The Ninefold Way: from Pauli to Gell-Mann and beyond

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1. Review

Much like Dirac's matrices **[1]**, Gell-Mann built-up his λ set from Pauli's σ matrices to fit into his 'Eightfold Way' scheme **[2]**; thus

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = 1/\sqrt{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

This set is traceless, Hermitian, normalized by the largest eigenvalue - $\sqrt{3}$ for λ_8 - but not involutive. Pauli's σ's can be seen clearly 'embedded' in most cases, in an 'upper left' or 'lower right' arrangement; σ_1 can be seen 'descending' on the main diagonal from λ_1 to λ_4 to λ_6; and σ_2 transits from λ_2 to λ_5 to λ_7 in the same way.

No such 'generating' motion is evident for λ_3; and while λ_8 may seem contrived, letting

$$\lambda_{81} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \lambda_{82} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

it is clear that actually

$$\lambda_8 = 1/\sqrt{3}(\lambda_{81} + \lambda_{82})$$

These observations have led naturally to a revision and enlargement: The **'Ninefold Way'.**

2. Enlargement

A nonet comprised of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\ \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9\}$ will be investigated here. This set, still traceless and Hermitian, is now:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The ‘descent’ by Pauli’s sigmas also appears more naturally:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \sigma_1 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \sigma_2 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \sigma_3 \end{pmatrix}$$

Even so, nothing like the elegant simplicity of Dirac’s $\{\Sigma, \rho, \alpha, \Gamma, \beta\}$ matrix table **[3]**

$\otimes$	$\mathbf{I}_2$	σ_1	σ_2	σ_3
$\mathbf{I}_2$	I_4	Σ_1	Σ_2	Σ_3
σ_1	ρ_1	α_1	α_2	α_3
σ_2	ρ_2	Γ_1	Γ_2	Γ_3
σ_3	ρ_3	β_1	β_2	β_3

is available here; Kronecker products do not help.

3. Generation

The method used to obtain Pauli’s matrices **[4]** can be altered to yield the nonet.

The image below displays results.

Matrix	$\theta = 0$	$\theta = \pi/2$
$\begin{pmatrix} \sin(\theta) & \cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}$	λ_1	λ_3
$\begin{pmatrix} \sin(\theta) & 0 & \cos(\theta) \\ 0 & 0 & 0 \\ \cos(\theta) & 0 & -\sin(\theta) \end{pmatrix}$	λ_4	λ_6
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin(\theta) & \cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix}$	λ_7	λ_9
$\begin{pmatrix} \cos(\theta) & -i\sin(\theta) & 0 \\ i\sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}$	λ_3	λ_2
$\begin{pmatrix} \cos(\theta) & 0 & -i\sin(\theta) \\ 0 & 0 & 0 \\ i\sin(\theta) & 0 & -\cos(\theta) \end{pmatrix}$	λ_6	λ_5
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & -i\sin(\theta) \\ 0 & i\sin(\theta) & -\cos(\theta) \end{pmatrix}$	λ_9	λ_8

While somewhat ad hoc, this method has (over)produced the entire nonet.

4. Tables

Preparatory to anything else, define the following (almost) identity matrices:

$$I_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$I_{12} + I_{13} + I_{23} = 2I_3$$

The nonet's Cayley table is immediately below; any number entry (real or imaginary) sub-indexed by a row/column pair, means the 3x3 matrix in the entry has all zeros except for that number positioned accordingly. For example

$$1_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Non-commuting off-diagonal λ's are inside blue borders; three pairs of commuting matrices are highlighted in green. A plethora of Hermitian pairings are evident elsewhere; just search on the cross-diagonals.

*	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
λ_1	I_{12}	$i\lambda_3$	$-i\lambda_2$	1_{23}	$-i_{23}$	1_{21}	1_{13}	$-i_{13}$	1_{12}
λ_2	$-i\lambda_3$	I_{12}	$i\lambda_1$	i_{23}	1_{23}	i_{21}	$-i_{13}$	-1_{13}	$-i_{12}$
λ_3	$i\lambda_2$	$-i\lambda_1$	I_{12}	1_{13}	$-i_{13}$	1_{11}	-1_{23}	i_{23}	-1_{22}
λ_4	1_{32}	$-i_{32}$	1_{31}	I_{13}	$i\lambda_6$	$-i\lambda_5$	1_{12}	i_{12}	-1_{13}
λ_5	i_{32}	1_{32}	i_{31}	$-i\lambda_6$	I_{13}	$i\lambda_4$	$-i_{12}$	1_{12}	i_{13}
λ_6	1_{12}	$-i_{12}$	1_{11}	$i\lambda_5$	$-i\lambda_4$	I_{13}	-1_{32}	$-i_{32}$	1_{33}
λ_7	1_{31}	i_{31}	-1_{32}	1_{21}	i_{21}	-1_{23}	I_{23}	$i\lambda_9$	$-i\lambda_8$
λ_8	i_{31}	-1_{31}	$-i_{32}$	$-i_{21}$	1_{21}	i_{23}	$-i\lambda_9$	I_{23}	$i\lambda_7$
λ_9	1_{21}	i_{21}	-1_{22}	-1_{31}	$-i_{31}$	1_{33}	$i\lambda_8$	$-i\lambda_7$	I_{23}

