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Existence of Mild Solution for Impulsive Fractional Differential Equations with Non-local Conditions in Banach Space

Hamdy M. Ahmed[1*], A. A. M. Hassan[1] and A. S. Ghanem[1]

[1] *Higher Institute of Engineering, El-Shorouk Academy, P.O. 3 El-Shorouk City, Cairo, Egypt.*

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Abstract

In this paper, by using the probability density function we introduce the mild solution of fractional differential equations with impulsive conditions and obtain various criteria on the existence of mild solutions by using fixed point theorem.

Keywords: Impulsive fractional differential equations, impulsive conditions, mild solution, semigroup theory, probability density function.

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1 Introduction

Recently, fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. indeed we can find many application in viscoelasticity, elctrochemistry, control, porous media and electromagnetic there has been a significant development in fractional differential equations in recent years see ([1-12]).

actually fractional differential equations are considered as an alternative model to integer differential equations for more details on fractional calculus theory one can see the excellent books [13,14].

In particular the non-local problems for impulsive fractional differential equations have been attractive to many researchers the advantage of impulsive fractional differential equations is that they can describe the model which at certain moments change their state rapidly and which can't be modeled by the classical differential equations (15).

Present work deals with fractional differential equations with impulsive conditions unlike ref [16] which deals with ordinary differential equations have no impulsive condition.

The main purpose of this paper is to prove the existence of mild solutions for the following impulsive fractional differential equations in a Banach space X:

*Corresponding author: E-mail: hamdy_17eg@yahoo.com

$$ {}^{c}D_0^q[x(t) - F(t, x(t), x(b_1(t)), ..., x(b_m(t))] = A[x(t) - F(t, x(t), x(b_1(t)), ..., x(b_m(t))] $$

$$ +G(t, x(t), x(a_1(t)), ..., x(a_n(t)))\ t \in J = [0, b], t \neq t_k, k = 1, 2, 3......, m \quad (1.1) $$

$$ x(0) + g(x) = x_0 \in X $$
$$ \Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, 3, ...m \qquad (1.2) $$

The linear operator A generates an analytic semigroup$(T(t))_{t\geq 0}$
where
$(T(t))_{t\geq 0}$ is a compact analytic semigroup of uniformly bounded linear operators $T(t)$ on X
$(T(t))_{t\geq 0}$ is a compact analytic semigroup of uniformly bounded linear
operators $(T(t))$ on X

$$ \Delta x|_{t=t_k} = I_k(x(t_k^-)) $$

where,
$x(t_k^+)$ is the right limit of $x(t)$ at $(t = t_k)$, $x(t_k^-)$ is the left limit of $x(t)$ at $(t = t_k)$. F, G and g are given functions to be specified later and ${}^{c}D_0^q$ is Caputo fractional derivative of order $0 < q < 1$

2 Preliminaries

Let X be a Banach space with norm $\|.\|$ and A :$D(A) \to X$ is the generator of a compact analytic semigroup of uniformly bounded linear operators $(T(t))$ on X.
there exist $M \geq 1$ such that $\|T(t)\| \leq M, t \geq 0$ (17)
We need some basic definitions and properties of the fractional calculus theory which are used in this paper
Definition 2.1 (15)
The fractional integral of order q with the lower limit 0 for a function f is defined as:

$$ I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, $$

$t > 0, \quad q > 0$
where Γ is the gamma function.
Definition 2.2 (15)
The Caputo derivative of order q with the lower limit 0 for a function f is defined as:

$$ {}^{c}D_0^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(s)}{(t-s)^{q+1-n}} ds, $$

$t > 0 \quad , 0 \leq n-1 < q < n$
We shall state some properties of the fractional integral I_0^q and fractional differential ${}^{c}D_0^q$ operators:
Properties. (18) For $q, r > 0$ and f as a suitable function we have:

1. $I_0^q I_0^r f(t) = I_0^{q+r} f(t)$
2. $I_0^q I_0^r f(t) = I_0^r I_0^q f(t)$
3. $I_0^q (f(t) + g(t)) = I_0^q f(t) + I_0^q g(t)$
4. $I_0^q\ {}^{c}D_0^q f(t) = f(t) - f(0)$, $0 < q < 1$
5. ${}^{c}D_0^q\ I_0^q f(t) = f(t)$
6. ${}^{c}D_0^q\ {}^{c}D_0^r f(t) \neq {}^{c}D_0^{q+r} f(t)$

7. ${}^{c}D_0^q\ {}^{c}D_0^r f(t) \neq {}^{c}D_0^r\ {}^{c}D_0^q f(t)$

For simplicity of notions we shall take ${}^{c}D_0^q f(t)$, $I_0^q f(t)$ as $D^q f(t)$,$I^q f(t)$.

