge \forall u \in V(G_t) \exists x \in \mathbb{N}\big(x = d_t(G_t,t,u,1)\big)\}$$

where

$$f \colon \mathbb{N}^* \times X \subseteq T(G_t) \times Y \subseteq]0,1] \to \mathbb{N}$$
$$(x,t,\tau) \qquad\qquad \mapsto y$$

be a **family** of unique time-varying graphs which shares $f(i,t,1) = D(G_t,t) = \mathbf{O}\big(\lg(i)\big)$, where i is the number of nodes, as common properties. The number of nodes of each graph in this family may vary from 1 to ∞.

Note 9.1.1. Note that the diameter might be much smaller indeed. For example, $D(G_t,t) = \mathbf{O}\big(\frac{\lg(N)}{\lg(\lg(N))}\big)$ or $D(G_t,t) = \mathbf{O}\big(\lg(\lg(N))\big)$ are covered by Definition 9.1 .

Note 9.1.2. As pointed in 3.4.1 and 12.1.1.1 , Corollary 9.1 also holds with a weaker assumption on family $\mathbb{G}_{sm}(f,t,1)$ in which

$$\mathbf{P}\left[\lim_{N \to \infty}\big(N = |V(G_t)| \implies D(G_t,t) = \mathbf{O}\big(\lg(N)\big)\big)\right] = 1$$

Definition 9.2. Let the algorithmic network $\mathfrak{N}_{BBsm}(N,f,t,1,j)$ denote the same algorithmic network $\mathfrak{N}_{BB}(N,f,t,\tau,j)$ defined in 3.13 except for replacing family $\mathbb{G}(f,t,\tau)$ with family $\mathbb{G}_{sm}(f,t,1)$.

Corollary 9.1. Let $w \in \mathbf{L_U}$ be a network input. Let $0 < N \in \mathbb{N}$.
Let $\mathfrak{N}_{BBsm}(N,f,t_{z_0},1,j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined. Then, for every non-decreasing total computable function $c \colon \mathbb{N} \to \quad \mathfrak{C}_{\mathfrak{BB}} \quad$ where $t_{z_0} \in T(G_t)$ and
$$x \mapsto c(x) = y$$
$c(z_0 + f(N,t_{z_0},1)+2) \geq c_0 + z_0 + f(N,t_{z_0},1)+2$ we will have that there are $t_{cen_2}(c)$ and $t_{cen_1}(c)$ such that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

Proof. The proof is analogous to Theorem 8.2. However, the average singleton diffusion density for a number of time intervals equal to the temporal diffusion diameter assumes its maximum value 1. And the small-diameter condition on family $\mathbb{G}_{sm}(f,t,1)$ will assure that the conditions of Theorem 8.2 are met. First, remember that Corollary 8.1.1 from Theorem 8.1 only depends on function $f(i,t,\tau)$ and not on it being equal to $CT(G_t,t,\tau)$. Second, the same is also valid for Theorem 8.2 that only depends on Corollary 8.1.1 and also not on $f(i,t,\tau) = CT(G_t,t,\tau)$. Thus, we just replace τ with 1 in Theorem 8.2 and the same proposition will also holds for

$$(19) \qquad\qquad f(i,t,1) = D(G_t,t) = \mathbf{O}\big(\lg(i)\big)$$

Since $G_t \in \mathbb{G}_{sm}(f,t,1)$, we will have from Definition 9.1 that

$$(20) \qquad z_0 + f(N,t_{z_0},1) = z_0 + D(G_t,t_{z_0}) = z_0 + \mathbf{O}\left(\lg(N)\right)$$

where $0 \leq z_0 \leq |T(G_t)| - 1$. As in steps 15 and 17, we will have from definitions 3.3.1, 5.2, 6.2.1.1 and 9.1 and step 20 that there are constants $\epsilon, \epsilon_2 > 0$ such that

(21)

$$-1 - \epsilon < 0 < C = \frac{\tau_{\mathbf{E}(max)}(N,f,t_{z_0},1)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},1)} - \Omega(w, c_0 + z_0 + f(N,t_{z_0},1)+2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N,t_{z_0},1)+2)} =$$

$$= \frac{1 - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, 1) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, 1) + 2)} \leq \frac{1}{\epsilon_2}$$

We have that if $\frac{1}{\epsilon_2} \geq C > 0$, then

$$(22) \qquad D(G_t, t_{z_0}) = \mathbf{O}\left(\lg(N)\right) \implies D(G_t, t_{z_0}) = \mathbf{O}\left(\frac{N^C}{\lg(N)}\right)$$

Then, from steps 21 and 22 and 19 we will have that there are $t_{cen_2}(c)$ and $t_{cen_1}(c)$ such that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

$\square$

Note 9.1. As mentioned in discussions 3.2 and 7.1, note that the result 9.1 to the above also holds for static graphs. These can be considered an instance of time-varying graphs in which every time snapshot $\mathcal{G}_t$ is equal. In this case, for strongly connected static graphs G the usual diameter of graphs $diam(G)$ [17] becomes equivalent to $D(G, t_0)$. Also, the notion of time centrality will be vacuous, since the graph's topology is always the same. Hence, for static graphs, the Corollary 9.1 states only that the respective algorithmic network has the property of EEOE.

10. RELATED WORK AND FUTURE RESEARCH

The results presented in this article link algorithmic information properties with network topological ones. As the reader might notice, there may be graph particular aspects that trigger AEOE (see Theorems 8.1 and 8.2 and Corollaries 8.1.1 and 9.1). These are embedded in the average singleton diffusion density (Definition 6.2.1.1) and function c that takes some time interval as input (see Corollaries 8.1.1 and 9.1 and Theorem 8.2). In fact, the AEOE phenomenon benefits from topologies capable of faster spreading or flooding [27]. Although building variations of our models to fit real complex networks measurements for scale-free networks in [61, 62] (through a susceptible-infected-susceptible (SIS) variation of the BBIG model and the correspondence between prevalence and average diffusion density) will be a fruitful application of our models in network science, there would be no necessary advantage in some particular heterogeneity on the network structure. Note that our result 9.1 only uses the temporal diffusion diameter. We suggest that our mathematical results might indeed corroborate with computational experimental approaches in complex networks in [73] [72] [75]. If an average optimization through diffusion corresponds somehow to average emergent complexity, graph sparseness (also discussed in [27]) and the small-diameter property might play a central role in achieving this optimization with minimum cost. A more accurate theoretical analysis of this potential corroborations is paramount for future research.

With the same focus on complex networks analysis, an statistical or probabilistic information-theoretic counterpart of our work might be developed with further investigation on [31, 56, 72]. The case more specific of an experimental approach centered on the relation between living systems and graph properties using statistical information is presented in [67, 80].

Establishing relations between algorithmic information and statistical information like in [30, 42, 77, 86] within the context of algorithmic networks is crucial to approximate or find equivalences between measures from algorithmic information and measures from statistical information. In the particular case of an approach

centered on emergence and self-organization, we suggest further investigation from [35, 64, 78]. Another interesting topic is whether the average (and the joint) emergent algorithmic complexity is sufficient to deal with synergistic and integrated information [40, 41, 59, 65, 82, 83] or not, as it directly gives a measure of irreducible information [19, 21, 50]. If one assumes the definition of synergy as the general phenomenon in which the whole system is irreducibly better in solving a common problem than the sum (or the union) of its parts taken separately, as is widely used in statistical information-theoretic approaches cited in the previous sentence, then there may be a possible interpretation of (joint or average) emergent algorithmic complexity as (joint or average) synergistic algorithmic information. For this, note that one may consider for example that the common goal of the algorithmic network is to increase the average fitness of each node/program (see Subsection 3.1).

The present model requires a hypercomputable function to deal with eventual nodes/programs that do not halt (see Definition 3.5). With the purpose of bringing this Busy Beaver imitation game to computer simulations or just to a fully computable model, the recursive relative incompressibility (or sub-incompressibility) as in [1, 3, 4] can be used to define a measure for resource-bounded algorithmic complexity [50] [6]. Or one can use a measure through asymptotic approximations to algorithmic complexity as in [32, 74, 86] in order to develop further investigations on algorithmic network complexity as an extension of an approach for graphs presented in [18, 84, 85]. This way one can aim to make the relative halting problem tractable while keeping the fitness function still relatively uncomputable.

