Discrete Isomorphic Completeness and a Unified Isomorphic Layout Format

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ABSTRACT

An isomorphic layout can be used to position pitches on a grid of hexagons. This has many beneficial musical properties such as consistent fingering and spatial harmonic consistency. A Unified Isomorphic Layout (UIL) format is presented in order to create a common specification for describing hexagonal isomorphic layouts. The UIL format provides an unambiguous description of relative pitch orientations and is easily visualized. The notion of complete and degenerate isomorphic layouts (along with a proof) is introduced to narrow down the number of valid isomorphic layouts used for exhaustive evaluations.

1. INTRODUCTION

There are many ways to arrange the available notes on a tone-centric musical instrument. A piano uses a linear layout of notes with a subset of notes (the accidentals) vertically offset. A guitar has a relatively consistent layout with the notes increasing by a semitone in one direction (along each string) and a perfect fourth in the other direction (from one string to the next) with the exception of one string at a major third. With these irregular note layouts, the musician has to learn a different set of fingerings for each key they play in. Piano students must practice scales in multiple keys, but the scales themselves are musically identical regardless of key, with the same pattern of musical intervals (tones and semitones, for example) from one note in the scale to the next. The difficulty of learning multiple scales stems from the note arrangement itself.

Some instruments (such as bass guitars) use a note layout that is *isomorphic*, which means that the distance (i.e. the number of keys) and direction of any musical interval is the same no matter which note you start on. A bass player can transpose to any key just by moving the fingerings being used to play a sequence of notes. This property of isomorphic layouts means that fingerings for playing a musical construct (such as a specific type of chord or scale) is independent of the root key. The "shape" of a major triad is the same for every major triad, which is why these layouts are called "isomorphic" (iso = same; morph = shape).

A hexagonal isomorphism is an isomoprhic arrangement of notes on a hexagonal grid rather than a rectangular grid. Each note has six adjacent tones allowing for more compact note layouts. Although hexagonal isomorphisms have been around for hundreds of years, there is not a lot of publicly available information on the many possible layouts and their properties. As well, only a few researchers are actively studying isomorphisms. In this paper, we present a unified framework for studying isomorphic layouts, which we call the *Unified Isomorphic Layout* (UIL) specification, This specification helps to identify and compare characteristics of layouts. We apply this framework to a number of "standard" isomorphic layouts, and present a method to guarantee completeness of any isomorphism. We also present a visualization system which allows detailed exploration of any isomorphism.

2. BACKGROUND

Hexagonal isomorphic layouts appear to have great potential, and many researchers have explored aspects of a set of specific layouts, but there is limited summative research bringing the field together as a whole. One of the biggest concerns with the existing hexagonal isomorphic research literature is that individual researchers have their favourite layouts, and commercial products tend to be focused on one particular isomorphism. There is, as yet, no public central repository of descriptions, evaluations, and visualizations of the many possible layouts. The research literature is sparse, and significant portions of the information are found within patents rather than research papers. We have created a tablet application that allows users to experiment with any possible isomorphic layout, and through public access to this application (Musix [1]) we have encountered many people who are interested in isomorphic layouts and are looking for more in-depth information. The focus of this paper, then, is a framework around which to centralize, summarize and visualize existing layouts, and to generate detailed analytical information about any hexagonal isomorphic note layout.

This paper continues with an itemized list of existing research (and researchers) into isomorphic layouts and their utility; a motivation of isomorphic research in a musical context; a proposal for a unified isomorphic layout notation to describe any layout and (more importantly) describe the relationships between different layouts; and a proof of isomorphic completeness both for and beyond western 12-tone scales; followed by a number of examples of common layouts described in the new UIL notation.

2.1 Research into Hexagonal Isomorphisms

Hexagonal isomorphic musical note layouts have been of interest for decades, although individual researchers have tended to focus on a specific layout or layouts, and much of the information available is presented in patents rather than research papers.

2.1.1 Harmonic Table

Peter Davies was awarded the patent for the Harmonic Table layout [2] which he filed in 1990. The harmonic table layout is equivalent to the Euler's Tonnetz [3] described in 1739, and is currently in use in the C-Thru AXiS commercial device. Davies discovered the layout during his analysis of notes contained in augmented and diminished chords. Although the patent describes information about his finding for the harmonic table, there is no known public material on his original analysis of other layouts.