Theorem 2.1. (Sadovskii)(17)

Let P be a condensing operator on a Banach space that is P is continuous and takes bounded sets into bounded sets; let $\alpha(P(B)) \leq \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$ of $P(H) \subset H$ for a convex, closed and bounded set H of X, then P has fixed point in H.

3 Main Results

In order to define the solution of the problem (1.1-1.2), we define the following space.

$$\Omega = \{x : J \to X, x_k \in C(J_k, X), k = 0, 1, ...m,$$

there exist

$$x(t_k^+), x(t_k^-), k = 0, 1, ..., m, x(t_k^-) = x(t_k); x(0) + g(x) = x_0\}$$

which is a Banach space with the norm $\|x\|_\Omega = max[\|x_k\|_{J_k}; k = 0, 1, ..., m]$ (17)

Definition 3.1 (15) A function $x \in \Omega$ is said to be a mild solution of the system (1.1)- (1.2) if:

(i)$x_0 + g(x) = x_0$

(ii)$\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, 3, ...m$

$$x(t) = S_q(t)[x_0 - g(x) - F(0, x(0), x(b_1(0)), ..., x(b_m(0)))] + F(t, x(t), x(b_1(t)), ..., x(b_m(t))$$
$$+ \int_0^t (t-s)^{q-1} T_q(t-s) G(s, x(s), x(a_1(s)), ..., x(a_n(s)) ds + \sum_{0<t_k<t} T_q(t - t_k) I_k(x(t_k^-))$$

, $t \in J$

where, $S_q(t)$ and $T_q(t-s)$ are defined by:

$S_q(t) = \int_0^\infty \phi_q(\theta) T(t^q \theta) d\theta$

$T_q(t-s) = q \int_0^\infty \theta \phi_q(\theta) T((t-s)^q\ \theta) d\theta$

where, $\phi_q(\theta)$ is the density function $\int_0^\infty \phi_q(\theta) d\theta = 1$, $\theta \in (0, \infty)$

Assume the following conditions (17):

(H1)there exist constant $\beta \in (0, 1)$ such that: $F : J \times X^{m+1} \to X$ is continuous function. $A^\beta F : J \times X^{m+1} \to X$ satisfy Lipschitz condition, that $\exists$ constant $L > 0$ such that:

$$\|A^\beta F(s_1, x_0, x_1, ..., x_m) - A^\beta F(s_2, x_0', x_1', ..., x_m')\| \leq L(|S_1 - S_2| + max_{i=0,1,...,m}\|x_i - x_i'\|)$$

for any $0 \leq s_1,\ s_2 \leq b$, $x_i, x_i' \in X$, $i = 0, 1, ..., m$

and there exist constant $L_1 > 0$ such that:
$\|A^\beta F(t, x_0, x_1, ..., x_m)\| \leq L_1(max\{\|x_i\| : i = 0, 1, ..., m\} + 1)$, holds for any $(t, x_0, x_1, ..., x_m) \in J \times X^{m+1}$

(H2) $G : J \times X^{n+1} \to X$, this function satisfy the following conditions:

(i) For each $t \in J$; $G(t, .) : X^{n+1} \to X$ is continuous, for

$$(x_0, x_1, ..., x_m) \times X^{n+1}$$

the function $G(., X_0, X_1, ..., X_n) : J \to X$ is strongly measurable.