There are further necessary investigations from a game-theoretical perspective. For example, in [57] it is shown a relation for reaching global consensus between a specific type of connectiveness of graphs and a local strategy (in our notation, protocols) for nodes. A transposition of this problem into an algorithmic network model would help to mathematically understand global certification of best solutions (see Subsection 3.1) from local ones for example. If fitness or payoff [44] is somehow connected to the complexity of the player's strategy, algorithmic networks would enable one to investigate game-theoretical consequences of randomly generated strategies for players without interaction or for networked players [7, 71] with global strategies like the one in the BBIG.

Another variation of the BBIG model in which the problem of consensus may be investigated is from synchronous distributed systems of failure-prone processors connected by small-diameter graphs as presented in [26]. Adding the possibility of failure or crash to the BBIG model will require modifications on the information sharing protocols (see Section 3 and Definition 3.9) in order to solve the respective problem of consensus [26] of choosing the fittest solution. A crash-prone system would introduce new difficulties on the plain diffusion that we have presented in this article; And a limit to the maximum expected number of failures in relation to the population size seems to be necessary to keep the phenomenon of expected emergent open-endedness in such algorithmic networks with crash-prone node/programs.

We suggest that other relations to the fields we have mentioned in this Section and in Section 1 may configure a long term horizon for future research. For example, one may relate the problem of evolvability versus robustness in evolutionary biology [12, 79] with intentional attack vulnerability and resilience to random failure in complex networks [39]. To this end, one can investigate a variation of the BBIG model (see Section 3) in which the diffused best solution may correspond

to an increase in the distributed redundancy [66]. Following this direction in the context of algorithmic networks, one may study why a network topology and an information-sharing protocol become relevant from an emergent open-ended evolutionary point of view that takes into account complexity and computational power in solving problems [45, 46]. While it is conjectured in complex networks theory that different types of real-world networks are indeed related by graph properties (e.g. the small-world phenomenon [48]), the theoretical approach we are developing may suggest (e.g. see Corollary 9.1) that they may be indeed [2], when one assumes that problem solving, complexity increasing and networks are deeply connected through the concept of computation.

11. Conclusion

In this article, we have presented definitions and theorems in order to investigate the problem of emergence and complexity when a population of computable systems is networked. In particular, we have investigated conditions that enable the triggering of emergent open-endedness, that is, these conditions trigger an unlimited increase of emergent complexity as the population size goes to infinity.

First, we have introduced the general definition of an algorithmic network $\mathfrak{N}$. This definition relies on a population of systems that runs on an arbitrarily chosen theoretical machine and relies on a multiaspect graph. This makes the population of systems to correspond to nodes in the respective multiaspect graph such that edges are communication channels that nodes/systems can use to send and receive information.

Second, we have defined a particular mathematical model of algorithmic networks $\mathfrak{N}_{BB}$ based on the simple imitation of the fittest neighbor: A type of algorithmic network that plays the Busy Beaver imitation game. In this model, the randomly generated population of nodes/programs of a universal Turing machine is synchronous and non-halting nodes/programs always return the "worst" fitness/payoff. This population is networked by a time-varying graph. Hence, topological measures like cover time and temporal diffusion diameter can be promptly well-defined. Moreover, we have defined a graph-topological measure of diffusion (the average diffusion density) and the cycle-bounded conditional halting probability. These algorithmic networks $\mathfrak{N}_{BB}$ can be seen as playing an optimization procedure where the whole is "trying" to increase the average fitness/payoff by diffusing through the network the best randomly generated solution.

Furthermore, we have presented the average (or expected) emergent algorithmic complexity of a node/program as the average (or expected) value obtained from the algorithmic complexity of what a node/program can do when running networked minus the algorithmic complexity of what the same node/program can do when running isolated. Thus, when this difference goes to infinity as the population size goes to infinity, this configures a phenomenon we have called as the *average (or expected) emergent open-endedness*, i.e., unlimited emergent complexity happens as an emergent phenomenon that occurs as the population grows. This is a phenomenon akin to evolutionary open-endedness, but in the latter unlimited complexity occurs as successive mutations and natural selection apply over time.

Since our network playing the Busy Beaver imitation game is linked by a time-varying graph, a centrality for dynamic networks is defined for the optimum time instant to trigger average emergent open-endedness in a way that it minimizes the

amount of cycles (or communication rounds) needed to do this triggering. Moreover, we presented our main theorem proving that there is a lower bound on the expected emergent algorithmic complexity for algorithmic networks $\mathfrak{N}_{BB}$ such that it depends on how much larger is the average diffusion density in a given time interval compared to the cycle-bounded conditional halting probability. This lower bound also depends on the parameter for which one is calculating the number of cycles. In fact, we prove a corollary showing that this parameter can be calculated for example from the cover time, so that our results hold even in the case of spending a computably bigger number of cycles compared to the cover time. Furthermore, from this corollary, we proved that there are asymptotic conditions on the increasing power of the cover time as a function of the population size such that they ensure that there is a central time to trigger expected emergent open-endedness.

Additionally, we have made a small modification on the family of time-varying graphs of $\mathfrak{N}_{BB}$ with the purpose of investigating what would happen if the networks were small-diameter. We replaced the cover time with the temporal diffusion diameter (i.e., the number of time intervals to reach every node from any node) — or the classical diameter (i.e., the maximum shortest path length) in the static case — in the definition of this family of graphs. Indeed, we proved that, in this case, a small-diameter (i.e., dominated by $\mathbf{O}(\lg(N))$, where N is the network size) is sufficient for existing a central time to trigger expected emergent open-endedness.

Finally, we have suggested future research and discussed that our results might be related to problems in network science, information theory, computability theory, distributed computing, game theory, evolutionary biology, and complex systems.

References

[1] Felipe S. Abrahão, *Metabiologia, Subcomputação e Hipercomputação: em direção a uma teoria geral de evolução de sistemas*, Ph.D. thesis, 2015.

[2] ———, *Emergent algorithmic creativity on networked Turing machines*, The 8th international workshop on guided self-organization at the fifteenth international conference on the synthesis and simulation of living systems (alife), 2016.

[3] ———, *Relativizing an incompressible number and an incompressible function through subrecursive extensions of Turing machines*, arXiv preprint (2016dec), available at `1612.05221`.

[4] ———, *The "paradox" of computability and a recursive relative version of the Busy Beaver function*, Information and complexity, 2016jan, pp. 3–15.

[5] Alyssa Adams, Hector Zenil, Paul C. W. Davies, and Sara Imari Walker, *Formal Definitions of Unbounded Evolution and Innovation Reveal Universal Mechanisms for Open-Ended Evolution in Dynamical Systems*, Scientific Reports **7** (2017dec), no. 1, 997.

[6] Eric Allender, Michal Koucký, Detlef Ronneburger, and Sambuddha Roy, *The pervasive reach of resource-bounded Kolmogorov complexity in computational complexity theory*, Journal of Computer and System Sciences **77** (2011), no. 1, 14–40.

[7] Robert Axelrod, *The Complexity of Cooperation: Agent-Based Models of Competition and Collaboration*, Vol. 1, Princeton University Press, 1997.

[8] ———, *The Evolution of Cooperation: Revised Edition*, Basic Books, 2006.

[9] Albert-László Barabási, *Scale-Free Networks: A Decade and Beyond*, Science **325** (2009jul), no. 5939, 412–413, available at `0803973233`.

[10] Albert-László Barabási and Eric Bonabeau, *Scale-Free Networks*, Scientific American **288** (2003may), no. 5, 60–69.

[11] George Barmpalias, Nan Fang, and Andrew Lewis-Pye, *Optimal asymptotic bounds on the oracle use in computations from Chaitin's Omega*, Journal of Computer and System Sciences **82** (2016dec), no. 8, 1283–1299.

[12] Patrick Bateson and Peter Gluckman, *Plasticity, Robustness, Development and Evolution*, Cambridge University Press, Cambridge, 2011.

[13] Mark A. Bedau, *Four Puzzles About Life*, Artificial Life **4** (1998apr), no. 2, 125–140.