2.1.2 Wicki-Hayden

Brian Hayden is credited with a patent for the Wick-Hayden layout [4] which was issued in 1986. The layout was developed for use on a concertina and was previously patented by Kaspar Wicki in 1896 (Swiss patent no. 13329). Several published conversations with Hayden can be found on the internet where he discusses concertina layouts [5] as well as describing a number of possible isomorphic layout combinations [6]. Hayden's research remains largely unpublished aside from his conversations with a few websites and magazines.

In Hayden's discussion with Woehr [6], he introduces an ordering of a largest absolute interval, smallest absolute interval, and the difference of the two as a method of describing layouts. Hayden concludes that only eleven interesting layouts exists and that their mirrors are not fundamentally different.

2.1.3 Notation and Alternate Tunings

Andrew Milne, William Sethares, and James Plamondon are important contributors to current research on isomorphic keyboards especially in the areas of isomorphic notation [7] and alternative tunings [8]. Their research goes into depths in regards to properties of layouts that make them good candidates for alternative tunings, compactness, and generic description. Tunings are described by periods, generators, syntonic commas, and temperament maps. A complete physical layout can be specified by a number of basis vectors and a series of matrix representations for button-lattices, layouts, and transformations. Proofs are also provided for their mappings in regards to linearity and transposition invariance [9]. Their description of isomorphic layouts is robust but does not easily allow the layouts to be visualized or implemented using the matrices.

Milne et al describe the isotone axis and the pitch axis of a layout. The isotone axis is a line that intersects all pitches of the same tone. The pitch axis is orthogonal to the isotone axis and shows the direction of uniform increasing pitch (from one isotone, say C to the next, say C♯). This allows a user to immediately visualize the "direction" of the layout without having to know which layout they are in. The pitch axis also has the property that the distance from the isotone axis along the pitch axis is equal to the pitch of the note.

2.1.4 Analysis and Reconfigurable Instruments

Brett Park, David Gerhard, Steven Maupin have been exploring hexagonal isomorphisms [10] based on their musical properties (melodic and harmonic) and fittings for specific musical styles. The analysis of layouts was conducted based on directions and distances for diatonic scales as well as major and minor triads. In addition to layout analysis, Park and Gerhard have been developing the commercially available isomorphic layout software called Musix [1] as well as creating a physical isomorphic keyboard with the ability to dynamically change isomorphic layouts while providing visual feedback. The device is named the Rainboard [11].

2.2 Why isomorphic layout research

Isomorphic note layouts have many potential advantages over non-isomorphic layouts such as transpositional invariance (fingerings are identical for different musical keys) and spatial / interval consistency (a relative interval is always in the same physical location relative to the base note). Hexagonal isomorphic layouts provide the tightest possible clustering of musical intervals [12]. Because of the many beneficial properties of hexagonal isomorphic layouts, they may provide the best opportunity for democratizing music creation.

Although many layouts have been "discovered" or analyzed, very few have empirical evidence to justify choosing one layout over another, and the benefits claimed by most researchers for their particular layout are, in fact, benefits of hexagonal isomorphisms in general. Most layouts are justified as being "good" because they group common music patterns in a close physical area. Although this may be true, there is no empirical evidence given that such groupings improve playability, learnability or other features of the instrument. As well, the layouts are generally considered to be unique based on the interval numbers that make up the layout. Additional properties, such as interval direction, are often not considered when evaluating layouts, however, the direction of the intervals can have a significant impact on playability and fingering. In fact, there exist some distinct traditional layouts are directional transpositions of each other, as will be shown later. Additional properties besides the identifying intervals should be considered as they may contribute to the ergonomic efficiency of the instrument.

2.3 Studying and evaluating all isomorphic layouts

There are two types of valid isomorphic layouts: complete and degenerate. Complete layouts contain at least one instance of every note in the given tonal system. For example: in a 12-tone musical system, all twelve tones will appear somewhere on the layout for the layout to be considered "complete". This does not guarantee playability or proximity of the notes, just that they will be present. Theoretical considerations can be made to prove completeness of a particular isomorphic layout, and therefore to list all complete layouts. Degenerate layouts, on the other hand, do not contain an instance of every tone in a musical system. Depending on the neighbouring intervals to a specific note, there may be no way to create all notes in the given tonal system. Even though some of the tones may be missing, however, the layout is still considered a valid isomorphism (based on the previous definition) in that fingerings are still identical in different keys and relative intervals are always in the same location.

