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On the Noneuclidean Geometries of the Kepler Problem

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Abstract
Noneuclidean geometries of constant and non-constant curvature are treated within the framework of the Beltrami and warped metrics. An example of non-constant curvature is equivalent to the Schwarzschild metric which yields the Einstein modification of the orbital equation for the precession of the perihelion. The static nature shows that time does not enter at all into the Schwarzschild solution. Introducing a parameter that describes the geodesic motion and then identifying it with the azimuthal angle confuses energy and angular momentum conservation; the latter does not preserve its Euclidean form. Gravitational repulsion is a figment of the indefinite metric, and has nothing to do with reality. The introduction of two times and the condition between the frequency and coefficient of the square of the time increment in the indefinite metric compensates the erred assumption that the angular momentum is given by its Euclidean form. The origin of the perihelion advance is to be found in the unequal velocities at the apsides causing the ellipse to rotate forward; the rosette shape of the orbit is its manifestation. Although it contributes to the total energy, a quadrupole moment cannot affect the shape of the elliptic orbit. Specifically, it cannot cause the orbit to precess. The sun's oblateness is treated as a rotational perturbation, derived from a constant curvature metric. Constant curvature rotational deformations are compared to non-constant tidal deformations. Both constitute completely static phenomena; neither warrants the introduction of time into the metric. The indefinite, hyperbolic, metric is a vestige of special relativity that is incompatible with the nature of the these perturbations.

Keywords
Noneuclidean geometries, Special relativity, Schwarzschild solution.

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Introduction

Projections have long been used to facilitate the description of motion which would otherwise seem more complicated. Historically, probably the oldest is the gnomonic projection used by cartographers which projects uniform motions on great circles of a sphere onto tangent planes, with nonuniform motion, that is the same as Euclidean geometry. The curvature affects the speed of the projected motion not the shape of the orbits. For closed elliptical orbits, a velocity imbalance, where the motion is slower at the perihelion and faster at the aphelion, can rotate the ellipse forward causing it to precess.

In contrast to stereographic projections from a sphere, such motions are hyperbolic. The relevant projections are from a hyperboloid, or 'pseudo-sphere' of imaginary radius, onto a tangent plane. What will be of interest to us is a model of non-constant curvature elements, like that of the Schwarzschild metric. Introducing time into the metric is completely irrelevant; time enters only in the expression for angular momentum in non-Euclidean spaces.

It is simpler to begin with a velocity space metric, and transform to configuration space using Kepler's laws. For any incremental change in the velocity $d\mathbf{v}$ the metric will be given by

$$ds^2 = \frac{1}{c^2}\frac{(d\mathbf{v})^2 - (\mathbf{v}\times d\mathbf{v})^2/c^2}{B^2(v)} \tag{1}$$

or

$$ds^2 = \frac{1}{c^2}\frac{B(v)(d\mathbf{v})^2 + (\mathbf{v}\cdot d\mathbf{v})^2/c^2}{B^2(v)}, \tag{2}$$

where c is a normalization constant, and becomes a unit of measurement in non-Euclidean geometries. The form factor, $B(v)$, produces a new geometric structure on the plane that is conformal to the old one. However, the resulting geometric surface has properties that are quite different from the Euclidean plane.

The form factor, $B(v)$, is chosen by the fact that the resulting motion in the plane should generate closed orbits. Only two such potentials are known where the central field generates closed, bounded orbits: the gravitational potential, and the harmonic oscillator potential. The general form of $B(v)$ is

$$B(v) = 1 \pm \frac{v^2}{c^2},$$

where the $\pm$ sign indicates that the projection is from a sphere and pseudo-sphere, respectively. Gravitational interactions will always involve the negative sign indicating that the geometry is hyperbolic.

For the Hookean potential,

$$V_H = \frac{1}{2}\kappa r^2, \tag{3}$$

where the source is located at the origin, $B(v)$ has the form

$$B(v) = 1 - \kappa r^2/c^2, \tag{4}$$

by using $v^2 = \kappa r^2$, to transform from velocity to configuration space, with κ the spring constant.

This can be converted into a non-constant curvature term,

$$B(v) = 1 - R_S/r \tag{5}$$

by introducing the square of the 'escape' velocity, $v^2 = 2\mu/r$, where μ is the gravitational parameter, and $R_S = 2\mu/c^2$ is the Schwarzschild radius. Unlike r which is not an actual radius, since it cannot approach the origin, R_S vanishes in the $c \to \infty$, non-relativistic, limit. Whereas the former form factor, (4), refers to a hyperbolic plane of constant curvature, the latter, (5), is one of non-constant curvature, and it coincides with the Schwarzschild factor. The former is known in the mathematical literature as the Poincaré disc model.