[14] Nils Bertschinger, Johannes Rauh, Eckehard Olbrich, Jürgen Jost, and Nihat Ay, *Quantifying unique information*, Entropy **16** (2014apr), no. 4, 2161–2183, available at `1311.2852`.

[15] Patrick Billingsley, *Probability and Measure*, Anniversar, John Wiley & Sons, Hoboken, 2012.

[16] Béla Bollobás and Oliver Riordan, *The diameter of a scale-free random graph*, Combinatorica **24** (2004jan), no. 1, 5–34.

[17] Béla Bollobás, *Random Graphs*, Cambridge University Press, Cambridge, 2001.

[18] Harry Buhrman, Ming Li, John Tromp, and Paul Vitányi, *Kolmogorov Random Graphs and the Incompressibility Method*, SIAM Journal on Computing **29** (1999jan), no. 2, 590–599.

[19] Cristian S. Calude, *Information and Randomness: An algorithmmic perspective*, 2nd ed., Springer-Verlag, 2002.

[20] Gregory Chaitin, *A Theory of Program Size Formally Identical to Information Theory*, Journal of the ACM **22** (1975), no. 3, 329–340.

[21] _____, *Algorithmic Information Theory*, 3rd ed., Cambridge University Press, 2004.

[22] _____, *Life as Evolving Software*, A computable universe, 2012dec, pp. 277–302.

[23] _____, *Proving Darwin: making biology mathematical*, Vintage Books, 2013.

[24] Gregory Chaitin, Virgínia Chaitin, and Felipe S. Abrahão, *Metabiología: los orígenes de la creatividad biológica*, Investigación y Ciencia (2014), 74–80.

[25] Alastair Channon, *Passing the ALife Test: Activity Statistics Classify Evolution in Geb as Unbounded*, Advances in artificial life, 2001, pp. 417–426.

[26] Bogdan S. Chlebus and Dariusz R. Kowalski, *Robust gossiping with an application to consensus*, Journal of Computer and System Sciences **72** (2006dec), no. 8, 1262–1281.

[27] A. E. F. Clementi, C. Macci, A. Monti, F. Pasquale, and R. Silvestri, *Flooding time of edge-Markovian evolving graphs*, SIAM Journal on Discrete Mathematics **24** (2010), 1694–1712.

[28] S. Barry Cooper, *Emergence as a computability-theoretic phenomenon*, Applied Mathematics and Computation **215** (2009oct), no. 4, 1351–1360.

[29] Eduardo Chinelate Costa, Alex Borges Vieira, Klaus Wehmuth, Artur Ziviani, and Ana Paula Couto da Silva, *Time Centrality in Dynamic Complex Networks*, Advances in Complex Systems **18** (2015), no. 07n08, available at `1504.00241`.

[30] Thomas M. Cover and Joy A. Thomas, *Elements of Information Theory*, John Wiley & Sons, Inc., Hoboken, NJ, USA, 2005.

[31] Matthias Dehmer, Frank Emmert-Streib, and Alexander Mehler (eds.), *Towards an Information Theory of Complex Networks*, Birkhäuser Boston, Boston, 2011.

[32] Jean Paul Delahaye and Hector Zenil, *Numerical evaluation of algorithmic complexity for short strings: A glance into the innermost structure of randomness*, Applied Mathematics and Computation **219** (2012sep), no. 1, 63–77.

[33] Ítala Maria Loffredo D'Ottaviano and Ettore Bresciani, *Sistêmica, Auto-Organização e Criação*, MultiCiência: revista interdisciplinar dos Centros e Núcleos da Unicamp **3** (2004), 1–23.

[34] Bernard Feltz, Marc Crommelinck, and Philippe Goujon, *Self-organization and emergence in life sciences* (Bernard Feltz, Marc Crommelinck, and Philippe Goujon, eds.), Kluwer Academic Publishers, Dordrecht, 2006.

[35] Nelson Fernandez, Carlos Maldonado, and Carlos Gershenson, *Information Measures of Complexity, Emergence, Self-organization, Homeostasis, and Autopoiesis*, 2013, pp. 19–51.

[36] David B. Fogel, *Evolutionary Computation: Toward a New Philosophy of Machine Intelligence*, IEEE Press Series on Computational Intelligence, John Wiley & Sons, Inc., Hoboken, NJ, USA, 2005.

[37] Carlos Gershenson and Nelson Fernández, *Complexity and information: Measuring emergence, self-organization, and homeostasis at multiple scales*, Complexity **18** (2012nov), no. 2, 29–44, available at `1205.2026`.

[38] Carlos Gershenson and Mikhail Prokopenko, *Complex Networks*, Artificial Life **17** (2011oct), no. 4, 259–261.

[39] Sam F. Greenbury, Iain G. Johnston, Matthew A. Smith, Jonathan P K Doye, and Ard A. Louis, *The effect of scale-free topology on the robustness and evolvability of genetic regulatory networks*, Journal of Theoretical Biology **267** (2010nov), no. 1, 48–61, available at `1005.4342`.

[40] Virgil Griffith, *Quantifying synergistic information*, Ph.D. Thesis, 2014.

[41] Virgil Griffith and Christof Koch, *Quantifying synergistic mutual information*, 2012, pp. 159–190.

[42] Peter Grunwald and Paul Vitanyi, *Shannon Information and Kolmogorov Complexity*, CoRR **cs.IT/0410** (2004oct), 54, available at `0410002`.

[43] Abraão Guimarães, Alex B Vieira, Ana Paula C Silva, and Artur Ziviani, *Fast Centrality-Driven Diffusion in Dynamic Networks*, Proceedings of the 22nd international conference on world wide web companion, 2013, pp. 821–828.

[44] Peter Hammerstein and Reinhard Selten, *Game theory and evolutionary biology*, Handbook of game theory with economic applications, 1994, pp. 929–993.

[45] Santiago Hernández-Orozco, Francisco Hernández-Quiroz, and Hector Zenil, *The Limits of Decidable States on Open-Ended Evolution and Emergence*, Proceedings of the artificial life conference 2016, 2016, pp. 200–207.

[46] Santiago Hernández-Orozco, Hector Zenil, and Narsis A Kiani, *Algorithmically probable mutations reproduce aspects of evolution such as convergence rate, genetic memory, modularity, diversity explosions, and mass extinction*, arXiv preprint (2017), 1–13, available at `1709.00268v3`.

[47] Vyacheslav V. Ivanishev, *The technology of modeling on a basis of visual representation*, Proceedings of the international conference on informatics and control (ici&c'97), 1997, pp. 448–450.

[48] Jon Kleinberg, *The small-world phenomenon*, Proceedings of the thirty-second annual acm symposium on theory of computing - stoc '00, 2000mar, pp. 163–170.

[49] Oleg Korolev, Vladimir Parfenov, and Semyon Potryasaev, *Program System for Solving Problems of Monitoring Social and Economic Development of Saint-Petersburg*, 2016, pp. 347–358.

[50] Ming Li and Paul Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications*, 2nd ed., Springer Science & Business Media, New York, 1997.

[51] David J. C. MacKay, *Information Theory, Inference, and Learning Algorithms David J.C. MacKay*, 11th ed., Vol. 100, Cambridge University Press, 2005.

[52] C C Maley, *Four Steps Toward Open-ended Evolution*, Proceedings of the 1st annual conference on genetic and evolutionary computation - volume 2, 1999, pp. 1336–1343.

[53] Margulis Lynn, *Symbiosis in Cell Evolution: Life and its Environment on the Early Earth.*, W.H Freeman & Co. (1981), 419.

[54] V. E. Marley, *Operations over algorithmic networks*, Proceedings of the international conference on informatics and control (ici&c'97), 1997, pp. 451–457.

[55] J H Miller and S E Page, *Complex Adaptive Systems: An introduction to computational models of social life*, Princeton University Press, 2007.

[56] Abbe Mowshowitz and Matthias Dehmer, *Entropy and the complexity of graphs revisited*, Entropy **14** (2012mar), no. 3, 559–570.

[57] Nabil H. Mustafa and Aleksandar Pekec, *Listen to Your Neighbors: How (Not) to Reach a Consensus*, SIAM Journal on Discrete Mathematics **17** (2004jan), no. 4, 634–660.