Degenerate isomorphic layouts have limited musical utility, although they should not be completely discounted. When a scale is degenerate, note intervals will be missing in regularized patterns, due to the underlying isomorphic nature of the system. For example, it is possible to create an isomorphism of a 12-tone equal tempered scale where only every third semitone is present, resulting in a sequence of minor thirds, or a diminished 7th chord. This makes sense in the context of isomorphic note layouts, because the diminished 7th chord is root-ambiguous. Most musical scales (i.e. specified subsets of a given tonal system) are not equally distributed (like the diminished 7th is), instead consisting of a pattern of whole tones and semitones (the counter-example of course being the whole tone scale itself).

2.3.1 Classes of Degenerate Layouts

The number of possible degenerate layouts depends on the intervals which are missing in the layout, or alternatively which present interval is the smallest. Because of the properties of isomorphisms, this interval must be a divisor of the number of tones in the system. This is not to say that the smallest interval is necessarily adjacent to the root note.

For a 12 tone system, there are 4 classes of degenerate layouts. If the smallest available interval is the semitone, the layout is complete. If the smallest interval is 2 semitones (i.e. the semitone interval is missing from the isomorphism), then the layout is a whole tone scale. If the semitone and whole tone are both missing, the result is a diminished chord layout; if the minor third is also missing, the result is the augmented triad; if all intervals but the tritone are missing, the result is a two-note layout, and if all intervals are multiples of the octave, then only one note is available. Because these degenerate layouts are missing some notes, there are also a number of sub-classes of each degenerate layout depending on which notes are present. For the whole tone scale, there are two subclasses: scales which include C and those which include C♯.

Since degenerate layouts provide a significant limitation to an isomorphism, it is useful to be able to test for completeness or to generate layouts that are complete. We therefore develop a proof of completeness, presented in Section 7, based on co-prime intervals. This proof also allows all possible complete layouts to be generated by using a series of increasing co-primes.

Theoretically, there are an infinite number of isomorphic layouts, since intervals greater than the octave can be represented. In order to compare, represent, and evaluate these layouts, it is important to have an unambiguous representation for each layout that allows it to be placed in context with other layouts. Brian Hayden has suggested a representation method which labels the greatest interval as *G*, the lowest non-negative interval as *L*, and the difference between the smallest and the largest as *D*. By using the intervals *G* and *L*, the intervals composing the layout can be determined, but this description does not specify the direction of *G* or the relative direction of *L*. In order to further disambiguate between layouts, we have developed a complete notation which can fully specify any hexagonal isomorphic layout. Given a disambiguated layout format, the location and orientation of relative pitches should be unambiguous, allowing comparison between different layouts.

Once an exhaustive list of layouts with a reasonable interval range (less than a few octaves) can be generated and represented in an unambiguous manner, it is possible to begin analyzing their properties in a more formalized manner, taking isomorphic research from individual conjecture to empirical truth.

3. A UNIFIED ISOMORPHIC LAYOUT (UIL) NOTATION

In order to unambiguously describe hexagonal isomorphic layouts, a Unified Isomorphic Layout (UIL) notation is presented , based on Hayden's initial *GLD* notation. The UIL format adds to Hayden's specification by also specifying interval listing order, rotation, mirroring, and shear, and allows for microtonal layouts and non-12-tone scales. The interval directions for a base representation of the *LGD* format, as well as a mirrored, rotated version, are shown in Fig. 1.

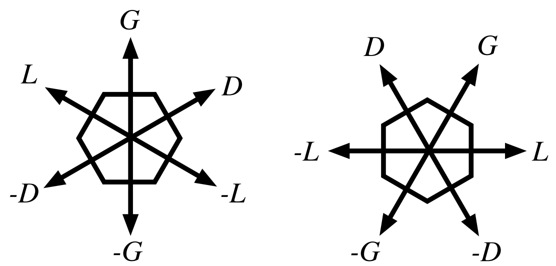

Figure 1. LGD format. (a) base representation; (b) mirrored and rotated by 30°.