(ii) For each $r \in N$ there is a positive function $g_r \in L^1(J)$ such that $sup_{\|x_0\|, ...\|x_n\| \leq r} \|G(t, x_0, x_1, ..., x_n)\| \leq g_r(t)$ and

$$\lim_{r\to\infty} inf \frac{\int_0^b (t-s)^{q-1} g_r(s) ds}{r} = \mu\Gamma(q) < (+\infty).$$

where μ is a constant.
(H3) $a_i, b_j \in C(J, J)$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$
where a_i, b_j are constants.
(H4) There exist positive constants $L_2, L_2^{'}$ such that $\|g(x)\| \leq L_2\|x\|_\Omega + L_2^{'}$ for all $x \in \Omega$ and $g : \Omega \to X$ is completely continuous.
(H5) $I_k : X \to X$ is completely continues and their exist continuous non-decreasing functions $L_k : R_+ \to R_+$ Such that for each $x \in X$.

$$\|I_k(x)\| \leq L_k(\|x\|), \lim_{r\to\infty} inf \frac{L_k(r)}{r} = \lambda_k\Gamma(q) \leq +\infty$$

where λ_k is a constant.
Also the present work inquires the following lemma (15):
lemma 3.1 For fixed $t \geq 0$, $(S_q(t), T_q(t))$ are linear and bounded operators
(i.e.,)

$$\|S_q(t)\| \leq M$$

$$\|T_q(t)\| \leq \frac{qM}{\Gamma(1+q)}$$

Proof.for any fixed $t \geq 0$,since $T(t)$ is linear operator,it is easy to see that $(S_q(t), T_q(t))$ are linear operators for $\xi \in [0, 1]$ according to[16] we find that

$$\int_0^\infty \frac{1}{\theta^\xi}\Psi_q(\theta)d\theta = \frac{\Gamma(1+\frac{\xi}{q})}{\Gamma(1+\xi)}$$

then we have

$$\int_0^\infty \theta^\xi\phi_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^{q\xi}}\Psi_q(\theta)d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+q\xi)}$$

in the case of $\xi = 1$ we have

$$\int_0^\infty \theta\phi_q(\theta)d\theta = \int_0^\infty \frac{1}{\theta^q}\Psi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}$$

for any $x \in X$ we have

$$|S_q(t)\ x| = |\int_0^\infty \phi_q(\theta)T(t^q\theta)xd\theta| \ \ \leq M|x|$$

and

$$|T_q(t)\ x| = |q\int_0^\infty \theta\phi_q(\theta)T(t^q\theta)xd\theta| \ \ \leq \frac{qM}{\Gamma(1+q)}|x|$$

lemma 3.2 The operators $(S_q(t))_{t\geq0}, (T_q(t))_{t\geq0}$ are strongly continuous, which means ,$\forall x \in X, (0 \leq t^{'} < t^{''} \leq b)$

$$\|S_q(t^{''})x - S_q(t^{'})x\| \to 0, \|T_q(t^{''})x - T_q(t^{'})x\| \to 0$$

as $(t^{'} \to t^{''})$
Proof. The operators $S_q(t)_{t\geq0}$ and $T_q(t)_{t\geq0}$ are strongly continues which means that:
For every $x \in X$ and $0 \leq t^{'} \leq t^{''} \leq a$ we have:

$$|T_q(t^{''})x - T_q(t^{'})x| = |q\int_0^\infty \theta\phi_q(\theta)[T((t^{''})^q\theta) - T((t^{'})^q\theta)]xd\theta|$$

$$\leq qM\int_0^\infty \theta\phi_q(\theta)|[T((t^{''})^q\theta-(t^{'})^q\theta)-I]x|d\theta$$

According to the strongly continuity of $T(t)_{t\geq 0}$

We note that $|T_q(t^{''})x-T_q(t^{'})x|$ tends to zero as $t^{''}-t^{'}\rightarrow 0$ which means that $\{T_q(t)\}_{t\geq 0}$ is strongly continuous. Using a similar method we can also obtain that $\{S_q(t)\}_{t\geq 0}$ is also strongly continues .

lemma 3.3 If $T(t)$ is compact operator for $t>0$ then $(S_q(t),T_q(t))$ are also compact operators for $t>0$

Our main results may be presented as the following theorem:

Theorem 3.1

Assume the conditions (H1)-(H5) then the system (1.1)- (1.2) has mild solution on J provided that:

$$L_0=L[(M+1)M_0]<1$$

$$M[L_2+M_0L_1+\mu+\sum_{k=1}^{m}\lambda_k]+M_0L_1<1$$

where $M_0=\|A^{-\beta}\|$

Proof. For simplicity we rewrite that

$$(t,x(t),x(b_1(t),......,x(b_m(t)))=(t,v(t))$$

and

$$(t,x(t),x(a_1(t),......,x(a_n(t)))=(t,u(t))$$

consider the operator $N:\Omega\rightarrow\Omega$ defined by

$$N(x)=\{\varphi\in\Omega:\varphi(t)=S_q(t)[x_0-g(x)-F(0,v(0))]+F(t,v(t))$$

$$+\int_0^t(t-s)^{q-1}T_q(t-s)G(s,u(s))ds+\sum_{0<t_k<t}T_q(t-t_k)I_k(x(t_k^-)),t\in J\}$$

The fixed points of N are mild solutions to the system (1.1)- (1.2) We shall show that N satisfies the hypotheses of Theorem (2.1) we will find the proof by the following steps.

Step1. There exists a positive integer $r\in N$ such that $N(B_r)\subset B_r$, where

$$B_r=\{x\in\Omega:\|x\|\leq r,0\leq t\leq b\}$$

For each positive number r,B_r is bounded,closed and convex set in Ω. We want to prove that $N(B_r)\subset B_r$, we use contradiction.

Let $N(B_r)$ is not subset of B_r then for each positive integer r, there exist the functions $x_r(.)\in B_r$ and $\varphi_r(.)\in N(x_r)$, but $\varphi_r(.)\notin B_r$, where $N(B_r)=\bigcup_{x\in B_r}N(x)$

that is

$$r<\|\varphi_r(t)\|=\|S_q(t)[x_0-g(x_r)-F(0,v_r(0))]+F(t,v_r(t))$$

$$+\int_0^t(t-s)^{q-1}T_q(t-s)G(s,u_r(s))ds+\sum_{0<t_k<t}T_q(t-t_k)I_k(x_r(t_k^-)))\|$$

$$\leq\|S_q(t)[x_0-g(x_r)-A^{-\beta}A^{\beta}F(0,v_r(0))]\|+\|A^{-\beta}A^{\beta}F(t,v_r(t))\|+\int_0^t\|(t-s)^{q-1}T_q(t-s)G(s,u_r(s))\|ds$$

$$+\sum_{0<t_k<t}\|T_q(t-t_k)I_k(x_r(t_k^-)))\|$$

$$r\le M[\|x_0\|+L_2r+L_2^{'}+M_0L_1(r+1)]+M_0L_1(r+1)+\frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}g_r(s)ds+\frac{M}{\Gamma(q)}\sum_{k=1}^{m}L_k(r)$$

dividing both sides on r and take the lower limit as $r\to\infty$ we get

$$1\le M[\frac{\|x_0\|}{r}+L_2+\frac{L_2^{'}}{r}+M_0L_1\frac{(r+1)}{r}]+M_0L_1\frac{(r+1)}{r}+\frac{M}{\Gamma(q)}inf\frac{\int_0^t(t-s)^{q-1}g_r(s)ds}{r}$$

$$+\frac{M}{\Gamma(q)}inf\sum_{k=1}^{m}\frac{L_k(r)}{r}$$

$$1\le M[L_2+M_0L_1]+M_0L_1+M\mu+M\sum_{k=1}^{m}\lambda_k$$

$$1\le M[L_2+M_0L_1+\mu+\sum_{k=1}^{m}\lambda_k]+M_0L_1 \quad (3.1)$$

the equation (3.1) contradict with the condition of theorem (3.1) then for positive integer $r\in N$ we find that

$$N(B_r)\subseteq B_r$$

step2 we will show that the operator $N=(N_1+N_2)$ is condensing this means thatN_1 is contraction and N_2 is compact the operators(N_1,N_2)are defined on B_r by:

$$(N_1x)(t)=F(t,v(t))-S_q(t)F(0,v(0))$$

$$N_2x=\{\varphi\in\Omega:\varphi(t)=S_q(t)[x_0-g(x)]+\int_0^t T_q(t-s)G(s,u(s))\,ds+\sum_{0<t_k<t}T_q(t-t_k)I_k(x(t_k^-)))\}$$

to prove that N_1 is contraction ,we take $x_1,x_2\in B_r$. then for each $t\in J$ and by using the first condition from the previous assuming conditions also we use this condition

$$L_0=L[(M+1)M_0]<1$$

.