5. Relationships

$$\lambda_1^{\ 2} = \lambda_2^{\ 2} = \lambda_3^{\ 2} = I_{\mathbf{12}}$$

$$\lambda_4^{\ 2} = \lambda_5^{\ 2} = \lambda_6^{\ 2} = I_{\mathbf{13}}$$

$$\lambda_7^{\ 2} = \lambda_8^{\ 2} = \lambda_9^{\ 2} = I_{\mathbf{23}}$$

so that a quadratic Casimir identity holds, a 'team' involution:

$$\sum_{1}^{9} \lambda_k^{\ 2} = 6I_3$$

Anticommuting pairs (inside blue borders) are listed next:

$$\lambda_1\lambda_2 = -\lambda_2\lambda_1 = i\lambda_3 \qquad \lambda_4\lambda_5 = -\lambda_5\lambda_4 = i\lambda_6 \qquad \lambda_7\lambda_8 = -\lambda_8\lambda_7 = i\lambda_9$$
$$\lambda_2\lambda_3 = -\lambda_3\lambda_2 = i\lambda_1 \qquad \lambda_5\lambda_6 = -\lambda_6\lambda_5 = i\lambda_4 \qquad \lambda_8\lambda_9 = -\lambda_9\lambda_8 = i\lambda_7$$
$$\lambda_3\lambda_1 = -\lambda_1\lambda_3 = i\lambda_2 \qquad \lambda_6\lambda_4 = -\lambda_4\lambda_6 = i\lambda_5 \qquad \lambda_9\lambda_7 = -\lambda_7\lambda_9 = i\lambda_8$$

Thus

$$\lambda_1\lambda_2\lambda_3 = i\lambda_3^{\,2} \qquad \lambda_4\lambda_5\lambda_6 = i\lambda_6^{\,2} \qquad \lambda_7\lambda_8\lambda_9 = i\lambda_9^{\,2}$$
$$\lambda_2\lambda_3\lambda_1 = i\lambda_1^{\,2} \qquad \lambda_5\lambda_6\lambda_4 = i\lambda_4^{\,2} \qquad \lambda_8\lambda_9\lambda_7 = i\lambda_7^{\,2}$$
$$\lambda_3\lambda_1\lambda_2 = i\lambda_2^{\,2} \qquad \lambda_6\lambda_4\lambda_5 = i\lambda_5^{\,2} \qquad \lambda_9\lambda_7\lambda_8 = i\lambda_8^{\,2}$$

hence

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_1 + \lambda_3\lambda_1\lambda_2 + \\ &\quad + \lambda_4\lambda_5\lambda_6 + \lambda_5\lambda_6\lambda_4 + \lambda_6\lambda_4\lambda_5 + \\ &\qquad + \lambda_7\lambda_8\lambda_9 + \lambda_8\lambda_9\lambda_7 + \lambda_9\lambda_7\lambda_8 = i\sum_1^9 \lambda_k^2 = 6i\mathrm{I}_3\end{aligned}$$

for another Casimir identity. The three commuting pairs (highlighted in green above) imply:

$$\lambda_3\lambda_6 = \lambda_6\lambda_3 = \mathbf{1}_{\mathbf{11}}$$
$$\lambda_3\lambda_9 = \lambda_9\lambda_3 = -\mathbf{1}_{\mathbf{22}}$$
$$\lambda_6\lambda_9 = \lambda_9\lambda_6 = \mathbf{1}_{\mathbf{33}}$$

leading to a partial ‘team’ involution:

$$\lambda_3\lambda_6 - \lambda_3\lambda_9 + \lambda_6\lambda_9 = \mathrm{I}_{\mathbf{3}}$$

References

[1] P. A. M. Dirac, ‘The Quantum Theory of the Electron’, Proceedings of the Royal Society of London, Vol. 117, No. 778 (1928), pp. 610-624.
[2] Wikipedia article on ‘Gell-Mann matrices’.
[3] A. Cusmariu, ‘Dirac’s alpha-gamma-beta, Pauli multiplication, and all that’, https://www.zenodo.org/record/8023850
[4] A. Cusmariu, ‘Whence Pauli matrices’, https://www.zenodo.org/record/7979146