The UIL notation format specification. $L, G, D; RMS; T$

L. *The lowest positive interval value. For the base layout, L is in the north-west direction and is to the left of G.*

G. *The greatest positive interval value. For the base layout, G will always point north. For all layouts (both complete and degenerate), the direction of G will be between the directions for L and D, since the interval values L and D must sum to G for the isomorphism to be valid.*

D. *The difference between G and L. It is possible for D = L. For the base layout, D is in the north-east direction and is to the right of G.*

R. *The clockwise rotation of the layout, in degrees (for both mirrored an non-mirrored layouts).* $R = 0$ *when the interval G is pointing directly north.*

M. *Indicates if the layout is mirrored. In a mirrored layout, the L and D axis are swapped, mirroring the layout about G.*

S. *Indicates the amount of shear after rotation as described by A. Prechtl et al [13]. In most cases, the shear will be 0 and may be omitted. A rotation can be applied to create the same result as a shear, with the only limitation being the shape of the note actuators (which can be stretched or otherwise modified under shear).*

T. *Indicates the number of tones in the scale. Other tuning parameters such as temperament can also be written directly after* T*. In the standard western 12-tone equal tempered case, (*$T12TET$*), the* T *value can be omitted.*

Base Representation. *A UIL layout representation where* $R = 0$ *and the layout is not mirrored.*

3.1 Special cases

Two special cases exists for UIL layouts. The first is when $G = D$, which only happens for intervals $0, 1, 1$. We refer to this as the zero case. In this case $L = 0$ and $D = 1$, and G is equal to the interval 1 that occurs between 0 and 1.

The second special case occurs where $L = D$ and is referred to as the equality case. The only equality case which results in a complete layout is $1, 2, 1$. In this case the interval direction chosen for L or D is irrelevant.

In both the zero case and the equality case, the mirror designation is meaningless.

4. REASONINGS FOR UIL FORMAT

One of the motivations for establishing a UIL format is to show when two layouts which may seem different are, in fact, within a rotation and mirroring of each other. It is therefore important to impose a restriction on the ordering of the *LGD* parameters so it is easier to identify related layouts. Without a strict ordering of intervals, rotation would be defined by both the degrees of rotation and interval order. The ordering restriction serves to disambiguate the rotation and mirroring of a layout, when given three intervals. Two layouts with the same *LGD* (within the same tuning) will have identical musical construct shapes (to within a rotation and mirroring). The use of *L*,*G*, and *D* also have some historic precedence from their use by Brian Hayden [6] although his definition was not order-restricted.

It is possible to represent the musical relations of the layout with only L and G since $D = G - L$. We chose to leave the D value in the format as it allows the adjacent intervals to be immediately visible without mentally performing the calculation for D. The inclusion of D also makes the interval directions visually similar to the interval directions in the base representation.

Although the inter-note relationships of a layout are completely specified by L, G, D, more information (mirroring and rotation) is required to fully define the physical layout.

Layout Name	UIL Format	L	G	D	R	M
Wicki-Hayden	2,7,5;R30M	2	7	5	30	1
Harmonic Table	3,7,4;R0	3	7	4	0	0
Gerhard	1,4,3;R60	1	4	3	60	0
Park	2,5,3;R90M	2	5	3	90	1
Janko	1,2,1;R90	1	2	1	90	0
C-System	1,3,2;R270M	1	3	2	270	1
B-System	1,3,2;R270	1	3	2	270	0
Bajan	1,3,2;R90M	1	3	2	90	1

Table 1. UIL notations for common isomorphic layouts. L = Least, G = Greatest, D = Difference, R = Rotation, M = Mirrored.

The mirroring and rotation parameters allow manipulation of the ergonomic aspects of the layout which may have a significant impact on playability. The size of the hexagons is not included in the UIL specification, since it simply introduces a scalar distance between intervals that is constant for all interval relations. It should be noted, however, that different layouts benefit from different hexagon sizes based on the compactness of the layout. A compact representation may need bigger hexagons to improve playability.

Scale intervals were chosen as the standard unit of *LGD* since it shows the musical relationship to the surrounding hexagons and allows for quick completion validation. If two of the interval values of the *LGD* are co-prime, then the layout will contain all intervals in the scale and be considered complete. A proof of this completeness is presented in Section 7.

4.1 Common UILs

Most interval sets that create a complete layout (with a reasonable interval size) have been named or patented. Some of the more common isomorphic layouts are listed in UIL format in Table 1, and are visualized in Section 6.1. An example of "different" layouts with the same *LGD* are the C-System, B-System, and Bajan layout. The difference between the three layouts can easily and clearly be seen by looking at the rotation and mirror properties of the three layouts.

4.2 Non 12-TET Scales

For the 12-tet scale the values for *LGD* are simple semitone intervals between 0 (unison / octave) and 11 (Major 7th), but nothing in the UIL format requires a 12-tone scale. This representation is useful for determining valid layouts and visualizing their relation, however, alternate equivalent representations of *LGD* can be given for different purposes. In these cases the values of *LGD* can be represented as cents, ratios, or roman numerals. The interval, roman, and shorthand format may be useful for musicians familiar with these notational systems. The cent and ratio representations are useful for comparing layouts across different tunings and will be suitable for microtonal music. Example alternate formats can be found in Table 2.