let $x_1,x_2\in B_r$ then for each $t\in J$ we have

$$\|(N_1x_1)(t)-(N_1x_2)(t)\|\le\|F(t,v_1(t))-F(t,v_2(t))\|+\|S_q(t)[F(0,v_1(0))-F(0,v_2(0))]\|$$

$$=\|A^{-\beta}[A^{\beta}F(t,v_1(t))-A^{\beta}F(t,v_2(t)]\|+\|S_q(t)A^{-\beta}[A^{\beta}F(0,v_1(0))-A^{\beta}F(0,v_2(0)]\|$$

$$\le M_0Lsup_{0\le s\le b}\|x_1(s)-x_2(s)\|+MM_0Lsup_{0\le s\le b}\|x_1(s)-x_2(s)\|$$

$$\le M_0L(M+1)sup_{0\le s\le b}\|x_1(s)-x_2(s)\|$$

$$\leq L_0 sup_{0\leq s\leq b}\|x_1(s)-x_2(s)\|$$

,

where

$M_0L(M+1)=L_0<1$ then $\|N_1x_1-N_1x_2\|\leq L_0\|x_1-x_2\|, 0<L_0<1$ then N_1 is contraction operator (8)to prove that N_2 is compact operator,firstly we prove that N_2 is continuous on B_r.

let $\{x_n\}_{n=0}^{n=\infty}$ with $x_n\rightarrow x$ in B_r,then by using the conditions of $(H2)(i)$ and $(H5)$.

$$(i)I_k, k=1,2,.....,m$$

is continuous .

$$(ii)G(s,u_n(s))\rightarrow G(s,u(s))$$

as $n\rightarrow\infty$

since ,$\|G(s,u_n(s))-G(s,u(s))\|\leq 2g_r(s)$

we have to prove that the operator N is continuous

$$\|N_2x_n-N_2x\|=sup\|S_q(t)[g(x)-g(x_n)]+\int_0^t T_q(t-s)[G(s,u_n(s))-G(s,u(s))]ds$$

$$+\sum_{0<t_k<t}T_q(t-t_k)[I_k(x_n(t_k^-))-I_k(x(t_k^-))]\|$$

$$\|N_2x_n-N_2x\|\leq M\|g(x_n)-g(x)\|+M\int_0^b\|G(s,u_n(s))-G(s,u(s))\|ds+M\sum_{0<t_k<t}\|I_k(x_n(t_k^-))-I_k(x(t_k^-))\|$$

$$(\|g(x_n)-g(x)\|\rightarrow 0, \|I_k(x_n(t_k^-))-I_k(x(t_k^-))\|\rightarrow 0, \|G(s,u_n(s))-G(s,u(s))\|\rightarrow 0) \quad as \quad n\rightarrow\infty$$

then

$$\|N_2x_n-N_2x\|\rightarrow 0 \;\; as \;\; n\rightarrow\infty$$

then N is continuous,

next we prove that $\{N_2x : x\in B_r\}$ is a family of equicontinuous functions

$$let \;\; x\in B_r and \;\; \tau_1,\tau_2\in J$$

then if $0<\tau_1\leq\tau_2\leq b$ and $\varphi\in N_2(x)$ then for each $t\in J$,

we have

$$\varphi(t)=S_q(t)[x_0-g(x)]+\int_0^t(t-s)^{q-1}T_q(t-s)G(s,u(s))ds+\sum_{0<t_k<t}T_q(t-t_k)I_k(x(t_k^-)))$$

then

$$\|\varphi(\tau_2)-\varphi(\tau_1)\|\leq\|S_q(\tau_2)-S_q(\tau_1)\|\|x_0-g(s)\|+\int_0^{\tau_1-\epsilon}\|T_q(\tau_2-s)-T_q(\tau_1-s)\|\|G(s,u(s))\|ds$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}\|T_q(\tau_2-s)-T_q(\tau_1-s)\|\|G(s,u(s))\|ds$$

$$+\int_{\tau_1}^{\tau_2}\|T_q(\tau_2-s)\|\|G(s,u(s))\|ds$$

$$+\sum_{0<t_k<\tau_1}\|T_q(\tau_2-t_k)-T_q(\tau_1-t_k)\|\|I_k(x(t_k^-))\|$$

$$+\sum_{\tau_1<t_k<\tau_2}\|T_q(\tau_2-t_k)\|\|I_k(x(t_k^-))\|$$

the right hand side of the previous equation is independent of $x\in B_r$ and tends to zero as $\tau_2-\tau_1\to 0$.