[58] Gregoire Nicolis and Catherine Nicolis, *Foundations of complex systems*, Vol. 17, World Scientific Publishing Co. Pte. Ltd., 2009.

[59] Masafumi Oizumi, Larissa Albantakis, and Giulio Tononi, *From the Phenomenology to the Mechanisms of Consciousness: Integrated Information Theory 3.0*, PLoS Computational Biology **10** (2014may), no. 5, e1003588.

[60] Raj Kumar Pan and Jari Saramäki, *Path lengths, correlations, and centrality in temporal networks*, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics **84** (2011jul), no. 1, 016105, available at `1101.5913`.

[61] Romualdo Pastor-Satorras and Alessandro Vespignani, *Epidemic spreading in scale-free networks*, Physical Review Letters **86** (2001apr), no. 14, 3200–3203, available at `0010317`.

[62] _____, *Immunization of complex networks*, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics **65** (2002feb), no. 3, 036104, available at `0107066`.

[63] Mikhail Prokopenko (ed.), *Guided Self-Organization: Inception*, Emergence, Complexity and Computation, vol. 9, Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[64] Mikhail Prokopenko, Fabio Boschetti, and Alex J. Ryan, *An information-theoretic primer on complexity, self-organization, and emergence*, Complexity **15** (2009sep), no. 1, 11–28, available at `arXiv:1103.4838v1`.

[65] Rick Quax, Omri Har-Shemesh, and Peter Sloot, *Quantifying Synergistic Information Using Intermediate Stochastic Variables*, Entropy **19** (2017feb), no. 2, 85.

[66] Martin Randles, David Lamb, E. Odat, and A. Taleb-Bendiab, *Distributed redundancy and robustness in complex systems*, Journal of Computer and System Sciences **77** (2011mar), no. 2, 293–304.

[67] N. Rashevsky, *Life, information theory, and topology*, The Bulletin of Mathematical Biophysics **17** (1955sep), no. 3, 229–235.

[68] Hartley Rogers Jr., *Theory of Recursive Functions and Effective Computability*, MIT Press, Cambridge, MA, USA, 1987.

[69] Kepa Ruiz-Mirazo, Juli Peretó, and Alvaro Moreno, *A Universal Definition of Life: Autonomy and Open-Ended Evolution*, Origins of life and evolution of the biosphere **34** (2004jun), no. 3, 323–346.

[70] Frank Schweitzer, Giordio Fagiolo, Didier Sornette, Fernando Vega-Redondo, and Douglas R. White, *Economic Networks: What Do We Know and What Do We Need To Know?*, Advances in Complex Systems **12** (2009aug), no. 04n05, 407–422.

[71] Yoav Shoham and Kevin Leyton-Brown, *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*, Cambridge University Press, New York, NY, USA, 2008.

[72] R Solé and Sergi Valverde, *Information theory of complex networks: On evolution and architectural constraints*, Complex networks, 2004aug, pp. 189–207.

[73] Ricard V. Solé, Ramon Ferrer-Cancho, Jose M. Montoya, and Sergi Valverde, *Selection, tinkering, and emergence in complex networks*, Complexity **8** (2002sep), no. 1, 20–33.

[74] Fernando Soler-Toscano, Hector Zenil, Jean-Paul Delahaye, and Nicolas Gauvrit, *Calculating Kolmogorov Complexity from the Output Frequency Distributions of Small Turing Machines*, PLoS ONE **9** (2014may), no. 5, e96223.

[75] Fernanda S H Souza, Geraldo R. Mateus, and Alexandre Salles Da Cunha, *Optimization in designing complex communication networks*, Handbook of optimization in complex networks, 2012, pp. 3–37.

[76] Russell K. Standish, *Open-ended artificial evolution*, International Journal of Computational Intelligence and Applications **03** (2003jun), no. 02, 167–175.

[77] Andreia Teixeira, Armando Matos, André Souto, and Luís Antunes, *Entropy measures vs. Kolmogorov complexity*, Entropy **13** (2011mar), no. 3, 595–611.

[78] Bernard Testa and Lemont B. Kier, *Emergence and dissolvence in the self-organisation of complex systems*, Entropy **2** (2000feb), no. 1, 1–25.

[79] A. Wagner, *Robustness and evolvability: a paradox resolved*, Proceedings of the Royal Society B: Biological Sciences **275** (2008jan), no. 1630, 91–100.

[80] Sara Imari Walker, Hyunju Kim, and Paul C. W. Davies, *The informational architecture of the cell Subject Areas : Author for correspondence :*, Philosophical Transactions A **374** (2016mar), no. 2063, 20150057.

[81] Klaus Wehmuth, Éric Fleury, and Artur Ziviani, *On MultiAspect graphs*, Theoretical Computer Science **651** (2016), 50–61, available at `1408.0943`.

[82] Michael Wibral, Joseph T. Lizier, and Viola Priesemann, *Bits from Brains for Biologically Inspired Computing*, Frontiers in Robotics and AI **2** (2015mar), no. March, 1–25, available at `1412.0291`.

[83] Paul L. Williams and Randall D. Beer, *Nonnegative Decomposition of Multivariate Information*, arXiv preprint (2010apr), available at `1004.2515`.

[84] Hector Zenil, Narsis A. Kiani, Francesco Marabita, Yue Deng, Szabolcs Elias, Angelika Schmidt, Gordon Ball, and Jesper Tegner, *An Algorithmic Information Calculus for Causal Discovery and Reprogramming Systems*, bioRxiv (2017).

[85] Hector Zenil, Fernando Soler-Toscano, Kamaludin Dingle, and Ard A. Louis, *Correlation of automorphism group size and topological properties with program-size complexity evaluations of graphs and complex networks*, Physica A: Statistical Mechanics and its Applications **404** (2014jun), 341–358, available at `1306.0322`.

[86] Hector Zenil, Fernando Soler-Toscano, Narsis A. Kiani, Santiago Hernández-Orozco, and Antonio Rueda-Toicen, *A Decomposition Method for Global Evaluation of Shannon Entropy and Local Estimations of Algorithmic Complexity* (2016sep), 1–48 pp., available at `1609.00110`.

12. Appendix

12.1. **Lemmas and Theorems.**

Lemma 12.1.1 (8.1).
Let $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be an algorithmic network.
Let $N = |\mathbf{L}_{\mathfrak{P}_{BB}(N)}|$.
Thus, on the average as N grows, we will have that there is a constant C_4 such that

$$A_{max} \geq \lg(N) - C_4$$

Note 12.1.1.1. Let $\mathbf{P}[X = a]$ be the usual notation for the probability of a random variable X assuming value a. Or $\mathbf{P}[statement\ S]$ denote the probability of a *statement S* be true. Thus, this theorem is formaly given by the strong law of large numbers [15] as: there is a constant C_4 such that

$$\mathbf{P}\left[\lim_{N\to\infty} A_{max} - \lg(N) + C_4 \geq 0\right] = 1$$

In fact, the main results 12.1.1 and 12.1.2 presented in this article can be translated into such probabilistic form by putting their last statements into the square brackets like we just did to the above. For the sake of simplifying the notation and shortening the formulas, we have chosen to state our results without using this probabilistic form.

Proof.
From AIT, we know that the algorithmic probability of occurring a program $p \in \mathbf{L_U}$ is

(23)
$$\frac{1}{2^{|p|}}$$

Let $\phi_N(p)$ be the frequency that p occurs in a random sample of size N. In the case, this random sample is the randomly generated population $\mathfrak{P}$.

Define a Bernoulli trial on a random variable $\mathbf{X}$ that assumes value 1 *iff* program p occurs and assumes value 0 *iff* otherwise. Since this random sample is identically distributed and/or define a binomial distribution where $\sum_{n=1}^{\infty} \frac{Var[X_n]}{n^2} < \infty$, we will have from the **strong law of large numbers** that

(24)
$$\mathbf{P}\left[\lim_{N\to\infty} \phi_N(p) = \frac{1}{2^{|p|}}\right] = 1$$

Thus, when N is big enough, one expects that p occurs $\frac{N}{2^{|p|}}$ times within N random tries. That is, since p was arbitrary, the probability distribution in N random tries tends to match the algorithmic probability distribution on $\mathbf{L_U}$ when N goes to ∞.