UIL Interval Format (12TET)	2,7,5;R30M
UIL Interval Format (7TET)	1,4,3;R30M;T7
UIL Interval Format (19TET)	3,11,8;R30M;T19
UIL Cent Format	200,700,500;R30M
UIL Ratio Format	9:8,3:2,4:3;R30M
UIL Roman Format	II,V,IV;R30M
UIL Shorthand Format	M2,P5,P4;R30M

Table 2. Alternate UIL LGD representations for interval representations and tunings

5. INFERENCES FROM UIL

5.1 Horizontal and Vertical alignment

Two common ways of visualizing a grid of hexagons is in a horizontal or vertical alignment [10]. The hexagon alignment can easily be discerned from the rotation angle of the UIL (Fig. 2). Vertical alignment occurs when the layout is rotated in increments of 60 degrees $(0, 60, 120, 180, 240, 300)$ and horizontal alignment occurs when the layout is rotated in increments of 60 degrees plus an initial 30 degree offset $(30, 90, 150, 210, 270, 330)$.

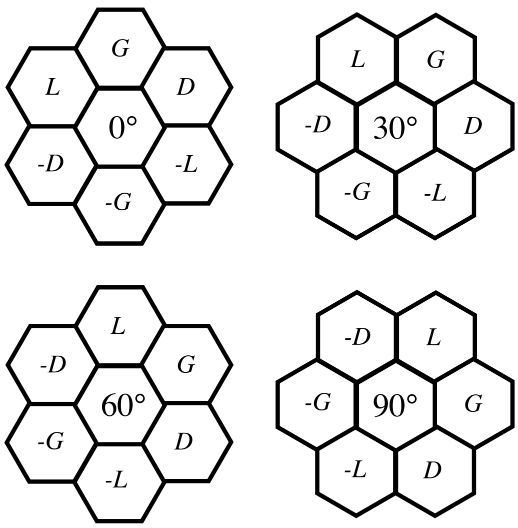

Figure 2. Vertical grid alignment (left) at 60 degree increments and Horizontal grid alignment (right) at 60 degree increments plus an initial 30 degree offset

5.2 Isotone and Pitch Axis

The *isotone axis* defines a line in an isomorphic layout which passes through all notes of a particular pitch [8]. If you draw a line between two instances of "A_4" and extend that line to infinity, all other instances of A_4 will appear only on that line. Further, all isotones are parallel. One important property that derives from the isotone axis is that the orthogonal distance of any note from this axis is directly related the pitch of the note. This orthogonal line, called the pitch axis [8], denotes the general direction in which pitches ascend.

Due to the strict interval order of the *LGD*, it is possible to infer information about the pitch axis and distance between isotones. In *LGD* base format (no rotation or mirroring), the pitch axis will always be between 0 and 30 degrees (Fig. 3). This results in the pitch axis being between R and $R+30$ degrees for non-mirrored layouts and the pitch axis being between $R-30$ and R for all mirrored layouts. The precise pitch axis angle (relative to R) can be can be calculated by $30 * \frac{D-L}{G}$. Since the pitch axis and the isotone axis are orthogonal, the isotone axis angle is equal to the pitch axis angle plus 90°.

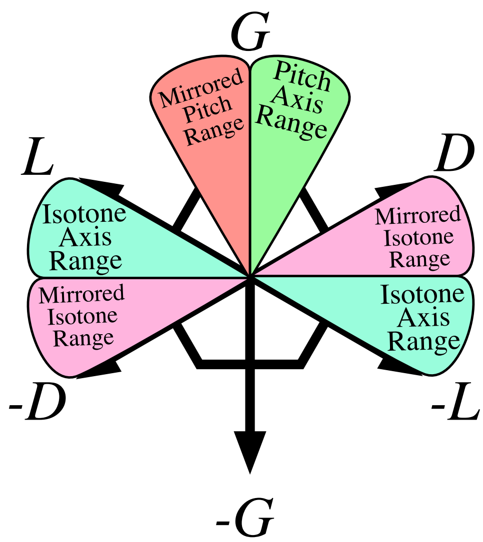

Figure 3. Normal and mirrored ranges for the pitch axis and isotone axis (without rotation).