Since the sphere and pseudo-sphere differ merely by a real and an imaginary radius, we may consider the the projection to a tangent plane of either. In the case of a sphere, the radial coordinate, $r = \tan\vartheta$, for $0 < \vartheta < \pi/2$, allows the Newtonian potential,

$$V_N = \frac{\mu}{r} \tag{6}$$

to be written as

$$V_N = -2\mu\cot\vartheta. \tag{7}$$

It is s harmonic function, satisfying the Laplace-Beltrami equation,

$$\Delta V = \sin^{-2}\vartheta\frac{\partial}{\partial\vartheta}\left(\sin^2\vartheta\frac{\partial V}{\partial\vartheta}\right) = 0.$$

Although the potentials, (7) and (3), are 'dual' to one another, insofar as the former can be transformed into the latter by a Bohlin transform, only the former satisfies the Laplace-Beltrami equation since (4) is

$$V_H(\vartheta) = \frac{1}{2}\kappa\tan^2\vartheta, \tag{8}$$

under the transform $r = \tan\vartheta$. That is to say squaring (7) and taking the inverse gives (8).

On the sphere of constant radius, the kinetic energy is

$$\mathbf{T} = \frac{1}{2}\left(\dot\vartheta^2 + \sin^2\vartheta\,\dot\varphi^2\right), \tag{9}$$

for spherical coordinates ϑ and φ. Although the radial coordinate is $r = \tan\vartheta$, the elliptic distance on a sphere is $\vartheta = \sin^{-1}\left(r/\sqrt{1+r^2}\right)$. Consequently, the conservation of angular momentum on the sphere is

$$\sin^2\vartheta\,\dot\varphi = h, \tag{10}$$

and not its Euclidean form as one would expect.

Thus, conservation of energy,

$$\frac{1}{2}\left(\dot\vartheta^2 + \frac{h^2}{\sin^2\vartheta}\right) = E - V(\vartheta),$$

where the second term is the centrifugal energy, can be expressed in terms of the inverse radial coordinate $u = 1/r$, and its derivatives, according to the Clairaut equation

$$\frac{1}{2}h^2\left(u'^2 + u^2\right) = E + V(\vartheta) - \frac{h^2}{2}.$$

The last term on the right-hand side is obtained by replacing $\csc^2\vartheta$ by $1+\cot^2\vartheta$. The only difference between Euclidean and spherical conservation of energy is that the total energy, E, acquires a constant term, $-h^2/2$.[1].

The equation of motion is

$$\left(\frac{\sin\vartheta}{h}\right)^2\ddot{\vartheta}+\cot\vartheta=\frac{\mu}{h^2},$$

where we have introduced the Newtonian potential, (6). This equation becomes the characteristic Clairaut equation for an elliptical orbit,

$$\frac{d^2u}{d\varphi^2}+u=\frac{\mu}{h^2}.$$

Its solution is

$$u=\cot\vartheta=\frac{1}{p}\left[1+\varepsilon\cos(\varphi-\varphi_0)\right],$$

where the parameters of the ellipse, ε and p, are the eccentricity and semi-latus rectum, h^2/μ, respectively, and φ_0 is an arbitrary phase which enters as a constant of integration.

Hence, ellipses are the same as in the Euclidean plane [cf. the note in [1]]. Although we expect the conservation of energy to hold in general, the conservation of angular momentum will not be given by its Euclidean form for metrics of the type (2). Physically speaking, the form factor, B, will distort the angular momentum from its Euclidean form.

For concrete sake, we will work with the Schwarzschild factor. The expression for the angular momentum,

$$\frac{r^2\dot{\varphi}}{(1-R_S/r)}=h, \tag{11}$$

differs from its Euclidean expression by the form factor in the denominator. Strangely enough, general relativity arrives at the identical expression for the angular momentum, (11).[2] Yet, so strong are the Euclidean binds that this term is neglected in order to reinstate Euclidean conservation of angular momentum at any cost, without realizing that the phenomenon to be derived (i.e., the angular increase in the orbit of Mercury) is precisely the same same order of magnitude as the one neglected in (11)!