Let $BB(k)$ be the Busy Beaver value for an arbitrary $k \in \mathbb{N}$ on machine $\mathbf{U}$.

From AIT [22], we know there are constants $C_\Omega, C_{BB} \geq 0$ and a program $p_{BB(k)}$ such that for every $w \in \mathbf{L_U}$

$$\mathbf{U}(p_{BB(k)} \circ w) = \mathbf{U}(p_{BB(k)}) = BB(k)$$

(25)   and

$$k - C_\Omega \leq A(BB(k)) \leq |p_{BB(k)}| \leq k + C_{BB}$$

and

$$\forall x > BB(k) \left(A(x) + C_{BB} \geq A(BB(k)) \right)$$

Since k was arbitrary, let $k = \lg(N) - C_{BB}$.

From 24 we will have that, when N is big enough, one should expect that $p_{BB(\lg(N)-C_{BB})}$ occurs

$$\frac{N}{2^{|p_{BB(\lg(N)-C_{BB})}|}} \geq \frac{N}{2^{\lg(N)}} = 1$$

times within N random tries. That is, from conditional probabilities,

(26) $$\mathbf{P}\left[\lim_{N \to \infty} \phi_N(p_{BB(\lg(N)-C_{BB})})N \geq \frac{N}{2^{\lg(N)}} = 1 \right] \geq$$

$$\geq \mathbf{P}\left[\lim_{N \to \infty} \phi_N(p_{BB(\lg(N)-C_{BB})})N = \frac{N}{2^{|p_{BB(\lg(N)-C_{BB})}|}} \right] = 1$$

Let $C_4 = C_\Omega + 2\,C_{BB}$.

From definitions 3.10 and 6.1.1 and step 25 , since any node/program count as isolated from the network when $c = 1$, we will have that, when $N \to \infty$:

if

$$\lim_{N \to \infty} \phi_N(p_{BB(\lg(N)-C_{BB})})N \geq 1$$

(27)   then

$$A_{max} \geq A(\mathbf{U}(P_{prot} \circ p_{BB(\lg(N)-C_{BB})})) - C_{BB} = A(\mathbf{U}(p_{BB(\lg(N)-C_{BB})})) - C_{BB} =$$
$$= A(BB(\lg(N) - C_{BB})) - C_{BB} \geq \lg(N) - C_{BB} - C_\Omega - C_{BB} = \lg(N) - C_4$$

Thus, from conditional probabilities, we will have that

$$\mathbf{P}\left[\lim_{N\to\infty} A_{max} - \lg(N) + C_4 \geq 0\right] \geq$$

$$\geq \mathbf{P}\left[\lim_{N\to\infty} \phi_N(p_{BB(\lg(N)-C_{BB})})N \geq 1\right] = 1$$

$\square$

Lemma 12.1.2 (8.2). *Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)}, w, c)$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$ and $N \in \mathbb{N}^*$ is arbitrary, there is a constant C_1 such that*

$$A(\boldsymbol{U}(p_{iso}(p_i, c))) \leq C_1 + |p_i| + A(w) + A(c)$$

Note 12.1.2.1. Note that from the Definition 4.1.1 this result is independent of any topology in which $\mathfrak{P}_{BB}(N)$ could be networked.

Proof.
Let $N \in \mathbb{N}^*$ be arbitrary. Remember the definition of $\mathbf{L}_{BB}$ in 3.10 . And note that p_i is a program in $\mathbf{L_U}$.
Then, there is at least one program p' such that

$$\mathbf{U}(p_{iso}(p_i, c)) = \mathbf{U}(p' \circ p_i \circ w \circ c)$$

is a well-defined value for every $p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)}, w, c)$.
Take the smallest such program p' and let $C_1 = |p'| + 5 - |p_{+1}|$, where p_{+1} is the smallest program that adds 1 to any computation (see Definition 3.5).
Then, from AIT and Definition 3.5 we will have that

$$A(\mathbf{U}(p_{iso}(p_i, c))) = A(\mathbf{U}(p' \circ p_i \circ w \circ c)) \leq C_1 + |p_i| + A(w) + A(c)$$

$\square$

Lemma 12.1.3 (8.3). *Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L_U}$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$, from AIT and Gibb's (or Jensen's) inequality, we will have that*

$$\frac{1}{\Omega(w, c)}\left(\lim_{N\to\infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N), w, c)} \frac{|p_i|}{2^{|p_i|}}\right) + \lg(\Omega(w, c)) \leq \lim_{N\to\infty} \lg(\Omega(w, c)N)$$

Proof.

Since $\mathbf{L_U}$ is self-delimited, from AIT and the Definition 5.2 , we will have that

$$(28) \qquad \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{1}{2^{|p_i|}} = \Omega(w,c) < 1$$

Hence,

$$(29)$$
$$\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{1}{\Omega(w,c)2^{|p_i|}} = \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{1}{2^{|p_i|+\lg(\Omega(w,c))}} = 1$$

Thus, from step 29 ,

$$(30) \qquad \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i| + \lg(\Omega(w,c))}{2^{|p_i|+\lg(\Omega(w,c))}} =$$

$$= \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}\Omega(w,c)} + \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{\lg(\Omega(w,c))}{2^{|p_i|}\Omega(w,c)} =$$

$$= \frac{1}{\Omega(w,c)} \left(\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \right) + \lg(\Omega(w,c))$$

Since 29 holds, from Gibb's (or Jensen's) inequality [30] [51], we will have that

$$(31) \qquad \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i| + \lg(\Omega(w,c))}{2^{|p_i|+\lg(\Omega(w,c))}} \leq \lg(|Halt_{iso}(\mathbf{L_U}(N),w,c)|)$$

We also have by Definition 5.2 that

$$(32) \qquad \lim_{N \to \infty} \lg(|Halt_{iso}(\mathbf{L_U}(N),w,c)|) = \lim_{N \to \infty} \lg(\Omega(w,c)N)$$

Therefore, from steps 30 , 31 and 32 we will have that

$$\frac{1}{\Omega(w,c)} \left(\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \right) + \lg(\Omega(w,c)) \leq \lim_{N \to \infty} \lg(\Omega(w,c)N)$$

$$\square$$

Lemma 12.1.4 (8.4). *Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L_U}$ and $P_{prot} \circ p_i \in \mathfrak{P}_{BB}(N)$, from Lemma 12.1.3 we will have that*

$$\lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}}{|\{b_j\}|} \leq \lim_{N\to\infty} \Omega(w,c)\lg(N)$$

Note 12.1.4.1. This theorem gives an upper bound on the algorithmic complexity of the randomly generated part of the elements of the population $\mathfrak{P}_{BB}(N)$. And it will be crucial to prove a lower bound on the emergent algorithmic complexity. However, the upper bound of Lemma 12.1.4 is overestimated since an algorithmic probability distribution is far from being uniform, which is the case where Gibb's inequality applies on entropies.

Proof.

From the definition of language $\mathbf{L}_{BB}$ in 3.10 we have that p_i is independent of any topology, so that

$$(33) \qquad \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}}{|\{b_j\}|} =$$

$$= \frac{\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}$$

And in the limit the ramdonly generated population $\mathbf{L}_{\mathfrak{P}_{BB}(N)}$ tends to include all programs in $\mathbf{L_U}$, so that

$$(34) \qquad \lim_{N\to\infty} \frac{\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N} = \lim_{N\to\infty} \frac{\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} |p_i|}{N}$$

Since one can define algorithmic probabilities in $\mathbf{L_U}$, by the Strong Law of Large Numbers, as in 12.1.1 , we have that in the limit $\mathbf{L_U}(N)$ tends to follow the same distribution, so that [53]

$$(35) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} |p_i| = \lim_{N\to\infty} \frac{1}{N} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{N|p_i|}{2^{|p_i|}} =$$

---

[53] Note that, since $|Halt_{iso}(\mathbf{L_U}(N),w,c)| < N$, then $\displaystyle\sum_{p_i \in Halt_{iso}(\mathbf{L_U},w,c)} \frac{1}{2^{|p_i|}} = \Omega(w,c) < 1$.

