Time and Space Wavefunction Related Entropy

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In a previous note (1) we considered a free particle entropy based on the real part of the wavefunction exp(ipx) i.e. |cos(px)|. |cos(px|) may be normalized over a certain length, just as exp(ipx) is. The idea is that there exists an uncertainty in space within a wavelength which should be reflected in an entropy expression. As p becomes very large, cos(px) becomes very narrow and has many peaks in L. Thus probability →0 so p ln(p)--> 0 using l'Hopital's rule. This coincides with a free classical particle having zero entropy.

In this note, we revisit arguments presented previously, suggesting that A= -Et+px, a Lorentz invariant and also the free particle classical relativistic and nonrelativistic action for v=x/t We suggested in the past that A is responsible for both the wavelength hbar/p and the time period hbar/E. Here we suggest uncertainty within a time period should be considered together with uncertainty within a wavelength. For instance, recent experiments (2) have shown that one may have quantum interference in time as well as space. Thus there is uncertainty for a free particle in both space and time. We add the entropies for these two.

Next, we consider the case of a bound system. Such a system creates a crest-trough pattern (with different heights and widths) for its spatial wavefunction, but E, average energy is constant and there is the free particle type exp(-iEnt). We suggest that the system is characterized by uncertainty in space and time with En being a known constant at each x. This suggests that various free particle p (momentum) values combine to create the wavefunction such that -1/2m d/dx d/dx W/W is average kinetic energy which in turn adds to V(x) to create a constant En. We are thus concerned with uncertainty in time and space and not in momentum. In the limit of high energy values, a crest fits within a tiny dx and prms is essentially the classical p value which is considered certain, even though in quantum mechanics the range of a(p) weights hold for all n (W(x)=Sum over p a(p)exp(ipx)). Thus we treat a high n limit as being completely certain and not having entropy. This differs from the approach of using C/v(x), where v(x) is velocity given by .5mv(x)v(x)+V(x)=En, as a probability and calculating Shannon's entropy using this.

Entropy and Uncertainty

We consider entropy from the point of view of uncertainty. Uncertainty, however, may be defined in different ways depending on what information one has. For example, following a classical particle in time and space is deterministic, but in a classical gas one does not have this information and speaks of probabilities of a gas particle to be at one x or another.

Here we consider a free classical particle with constant p momentum to be completely deterministic and certain and have no entropy. The quantum mechanical analogue, however, has uncertainty in both time and space due to a frequency hbar/E from exp(-iEt) and a wavelength hbar/p from exp(ipx). Both of these are physical as two slit spatial interference is well-known and recent experiments (2) show there is also interference in time.

For a constant p, exp(ipx)/sqrt(L) corresponds to a density of 1/L which yields a constant Shannon's entropy for all p. We argue in (1), however, that if one considers |cos(px)|, normalized over the length L, then different p values lead to different entropies which reflect the different spatial uncertainty due to the different wavelengths.

Here we suggest one must also consider uncertainty in time as well, which is classically usually not considered in entropy expressions. We suggest that for a given p=mov (nonrelativistically):

L= v T so T is the time range corresponding to L ((1))

One may define a Shannon entropy for C1|cos(Et)| normalized over L/v. Combining time and spatial entropies, one may create an overall entropy for a free quantum particle:

S = Integral dt (0, L/v) C1|cos(Et)| ln{ C1|cos(Et)| } + Integral dx (0,L) C2 |cos(px)| ln{C2 |cos(px)|} ((2))

High E, p Limits

What happens to the entropy ((2)) in high E, p limits? Momentum p becoming large implies hbar/p is very small so that probability in L approaches 0 in each dx region. Thus:

Lim prob->0 Prob ln(Prob) = 0 using l'Hopital's rule. ((3))

A similar argument holds for E becoming large. As a result, we suggest that in the high E,p limit, quantum free particle uncertainty shrinks and the entropy ((2)) approaches 0.

Bound State

Next we try to extend the entropy of ((2)) to a bound state. Such a state is associated with:

((A)) A pattern of spatial crests and troughs of different heights and weights (e.g. harmonic oscillator) i.e. spatial uncertainty.

((B)) exp(-iEn t) i.e. time uncertainty identical to a free particle, but an exactly known En (average energy) at each x

((C)) A spread in free particle momentum because W(x)=wavefunction= Sum over p a(p)exp(ipx)

For the case of the free particle with exact p and E, we focused on spatial and time uncertainties. In the bound case, spatial and time uncertainties exist, E is exact, but there is a spread in free particle momenta. The spread in free particle momenta, however, is required to allow for interference such that -1/2m d/dx dW/dx / W yields a kinetic energy which when added to potential energy V(x) yields a constant En at each x. In other words, if one does observe the free particle p values, one may consider this to be again a system with spatial and time uncertainties and create a corresponding entropy. This is the approach we adopt here.

One must then choose limits for the length L and time T. We choose L to be the length between classical turning points for a given En and T to be given by:

T= Integral dx / vrms(x)   where .5m vrms(x)vrms(x) = -1/2m d/dx dW/dx / W ((4))

Then:

Time probability = C1 |cos(En t)| where C1 is the normalization constant over (0,T) ((5a))

Spatial probability = C2 |Wn(x)| where C2 is normalized over L ((5b))

Using spatial and time probabilities, one may create the entropy:

S bound = Stime + S spatial ((6))

using Shannon's recipe: Sum over i Pi) ln{P(i)}   ((7)) and the probabilities ((5a)) and ((5b)) integrated over dx and dt instead of summed.

In the En limit, the time entropy becomes 0 as in the free particle case. For the spatial case, the crests and troughs should also become very narrow, even though the envelope describing W*(x)W(x) = C/v(x). As a result, there is no uncertainty in x and ((5b)) should approach 0 yielding zero for spatial entropy as well.

We note that in some cases in the literature W*(x)W(x) is used as a probability to calculate spatial Shannon's entropy. In the classical limit, this is approximated by C/v(x)=P(x). This yields a different entropy than the one we propose here.

Conclusion

In conclusion, due to recent experiments (2) which demonstrate quantum interference in time, we suggest that a wavefunction based entropy which holds even for free quantum particles should be extended to include a time portion. In such a case, one must choose a length L in space (the same L chosen in normalizing exp(ipx)/sqrt(L)) and a corresponding time T= L/v where p=mv (nonrelativistically). The probabilities are then C1 |cos(Et)| and C2 |cos(px)| normalized over (0,L/v)and (0,L). In the high E,p limit, we argue that Shannon's entropy tends to 0.

We next consider the bound case for which there is again time entropy due to exp(-iEnt) and a spatial entropy due to the wavefunction W(x) which has crests and troughs. There is also a spread in momenta p, i.e. W(x) = Sum over p a(p)exp(ipx), but this spread creates the certain value En and one does not measure free particle p values. Thus we consider only spatial and time entropy. We use L as the length between classical turning points for En and T = Integral dx/vrms(x) where .5mvrms(x)vrms(x)= -1/2m d/dx dW/dx / W. We then normalize C1 |W(x)| and C2 |cos(Et)| within these lengths and create the two corresponding Shannon's entropies which are summed.

References

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