since the compactness of $\{T(t)\}_{t\geq 0}$ implies the continuity in the uniform operator topology, similarly using the compactness of the set $g(B_r)$ we can prove that the functions$N_2x, x\in B_r$ are equi-continuous at $t=0$. hence N_2 maps B_r into a family of equi-continuous functions. it remains to prove that$(N_2B_r)(t)$ is relatively compact for each $t\in J$

where

$$(N_2B_r)(t)=\{\varphi(t):\varphi\in N_2(B_r)\}, t\in J$$

obviously by using condition $(H4)$,$(N_2B_r)(t)$ is relatively compact in Ω for $t=0$.

Let $0<t\leq b$ be fixed and $0<\epsilon<t$ for $x\in B_r$ and $\varphi\in N_2(x)$ we have

$$\varphi(t)=S_q(t)[x_0-g(x)]+\int_0^t(t-s)^{q-1}T_q(t-s)G(s,u(s))ds+\sum_{0<t_k<t}T_q(t-t_k)I_k(x(t_k^-))\ , t\in J$$

define

$$\varphi_\epsilon(t)=S_q(t)[x_0-g(x)]+\int_0^{t-\epsilon}(t-s)^{q-1}T_q(t-s)G(s,u(s))ds+\sum_{0<t_k<t-\epsilon}T_q(t-t_k)I_k(x(t_k^-))\ , t\in J$$

$$=S_q(t)[x_0-g(x)]+T_q(\epsilon)\int_0^{t-\epsilon}(t-s)^{q-1}T_q(t-s-\epsilon)G(s,u(s))ds+T(\epsilon)\sum_{0<t_k<t-\epsilon}T_q(t-t_k-\epsilon)I_k(x(t_k^-)), t\in J$$

since $\{T_q(t)\}_{t\geq 0}$ is compact, the set $V_\epsilon(t)=\{\varphi_\epsilon(t):\varphi\in N_2(B_r)\}$ is relatively compact in Ω for every ϵ, $0<\varepsilon<t$

for every $\varphi\in N_2(B_r)$

$$\|\varphi(t)-\varphi_\epsilon(t)\|\leq\int_{t-\epsilon}^t\|(t-s)^{q-1}T_q(t-s)G(s,u(s))\|ds+\sum_{t-\epsilon<t_k<t}\|T_q(t-t_k-\epsilon)I_k(x(t_k^-))\|$$

$$\leq\frac{M}{\Gamma(q)}\int_{t-\epsilon}^t(t-s)^{q-1}g_r(s)ds+\frac{M}{\Gamma(q)}\sum_{t-\epsilon<t_k<t}L_k(r)$$

Therefore, let $\epsilon\to 0$ we can see that there are relatively compact sets close to the set $\{\varphi(t):\phi\in(N_2B_r)\}$ is relatively compact in Ω.

As a sequence of the above steps and the Arzela-Ascoli theorem we can conclude that N_2 is a compact operator. These arguments enable us to conclude that $N=N_1+N_2$ is a condensing map on B_r and by using the fixed point theorem of Sadoviskii, there exist a fixed point $x(.)$ for N on B_r. Therefore the non-local system (1.1)-(1.2) has a mild solution (16).

4 Applications

consider the following nonlinear integro- partial differential equation of fractional order(17):

$$\frac{\partial^q}{\partial t^q}[z(t,x) - \int_0^\pi b(y,x)z(te^t,y)dy] = \frac{\partial^2}{\partial x^2}[z(t,x) - \int_0^\pi b(y,x)z(te^t,y)dy] + h(t, z(te^t,x))$$