Thus, $\displaystyle\lim_{N\to\infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}}$ is not a proper Shannon entropy.

$$= \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}}$$

From Lemma 12.1.3 we will have that

(36)
$$\lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \leq \lim_{N \to \infty} \Omega(w,c) \lg(\Omega(w,c)N) - \Omega(w,c) \lg(\Omega(w,c))$$

And

(37)
$$\Omega(w,c) \lg(\Omega(w,c)N) - \Omega(w,c) \lg(\Omega(w,c)) =$$
$$= \Omega(w,c) \lg(\Omega(w,c)) + \Omega(w,c) \lg(N) - \Omega(w,c) \lg(\Omega(w,c)) =$$
$$= \Omega(w,c) \lg(N)$$

So, from steps 33 , 34 , 35 , 36 and 37

$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in Halt_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} |p_i|}{N}}{|\{b_j\}|} =$$
$$= \lim_{N \to \infty} \frac{\sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} |p_i|}{N} =$$
$$= \lim_{N \to \infty} \sum_{p_i \in Halt_{iso}(\mathbf{L_U}(N),w,c)} \frac{|p_i|}{2^{|p_i|}} \leq \Omega(w,c) \lg(N)$$

$\square$

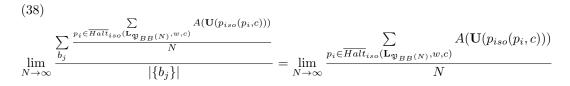**Lemma 12.1.5** (8.5). *Given a population $\mathfrak{P}_{BB}(N)$ defined in 3.11 , where $p_i \in \mathbf{L_U}$ and $P_{prot} \circ p_i = o_i \in \mathfrak{P}_{BB}(N)$, there is a constant C_0 such that*

$$\lim_{N \to \infty} \frac{\sum_{b_j} \frac{\sum_{p_i \in \overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = C_0(1 - \Omega(w,c))$$

Note 12.1.5.1. Note that every lemma until here deals with the behavior of $A(\mathbf{U}(p_{iso}(p_i,c)))$, so that they give tools to obtain an upper bound on the algorithmic complexity of what each node can do when isolated. Thus, these results are independent of any topological feature that the algorithmic network $\mathfrak{N}_{BB}(N,f,t,\tau,j)$ might have.

Proof.

As in step 33 , $A(\mathbf{U}(p_{iso}(p_i,c)))$ is independent of any topology, so that

(38)
$$\lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = \lim_{N\to\infty} \frac{\displaystyle\sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}$$

From the Definition 5.1.1 , as in step 35 , we will have that

(39)
$$\lim_{N\to\infty} \frac{\displaystyle\sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N} = \lim_{N\to\infty} \sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)} \frac{A(\mathbf{U}(p_{iso}(p_i,c)))}{2^{|p_i|}}$$

Now, let $A(0) = C_0$.

Since $p_i \in \overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)$ and $\mathfrak{P}_{BB}(N)$ is sensitive to oracles 3.11 , by the definition of p_{iso} in 4.1.1 we will have that, for every $p_i \in \overline{Halt}_{iso}(\mathbf{L_U},w,c)$,

(40)
$$A(\mathbf{U}(p_{iso}(p_i,c))) = A(0) = C_0$$

Thus, by the definition of $\Omega(w,c)$ in 5.2 ,

(41)
$$\lim_{N\to\infty} \sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)} \frac{A(\mathbf{U}(p_{iso}(p_i,c)))}{2^{|p_i|}} = \lim_{N\to\infty} \sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)} \frac{C_0}{2^{|p_i|}} =$$

$$= \lim_{N\to\infty} C_0 \sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)} \frac{1}{2^{|p_i|}} = C_0(1 - \Omega(w,c))$$

And we conclude from steps 38 , 39 and 41 that

$$\lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L}_{\mathfrak{P}_{BB}(N)},w,c)} A(\mathbf{U}(p_{iso}(p_i,c)))}{N}}{|\{b_j\}|} = \lim_{N\to\infty} \sum_{p_i\in\overline{Halt}_{iso}(\mathbf{L_U}(N),w,c)} \frac{A(\mathbf{U}(p_{iso}(p_i,c)))}{2^{|p_i|}} =$$

$$= C_0(1 - \Omega(w,c))$$

$\square$

Lemma 12.1.6 (8.6). *Let $\mathfrak{P}_{BB}(N)$ be a population in an arbitrary algorithmic network $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ as defined in 3.13 and 3.11.*
Let $t_0 \leq t \leq t' \leq t_{|T(G_t)|-1}$.
Let $c \in \mathfrak{C}_{\mathfrak{B}\mathfrak{B}}$ be an arbitrary number of cycles where $c_0 + t' + 1 \leq c$.
Then, there is a constant C_2 such that

$$\frac{\sum\limits_{b_j} \frac{\sum\limits_{p_i \in \mathfrak{P}_{BB}(N)} A(\boldsymbol{U}(p_{net}^{b_j}(o_i, c)))}{N}}{|\{b_j\}|} \geq (A_{max} - C_2)\, \tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} + C_2$$

Proof.

Let $\mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}$ denote the set of nodes/programs that belong to fraction $\tau_{max}(N, f, t, \tau, j)|_t^{t'}$ as defined in 6.2.1. Hence,

$$|\mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}| = N\tau_{max}(N, f, t, \tau, j)|_t^{t'}$$

Let $C_2 = \min\{A(w) \mid \exists x \in \mathbf{L}_{\mathbf{U}}(\mathbf{U}(x) = w)\}$. [54]

From the Definition 6.2.1 of $\tau_{max}(N, f, t_i, \tau, j)|_t^{t'}$ we will have that

$$
\begin{aligned}
(42) \quad & \frac{\sum\limits_{b_j} \frac{\sum\limits_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{N}}{|\{b_j\}|} = \\[2em]
& = \frac{\sum\limits_{b_j} \left(\frac{\sum\limits_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}} A(\mathbf{U}(p_{net}^{b_j}(o_i,c)))}{\tau_{max}(N,f,t,\tau,j)|_t^{t'} N} \tau_{max}(N, f, t, \tau, j)|_t^{t'} \right)}{|\{b_j\}|} + \\[2em]
& + \frac{\sum\limits_{b_j} \left(\frac{\sum\limits_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)}}} A(\mathbf{U}(p_{net}^{b_j}(o_i,c)))}{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)} N} \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} \right)}{|\{b_j\}|}
\end{aligned}
$$

Note that in the case the temporal diameter $D(G_t, t)$ is not well-defined one can replace $\tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)}$ with fraction

$$\frac{\left| \mathfrak{P}_{BB}(N) \setminus \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}} \right|}{N}$$

---

[54] Note that depending on the choice of the programming language one may have $C_2 \leq A(0)$ for example .

in the rest of this proof below without loss of generality.

Since fraction $\tau_{max}(N, f, t, \tau, j)|_t^{t'}$ is centered on a node from which A_{max} is diffused — see Definition 6.2 —, by definitions 3.10 and 6.1.1 we will have that

(43)
$$\frac{\displaystyle\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{\tau_{max}(N, f, t, \tau, j)|_t^{t'} N} \tau_{max}(N, f, t, \tau, j)|_t^{t'} \geq A_{max} \, \tau_{max}(N, f, t, \tau, j)|_t^{t'}$$

and, analogously, the following always holds despite on which node fraction $\tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)}$ is centered and whenever it starts to diffuse

(44)
$$\frac{\displaystyle\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)}}} A(\mathbf{U}(p_{net}^{b_j}(o_i, c)))}{\tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} N} \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} \geq C_2 \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)}$$

Thus, since we have that $\tau_{max}(N, f, t, \tau, j)|_t^{t'} + \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)} = 1$, then by steps 42 , 43 and 44

(45)
$$\frac{\displaystyle\sum_{b_j}\left(\frac{\displaystyle\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_t^{t'}}} A(\mathbf{U}(p_{net}^{b_j}(o_i,c)))}{\tau_{max}(N,f,t,\tau,j)|_t^{t'} N} \tau_{max}(N, f, t, \tau, j)|_t^{t'}\right)}{|\{b_j\}|} +$$

$$+ \frac{\displaystyle\sum_{b_j}\left(\frac{\displaystyle\sum_{o_i \in \mathbf{X}_{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)}}} A(\mathbf{U}(p_{net}^{b_j}(o_i,c)))}{\tau_{max}(N,f,t,\tau,j)|_{t'}^{D(G_t,t)} N} \tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)}\right)}{|\{b_j\}|} \geq$$

$$\geq \frac{\displaystyle\sum_{b_j}\left(A_{max}\tau_{max}(N, f, t, \tau, j)|_t^{t'} + C_2\tau_{max}(N, f, t, \tau, j)|_{t'}^{D(G_t,t)}\right)}{|\{b_j\}|} =$$

$$= \frac{\displaystyle\sum_{b_j}\left((A_{max} - C_2)\tau_{max}(N, f, t, \tau, j)|_t^{t'} + C_2\right)}{|\{b_j\}|}$$

$\square$

Theorem 12.1.1 (8.1)**.**

Let $w \in \mathbf{L_U}$ be a network input.