6. VISUALIZATION OF LAYOUTS

Using the codebase we initially developed for the Musix iOS app, we constructed a system that allows the visualization of any isomorphic layout, including the isotone and the pitch axis, as well as the parallelogram which contains a single complete 12-note octave. This visualization is particularly useful for judging alignment of pitch axis, compactness of the representation, and similarity to other isomorphisms. The results of this visualization, as well as a set of example layouts and their analysis, are presented here.

6.1 Examples

The following figures present visualizations of some of the more common hexagonal isomorphic layouts in use today, as well as representations in the base UIL format. In these figures, notes are coloured with the root note of the scale in red, intervals in the major scale of that key coloured in white, and the other intervals coloured in black. Notes are labeled as N_o^i, where N is the the number of semitones from the root note, o is the octave of that note, and i is the common interval abbreviation. For example, 6_4^{tt} is the tritone in the 4th octave, 6 semitones from the root.

Figure 4 shows Wicki-Hayden, a popular layout discussed in Section 2.1.2. This layout collects "white" notes together, making whole tone and pentatonic melodies easy to play. Figure 5 shows the Janko keyboard, an early isomorphic layout related to the piano. Like the piano, pitches ascend to the right. The base UIL format visualization shows the pitch axis rising to the north. The harmonic table layout (Figure 7), discussed in Section 2.1.1, is already in base format, and makes plain one of the complaints about this layout: while major and minor triads are compact, whole tones are quite distant, and the layout as a whole is not as compact as, for example, the Bajan.

Figure 6 shows that three traditional isomorphic layouts, the Bajan, C-system, and B-system, are in fact mirrored and rotated versions of the same base layout. Figure 8 shows two additional layouts, the Gerhard and the Park, which have been studied in detail by the authors.

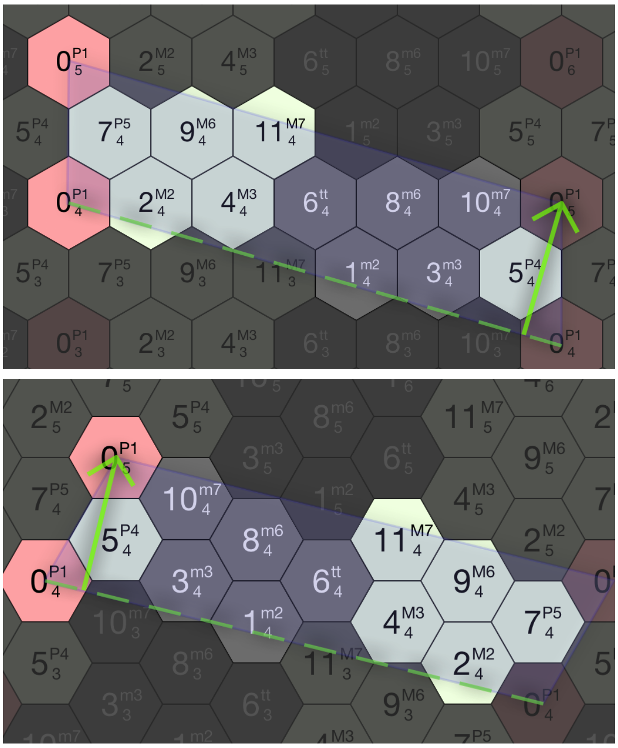

Figure 4. Wicki-hayden layout: 2,7,5;R30M (top) and the corresponding base representation 2,7,5;R0

7. PROOF OF ISOMORPHIC COMPLETENESS

In the following proof, hexagonal isomorphic intervals are listed as x, y, and z. The UIL format of *LGD* is a subspecification of any hexagonal isomorphism, and L, G, and D can each be any one of x, y, and z.

In order to create a valid hexagonal isomorphism (H), three note intervals x, y, and z are chosen that represent the neighbouring interval distances for each linear direction. The sum of these three intervals must be zero,