The non-relativistic, non-Euclidean, conservation of energy is

$$\mathbf{T}=\frac{1}{2}\left\{\frac{\dot{r}^2}{(1-R_S/r)^2}+\frac{r^2\dot{\varphi}^2}{(1-R_S/r)}\right\}=E+\frac{\mu}{r}. \tag{12}$$

It differs from the relativistic expression,

$$\mathbf{T}=\left(E+\frac{\mu}{r}\right)+\frac{1}{2c^2}\left(E+\frac{\mu}{r}\right)^2, \tag{13}$$

by the factor

$$\frac{1}{2c^2}\left(E+\frac{\mu}{r}\right)^2,$$

which depends only on the radial coordinate.

The equation of motion is given by

$$\frac{d}{dt}\frac{\partial\mathbf{T}}{\partial\dot{r}}-\frac{\partial\mathbf{T}}{\partial r}=-g$$

where $g=\mu/r^2$, the gravitational acceleration. Explicitly, we obtain

$$\ddot{r}-r\dot{\vartheta}^2=-g\left\{1-\frac{R_S}{r}+3v_S^2+2\frac{h^2}{r^2}\right\}, \tag{14}$$

where $v_S=r\dot{\varphi}$ is the angular velocity in the plane, $\vartheta=\pi/2$.

The equation of motion, (14), differs from that usually obtained by the absence of a term that gives a positive contribution to the radial acceleration.[3] This is because the time component of the Schwarzschild metric imposes a relation between coordinate time, t, and a geodesic parameter, λ, describing the trajectory[4]

$$\frac{d\lambda}{dt}=B(v)=1-\frac{R_S}{r}. \tag{15}$$

Subsequently λ is identified as φ, and this destroys the conservation of angular momentum in any form. Thus, (15) neither expresses the general relativistic conservation of energy,

$$\left(1-\frac{R_S}{r}\right)\frac{dt}{d\lambda}=E$$

nor is it the conservation of angular momentum, (11). For the identification of λ with φ would lead to

$$\frac{r^2\dot{\varphi}}{(1-R_S/r)}=r^2/E\neq h.$$

However, like so many other cases in general relativity, the two errors, miraculously, cancel one another. Hence, gravitational repulsion is a figment of the unwarranted use of an indefinite metric involving coordinate and local times. General relativity is simply not a generalization of special relativity where the metric coefficients are allowed to become functions of space and time.

1. The Precession of the Orbit of Mercury

One of the pillars of general relativity is the derivation of the 'missing' $43''$ per century of the full excess motion of Mercury's perihelion. Since it was first observed by Le Verrierin 1859, there have been many attempts to identify the possible sources of this discrepancy. Among the possibilities were the prediction of an unobserved planet, Vulcan, interplanetary dust, a modification of Newton's universal law, and the quadrupole moment of the solar oblateness. Only the last has remained a feasible possibility.[5] In this section we will show that a quadrupole moment cannot affect the orbit of an ellipse, and show that it is caused by the unequal velocities at the apsides; the velocity of the orbiting satellite is smaller at the perihelion than at the aphelion.[1]

The old quantum theory introduces quantum numbers: the principal number, n, and the subsidiary, or azimuthal, number k. The square of their ratio defines the eccentricity of the orbit,

$$\varepsilon = \sqrt{1 - \frac{k^2}{n^2}},$$

since $0 \leq k \leq n$. And the ratio of the semi-minor axis of the ellipse to the semi-major axis is in the same ratio, k/n. For $k = 0$, the orbits are 'pendulum' orbits; that is, straight paths through the center of force.[6] For large k, or equivalent small ε, the orbits will be hydrogen-like, being nearly circular.

In terms of the velocity circle of radius, v, and center, $\mathbf{c}$, the total energy will be given by

$$-2E = \mathbf{c}\cdot\mathbf{c} - v^2 = -(1-\varepsilon^2)\frac{\mu^2}{h^2} = -\mu/a, \qquad (16)$$

where a is the semi-major axis of the ellipse. This shows that the eccentricity is a relative velocity, $\varepsilon = v_H/c$. From the second equality in (16), it follows that

$$\frac{h^2}{\mu} = a(1-\varepsilon^3) = p,$$

is the semi-latus rectum, and the Bohr velocity is

$$v_H = \frac{\mu}{h}. \qquad (17)$$

This makes (16) proportional to the kinetic energy,

$$-2E = \left(\frac{k}{n}\right)^2 v_H^2.$$

What Born called the radius of a hydrogen-like atom, a_H is really the semi-latus rectum, $p = a(k/n)^2$.