Let $0 < N \in \mathbb{N}$.

Let $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined.

Let $t_0 \le t \le t' \le t_{|T(G_t)|-1}$.

Let $c \colon \mathbb{N} \to \quad \mathfrak{C}_{\mathfrak{BB}} \quad$ be a total computable function where $c(x) \ge c_0 + t' + 1$.
$\quad\quad\quad x \mapsto c(x) = y$

Then, we will have that:

$$\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N, f, t, \tau)} \left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(x)) \right) \ge \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) -$$
$$- \Omega(w, c(x)) \lg(x) - \Omega(w, c(x)) \lg(\lg(x)) - A(w) - C_5$$

Note 12.1.1.1. Thus, note that for bigger enough values of $c(x)$ compared to $\lg(N)$ one can make this lower bound always negative for example. One of the main ideas behind forthcoming results in this paper is to find optimal conditions where this lower bound is not only positive, but also goes to ∞.

Note 12.1.1.2. Note that this lower bound on the expected emergent algorithmic complexity is dependent on the value in the domain of the function c and not on function c itself, even if it grows fast. In fact, one may want to obtain this theorem for fixed values of c in which it is not a function but an arbitrary value. And the same result also hold in this case. The reader is invited to check that a simple substitution of $c(x)$ for c and of x inside the logarithms for c is enough.

Note 12.1.1.3. The same result also holds if only one possible function b_j is defined for each member of the family $\mathbb{G}(f, t, \tau)$. However, the notion of expected emergent algorithmic complexity $\tau_{\mathbf{E}(max)}$ becomes redundant, as only one function b_j will be taken into account within the sum in order to give the mean. Thus, in this case one can replace $\tau_{\mathbf{E}(max)}$ with τ_{max} not only in Theorem 12.1.1 but also in 8.1.1 and 8.2 . Such variation of these theorems becomes useful when one has algorithmic networks $\mathfrak{N}_{BB}(N, f, t, \tau, j)$ built upon a historical population size growth in which each new node/program is linked (or not) to the previous existing algorithmic network.

Proof.

We have from AIT that there is $C_c \in \mathbb{N}$ for every $x \in \mathbb{N}$ such that

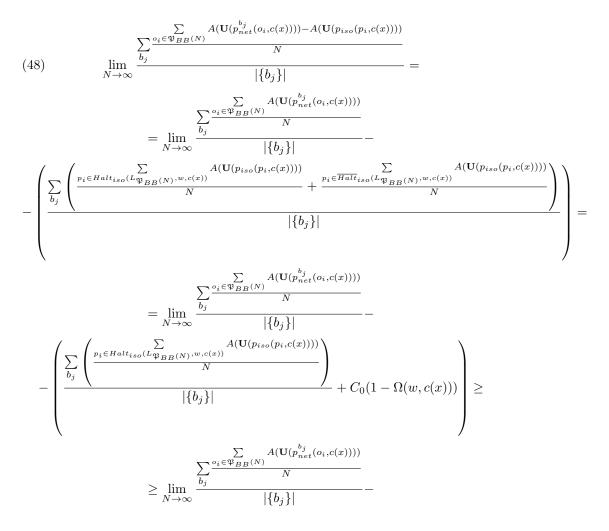$$(46) \qquad\qquad\qquad A(c(x)) \le C_c + A(x)$$

Let $C_5 = C_c + C_L + C_1 + C_4$.

From Definition 4.4, we have that the expected emergent algorithmic complexity of a node/program for $\mathfrak{N}_{BB}(N, f, t, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$, where $0 < j \leq |\{b_j\}|$ is given by

(47)

$$\mathbf{E}_{\mathfrak{N}_{BB}(N,f,t,\tau)}\left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(x))\right) = \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x)))) - A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}}{|\{b_j\}|}$$

And from definitions 5.1 , 5.1.1 , 5.2 , 6.2.1.1 , 3.5 and Lemmas 12.1.2 , 12.1.5 , 12.1.4 , 12.1.6 , 12.1.1 and step (46) we will have that

(48)
$$\lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x)))) - A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}}{|\{b_j\}|} =$$

$$= \lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x))))}{N}}{|\{b_j\}|} -$$

$$- \left(\frac{\displaystyle\sum_{b_j}\left(\frac{\displaystyle\sum_{p_i \in Halt_{iso}(L_{\mathfrak{P}_{BB}(N)},w,c(x))} A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N} + \frac{\displaystyle\sum_{p_i \in \overline{Halt}_{iso}(L_{\mathfrak{P}_{BB}(N)},w,c(x))} A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}\right)}{|\{b_j\}|}\right) =$$

$$= \lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x))))}{N}}{|\{b_j\}|} -$$

$$- \left(\frac{\displaystyle\sum_{b_j}\left(\frac{\displaystyle\sum_{p_i \in Halt_{iso}(L_{\mathfrak{P}_{BB}(N)},w,c(x))} A(\mathbf{U}(p_{iso}(p_i,c(x))))}{N}\right)}{|\{b_j\}|} + C_0(1 - \Omega(w,c(x)))\right) \geq$$

$$\geq \lim_{N\to\infty} \frac{\displaystyle\sum_{b_j} \frac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i,c(x))))}{N}}{|\{b_j\}|} -$$

$$
- \left(\frac{\displaystyle\sum_{b_j} \left(\dfrac{\displaystyle\sum_{p_i \in Halt_{iso}(L_{\mathfrak{P}_{BB}(N)}, w, c(x))} C_1 + |p_i| + A(w) + A(c(x))}{N} + C_0(1 - \Omega(w, c(x))) \right)}{|\{b_j\}|} \right) =
$$

$$
= \lim_{N \to \infty} \frac{\displaystyle\sum_{b_j} \dfrac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i, c(x))))}{N}}{|\{b_j\}|} -
$$

$$
- \left(\frac{\displaystyle\sum_{b_j} \left(\dfrac{\displaystyle\sum_{p_i \in Halt_{iso}(L_{\mathfrak{P}_{BB}(N)}, w, c(x))} |p_i|}{N} \right)}{|\{b_j\}|} + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big) \right) \geq
$$

$$
\geq \lim_{N \to \infty} \frac{\displaystyle\sum_{b_j} \dfrac{\displaystyle\sum_{o_i \in \mathfrak{P}_{BB}(N)} A(\mathbf{U}(p_{net}^{b_j}(o_i, c(x))))}{N}}{|\{b_j\}|} -
$$

$$
- \left(\Omega(w, c(x)) \lg(N) + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big) \right) \geq
$$

$$
\geq \lim_{N \to \infty} (A_{max} - C_2) \, \tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} + C_2 -
$$

$$
- \left(\Omega(w, c(x)) \lg(N) + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big) \right) \geq
$$

$$
\geq \lim_{N \to \infty} (\lg(N) - C_4 - C_2) \, \tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} + C_2 -
$$

$$
- \left(\Omega(w, c(x)) \lg(N) + \Omega(w, c(x))\big(C_1 + A(w) + A(c(x))\big) + C_0\big(1 - \Omega(w, c(x))\big) \right) =
$$

$$
= \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) - \Omega(w, c(x)) A(c(x)) -
$$

$$
- (C_4 + C_2) \tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) C_1 + \Omega(w, c(x)) C_0 + C_2 - C_0 - \Omega(w, c(x)) A(w) \geq
$$

$$
\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) - \Omega(w, c(x)) A(c(x)) -
$$

$$
- (C_4 + C_2) - C_1 + C_0 + C_2 - C_0 - A(w) =
$$

$$
= \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) - \Omega(w, c(x)) A(c(x)) -
$$

$$
- C_4 - C_1 - A(w) \geq
$$

$$
\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) -
$$

$$
- \Omega(w, c(x)) \lg(x) - \Omega(w, c(x))(1 + \epsilon) \lg(\lg(x)) - \Omega(w, c(x)) C_L -
$$

$$
- C_c - C_4 - C_1 - A(w)
$$

$$
\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t, \tau)\big|_t^{t'} - \Omega(w, c(x)) \right) \lg(N) -
$$

$$
- \Omega(w, c(x)) \lg(x) - \Omega(w, c(x)) \lg(\lg(x)) - C_5 - A(w)
$$

$$
\square
$$

Theorem 12.1.2 (8.2)**.**

Let $w \in \mathbf{L_U}$ be a network input.