$0 \le t \le b$, $0 \le x \le \pi$, $t \ne t_k$,$k = 1, 2, ..., m$ (4.1)
$z(t, 0) = z(t, \pi) = 0$ (4.2)
$z(t_k^+) - z(t_k^-) = I_k(z(t_k^-)), k = 1, 2, ..., m,$ (4.3)
$z(0, x) + \sum_{i=0}^p \int_0^\pi k(x, y)z(t_i, y)dy = z_0x, \quad 0 \le x \le \pi$ (4.4)

where p is a positive integer,$0 < t_0 < .... < t_p < 1$, and $0 < t_1 < t_2 < ....t_m < b$. the function $z_0x \in X = L^2([0, \pi])$ and A is defined by $Af = f''$ with the domain $D(A) = \{f(.) \in X : f', f'' \in X, f(0) = f(\pi) = 0\}$.

then A generates a strongly continuous semigroup T(.) which is compact ,analytic and self adjoint .

Furthermore A has a discrete spectrum and has an eigenvalues $-n^2, n \in N$, with the corresponding normalized eigenvectors.

$Z_n(x) = \sqrt{\frac{2}{\pi}}\, sin(nx)$ .then the following properties hold :
(a)if $f \in D(A)$, then
$Af = \sum_{n=1}^\infty n^2 < f, z_n > z_n.$
(b) the operator $A^{\frac{1}{2}}f = \sum_{n=1}^\infty n < f, z_n > z_n$
on the space $D(A^{\frac{1}{2}}) = \{f(.) \in X, \sum_{n=1}^\infty n < f, z_n > z_n \in X\}$.
We assume that the following conditions hold:
(i)The function b is measurable and
$\int_0^\pi \int_0^\pi b^2(y, x)dydx < \infty$.
(ii) the function $\frac{\partial}{\partial x}b(y, x)$ is measurable ,$b(y, 0) = b(y, \pi) = 0$ and let
$N_1 = [\int_0^\pi \int_0^\pi (\frac{\partial}{\partial x}b(y, x))^2dydx]^{\frac{1}{2}} < \infty$
(iii) For the function $h : J \times R \to R$ the following three conditions are satisfied :
(1) for each$t \in J, h(t, .)$ is continuous
(2) for each $z \in X, h(., z)$ is measurable
(3) There are positive functions $h_1, h_2 \in L^1(J)$ such that
$|h(t, z)| \le h_1(t)|z| + h_2(t)$, for every $(t, z) \in J \times X$.
(iv) the function $I_K(x, x), k = 1, 2, ..., m$ and there exist nondecreasing functions
$L_k \in (J, R_+), k = 1, 2, ..., m$ such that for each $x \in X$ $\|I_k(x)\| \le L_k(\|x\|)$
We define $F, G : X \times X \to X$ and $g : \Omega \to X$ by
$F(t, z) = Z_1(z),$
$G(t, z)(x) = h(t, z(x)),$
$g((w(t))) = \sum_{i=0}^p Kw(t_i)$,$w \in \Omega$ (Ω is defined in section 3) , where
$Z_1(z)(x) = \int_0^\pi b(y, x)z(y)dy$
$K(z)(x) = \int_0^\pi k(x, y)z(y)dy$

Then G satisfies condition (2)in section (3) while g verifies condition (4) in section (3) (noting that $K : X \to X$ is completely continuous).

from(i) it is clear that Z_1 is a bounded linear operators on X

furthermore , $Z_1(z) \in D[A^{\frac{1}{2}}]$ and $\|A^{\frac{1}{2}}Z_1\| \le N_1$. in fact from the definition of Z_1 and (ii) it follows that

$< Z_1(z), z_n >= \int_0^\pi z_n(x)[\int_0^\pi b(y, x)z(y)dy]dx = \frac{1}{n}\sqrt{\frac{2}{\pi}} < Z(z), \cos(nx) >$,where$Z$ is defined by:
$Z(z)(x) = \int_0^\pi \frac{\partial}{\partial x}b(y, x)z(y)dy$
from (ii) we know that $Z : X \to X$ is abounded linear operator with $\|Z\| \le N_1$.

hence $\|A^{\frac{1}{2}}Z_1(z)\| = \|Z(z)\|$ which implies the assertion. Therefore the condition (H1)-(H5) are all satisfied .

hence from theorem (3.1) the system (4.1) -(4.4) admits a mild solution on J under the above assumptions additionally provided that (3.1) and (3.2) hold.

Competing Interests

The authors declare that no competing interests exist.

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