$$H = \{x, y, z \in \mathbb{Z} \mid x + y + z = 0\} \quad (1)$$

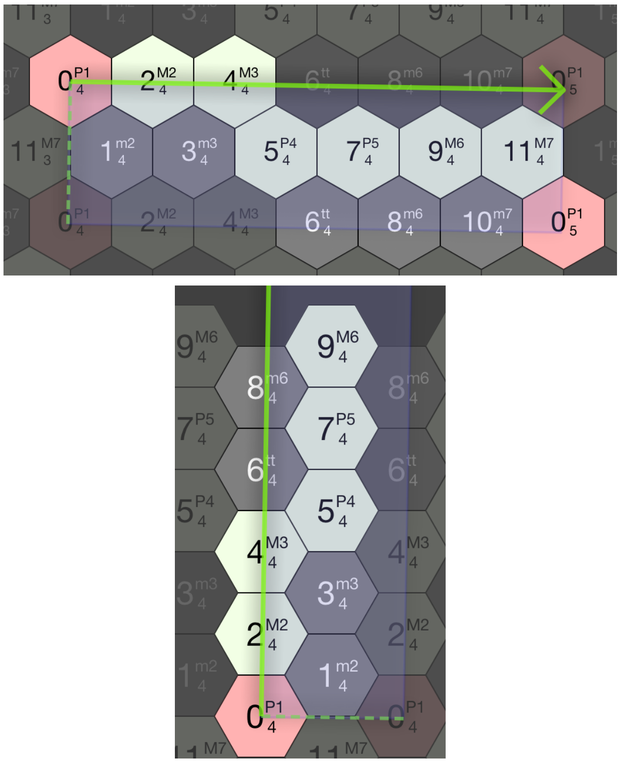

Figure 5. Janko layout: 1,2,1;R90 (top) and the corresponding base representation: 1,2,1;R0

since progressing once in each of three directions will return to the original note. By simplification, $z = -x - y$.

Theorem 7.1. *Isomorphic Completeness: Given two interval vectors x, y that define a unidirectional isomorphism, the isomorphism is complete (contains all note intervals) if and only if x and y are co-prime.*

Proof of Isomorphic Completeness. An isomorphism is complete if and only if some scalar combination of x, y, z exists such that all integers can be produced

$$(\forall n \in \mathbb{Z})(\exists a, b, c \in \mathbb{Z}) \mid ax + by + cz = n \quad (2)$$

First lets consider the case where $n = 1$.

$$ax + by + cz = 1 \quad (3)$$

We can simplify equation 2 to $hx + iy = 1$ by the following process:

$$\begin{aligned}
1 &= ax + by + cz \\
1 &= ax + by + c(-x - y) \\
1 &= ax + by - cx - cy \\
1 &= x(a - c) + y(b - c) \quad (4) \\
\text{Let } h &= a - c \\
\text{Let } i &= b - c \\
1 &= hx + iy
\end{aligned}$$

Two cases then exists: x and y are coprime (they do not share a positive factor other than 1) or they are not coprime.

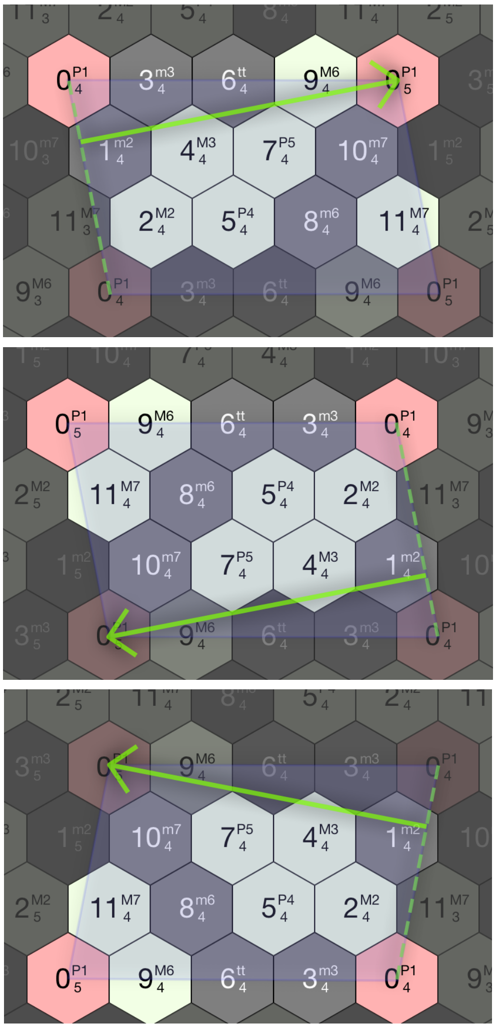

Figure 6. Bajan (top), C-system, and B-system, all of which use the intervals 1,3,2 with different rotations and mirrorings. C-system and B-system are mirrored versions of each other.