Let $0 < N \in \mathbb{N}$.

Let $\mathfrak{N}_{BB}(N, f, t_z, \tau, j) = (G_t, \mathfrak{P}_{BB}(N), b_j)$ be well-defined.

If there is $0 \leq z_0 \leq |T(G_t)| - 1$ and $\epsilon, \epsilon_2 > 0$ such that

$$z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^C}{\lg(N)}\right)$$

where

$$-1-\epsilon \leq C = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)} \leq \frac{1}{\epsilon_2}$$

Then, for every non-decreasing total computable function $c \colon \mathbb{N} \to \mathfrak{C}_{\mathfrak{B}\mathfrak{B}}$ where $x \mapsto c(x) = y$ $t_{z_0} \in T(G_t)$ and $c(z_0 + f(N, t_{z_0}, \tau) + 2) \geq c_0 + z_0 + f(N, t_{z_0}, \tau) + 2$ we will have that there are $t_{cen_2}(c)$ and $t_{cen_1}(c)$ such that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

Proof.
We know from Corollary 8.1.1 that

(49)
$$\lim_{N \to \infty} \mathbf{E}_{\mathfrak{N}_{BB}(N,f,t_z,\tau)}\left(\overset{net}{\underset{iso}{\Delta}} A(o_i, c(z + f(N, t_z, \tau) + 2))\right) \geq$$

$$\geq \lim_{N \to \infty} \left(\tau_{\mathbf{E}(max)}(N, f, t_z, \tau)|_{t_z}^{t_z+f(N,t_z,\tau)} - \Omega(w, c(z + f(N, t_z, \tau) + 2))\right) \lg(N) -$$
$$- \Omega(w, c(z + f(N, t_z, \tau) + 2)) \lg(z + f(N, t_z, \tau) + 2) -$$
$$- \Omega(w, c(z + f(N, t_z, \tau) + 2)) \lg(\lg(z + f(N, t_z, \tau) + 2)) - A(w) - C_5$$

Suppose that there is $t_{z_0} \in T(G_t)$, where $0 \leq z_0 \leq |T(G_t)| - 1$, and $\epsilon > 0$ such that

(50)
$$z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^C}{\lg(N)}\right)$$

where

$$C = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)}$$

From the Definition 5.2 we have that, for every $y \in \mathbb{N}$, if $y \geq c_0 + z_0 + f(N, t_{z_0}, \tau) + 2$, then

$$(51) \qquad\qquad \Omega(w, y) \leq \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)$$

Thus, since we are assuming $c(z_0 + f(N, t_{z_0}, \tau) + 2) \geq c_0 + z_0 + f(N, t_{z_0}, \tau) + 2$, for fixed values of $\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)}$ and ϵ we will have from step 51 that

$$(52) \qquad \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) - \epsilon}{\Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))} \geq$$

$$\geq \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2) - \epsilon}{\Omega(w, c_0 + z_0 + f(N, t_{z_0}, \tau) + 2)} = C$$

Let

$$C' = \frac{\tau_{\mathbf{E}(max)}(N, f, t_{z_0}, \tau)|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2)) - \epsilon}{\Omega(w, c(z_0 + f(N, t_{z_0}, \tau) + 2))}$$

Remember that for every $x > 0$ and $t, t' \in T(G_t)$ there is ϵ_2 such that[55]

$$(53) \qquad\qquad 0 < \epsilon_2 \leq \Omega(w, x) \leq 1$$

and thus, from the Definition 6.2.1.1 , we will also have that

$$(54) \qquad -1 - \epsilon \leq \frac{\tau_{\mathbf{E}(max)}(N, f, t, \tau)|_t^{t'} - \Omega(w, x) - \epsilon}{\Omega(w, x)} \leq \frac{1}{\epsilon_2}$$

Hence, from steps 52 and 54 we will have that

$$z_0 + f(N, t_{z_0}, \tau) = \mathbf{O}\left(\frac{N^{C'}}{\lg(N)}\right)$$

where

---

[55] Remember that one can always have a program that halts for every input, so that it will also halts for every partial output and, hence, halt on every cycle — see 5.2 .

$$-1-\epsilon \le C' = \frac{\tau_{\mathbf{E}(max)}(N,f,t_{z_0},\tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2)) - \epsilon}{\Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))} \le \frac{1}{\epsilon_2}$$

And since $z_0 + f(N,t_{z_0},\tau)$ is now assymptotically dominated by $\frac{N^{C'}}{\lg(N)}$, then by definition we will have that there is a constant C_6 such that

(55)
$$\lim_{N\to\infty} \left(\tau_{\mathbf{E}(max)}(N,f,t_{z_0},\tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\right)\lg(N) -$$

$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg(z_0+f(N,t_{z_0},\tau)+2) -$$
$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg(\lg(z_0+f(N,t_{z_0},\tau)+2)) - A(w) - C_5 \ge$$

$$\ge \lim_{N\to\infty} \left(\tau_{\mathbf{E}(max)}(N,f,t_{z_0},\tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\right)\lg(N) -$$

$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg(C_6\frac{N^{C'}}{\lg(N)}) -$$

$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg(\lg(C_6\frac{N^{C'}}{\lg(N)})) - A(w) - C_5 \ge$$

$$\ge \lim_{N\to\infty} \left(\tau_{\mathbf{E}(max)}(N,f,t_{z_0},\tau)\big|_{t_{z_0}}^{t_{z_0}+f(N,t_{z_0},\tau)} - \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\right)\lg(N) -$$
$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\left(\lg(C_6) + C'\lg(N) - \lg(\lg(N))\right) -$$
$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg\left(\lg(C_6) + \lg(N^{C'}) - \lg(\lg(N))\right) - A(w) - C_5 \ge$$

$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\left(\lg(C_6) - \lg(\lg(N))\right) -$$
$$- \Omega(w,c(z_0+f(N,t_{z_0},\tau)+2))\lg\left(\lg(C_6) + \lg(N^{C'}) - \lg(\lg(N))\right) - A(w) - C_5 \ge$$

$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \left(\lg(C_6) - \lg(\lg(N))\right) - \lg\left(\lg(C_6) + \lg(N^{C'}) - \lg(\lg(N))\right) -$$
$$- A(w) - C_5 \ge$$

$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \lg(C_6) + \lg(\lg(N)) - \lg\left(2\lg(N^{C'})\right) - A(w) - C_5 \ge$$
$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \lg(C_6) + \lg(\lg(N)) - \lg(2) - \lg(C'\lg(N)) - A(w) - C_5 \ge$$
$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \lg(C_6) + \lg(\lg(N)) - \lg(2) - \lg(C') - \lg(\lg(N)) - A(w) - C_5 \ge$$
$$\ge \lim_{N\to\infty} (\epsilon)\lg(N) - \lg(C_6) - \lg(2) - \lg(\frac{1}{\epsilon_2}) - A(w) - C_5 = \infty$$

Thus, from steps 49 and 55, we will have that

(56)
$$\lim_{N\to\infty} \mathbf{E}_{\mathfrak{N}_{BB}(N,f,t_{z_0},\tau)}\left(\overset{net}{\underset{iso}{\Delta}}A(o_i,c(z_0+f(N,t_{z_0},\tau)+2))\right) = \infty$$

Then, directly from the definitions 7.1 and 7.2 and step 56 , since t_{z_0} satisfy these definitions, we will have that

$$t_{cen_2}(c) = t_{cen_1}(c) \leq t_{z_0}$$

$\square$