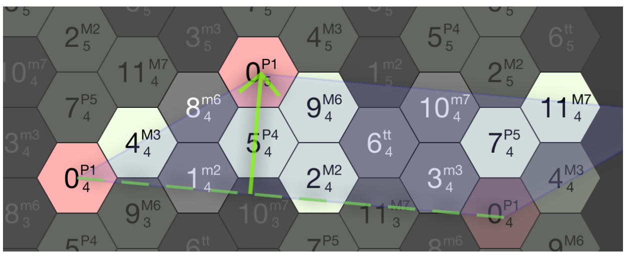

Figure 7. Harmonic Table layout: 3,7,4;R0

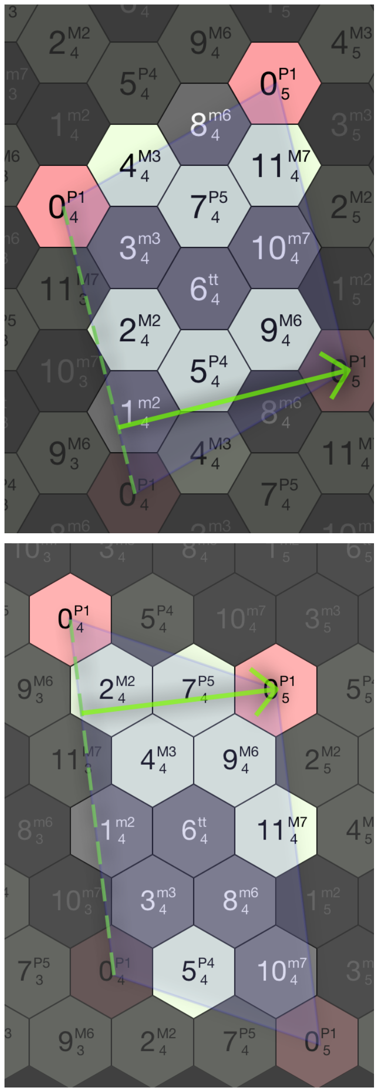

Figure 8. Gerhard (top): 1,4,3;R60; Park: 2,5,3;R90M

Case 1: (x and y are coprime)
If $x \perp y$ then hx + iy = 1 by Bézout's identity [1] .

Case 2: (x and y are not coprime)
If x and y are not coprime they must share a prime factor e such that $x = ej$ and $y = ek$ where j and k are integers.

$$(\exists e, j, k \in \mathbb{Z}) \mid x = ej, y = ek \tag{5}$$

If we assume this is true for the case where $n = 1$:

$$\begin{aligned} 1 &= hej + iek \\ 1 &= e(hj + ik) \\ \text{Let } m &= hj + ik \\ 1 &= me \\ e &= 1/m \\ \text{since } e \in \mathbb{Z}, m &= \pm 1 \\ e &= \pm 1 \end{aligned} \tag{6}$$

Then we can substitute and simplify to determine $e = \pm 1$. Since the only positive common factor is $e = 1$, x and y are coprime which contradicts the assumption.

We must now extend the proof for all n. Equation 3 can be multiplied by an integer scalar t in order to produce the entire range of integers for n.

$$(\forall n \in \mathbb{Z})(\exists t \in \mathbb{Z}) \mid t(ax + by + cz) = t * 1 = n \tag{7}$$

If x, y, z can be multiplied by integer scalars to equal 1, then the scalars can also be multiplied by any integer t in order to produce the entire integer set n.

□

8. CONCLUSIONS AND FUTURE WORK

By considering the notion of complete and degenerate layouts, along with formalization of the criteria for each type of layout, it is possible to iterate through intervals that create complete layouts. These intervals can then be represented in UIL notation in order to disambiguate musical properties and pitch orientation. The Unified Isomorphic Layout notation provides an unambiguous textual representation of an isomorphic layout that can be easily visualized, resulting in a useful tool for isomorphic research.

Now that the UIL is established, we plan to iterate through all of the non-degenerate base layouts and explore their properties independent of rotation and mirroring. Such properties include pitch axis angle, isotone axis angle, isotone axis length (between two notes), pitch axis length (orthogonal distance for an octave), octave parallelogram area, and parallelogram squareness. After these properties are calculated, ergonomics of the layouts will be evaluated for various intervals, rotations, and mirrors. The ergonomic data will be used to develop a suggested fingering for playing in various UILs, as well as a recommendation system for which specific isomorphic layout would be best suited to any particular musical context or task.

[1] `http://en.wikipedia.org/wiki/Bezout's_identity`

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