The most important parameter, whose square gives all corrections to the Kepler motion, is the gravitational fine-structure constant,

$$\alpha = \frac{v_H}{c} = \frac{h}{a_H c}. \qquad (18)$$

For instance, the lowest-order approximation to the precession per orbit of Mercury is

$$6\pi\frac{\mu}{p^2} = 6\pi\alpha^2. \qquad (19)$$

This give the gravitational fine-structure constant a value of

$$\alpha_M = 2.65\cdot 10^{-8},$$

which is some five orders of magnitude smaller than the electromagnetic fine-structure constant, $7.29\cdot 10^{-3}$.

We now proceed to show that only dipole interactions can affect the shape of the elliptical trajectory. Expanding the central potential in powers of $1/r$ we get

$$V(r) = -\frac{\mu}{r}\left(1 + J_1\frac{a_H}{r} + J_2\left(\frac{a_H}{r}\right)^2 + \cdots\right)$$

where the dimensionless constant, J_1, is a measure of the gravitational dipole moment, does not vanish when the center of mass does not coincide with the origin, and the second dimensionless constant, J_2, is a measure of the quadrupole moment. We have identified the characteristic length with a_H, the first Bohr radius in the expansion of the potential, $V(r)$.

The orbital equation is

$$\frac{d\varphi}{dr} = \frac{1}{\gamma}\frac{\sqrt{C}}{r^2\sqrt{-A + \frac{2B}{r} - \frac{C}{r^2}}}, \qquad (20)$$

where

$$A = -2E$$

$$B = \mu$$

$$C = h^2\left(1 - \frac{J_1}{k^2}\right) \qquad (21)$$

$$D = \mu J_2 p^2 \qquad (22)$$

and the precession parameter is

$$\gamma = \sqrt{1 - \frac{J_1}{k^2}}. \qquad (23)$$

The orbital equation, (20), has the same form as a Keplerian ellipse with the exception that C, instead of being given by the square of the angular momentum, h^2, has a modified form given by (21). In accordance with Newton's theorem of revolving orbits, the ellipse is now given by

$$r = \frac{C/B}{1 + \varepsilon\cos\gamma(\varphi - \varphi_0)}. \qquad (24)$$

In addition, the semi-latus rectum has been decreased by

$$\frac{C}{B} = \frac{h^2(1 - J_1/k^2)}{\mu} = p,$$

and the eccentricity increased by

$$1 - \frac{AC}{B^2} = 1 - 2|E|\left(\frac{h}{\mu}\right)^2\left(1 - \frac{J_1}{k^2}\right) = \varepsilon^2. \qquad (25)$$

Consequently, the total energy suffers a decrease,

$$-v_H^2\frac{k^2}{n^2}\bigg/\left(1 - \frac{J_1}{k^2}\right) = 2E.$$

[1] This idea was probably originated by Tom Van Flandern (1999). *Meta Research Bulletin* **8** 24-30, but his analysis proved to be faulty. Moreover, the slowing down of the velocity by the Lorentz contraction factor is not the reason for the velocity imbalance.

It is important to observe that the term (22) will affect the total energy, but neither will it affect the shape nor the precession of the elliptical orbit.[6], p. 159. This is, essentially, Newton's theory of revolving orbits. As r goes through one libration, the true anomaly, φ, increases by a factor $2\pi/\gamma$. As $J_1 \to 0$, the orbit degenerates to an ellipse, while for small J_1 the perihelion slowly rotates with an angular velocity $\omega J_1/k^2$, where ω is the mean motion.

The dipole term, J_1, is equivalent to Newton's adding a term $1/r^3$ to the force. It cause the orbit to precess in a rosette fashion. This clearly shows that the quadrupole term, (22), has no effect on the precession of the ellipse, but, rather, affects only the expression for the total energy.[[6], p. 159].

Yet, it would appear from Einstein's modified orbital equation,

$$\frac{d^2u}{d\varphi^2} + u = \frac{h^2}{\mu} + 3\frac{\mu}{c^2}u^2, \tag{26}$$

that the quadrupole *does* have an effect upon the orbital motion. However, it is not (26) which can be solved in closed form: Rather, it is the *linearized* equation which is employed,

$$\frac{d^2u}{d\varphi^2} + \left(1 - 6\frac{\mu}{c^2}u_0\right)u = \frac{h^2}{\mu},$$

on the strength that the gravitational field is weak. But weak with respect to what?

Setting $u_0 = \mu/h^2$, the inverse of the semi-latus rectum, and dividing through by the coefficient of the linear term gives

$$\frac{d^2u}{d(\gamma\varphi)^2} + u = \frac{h^2}{\gamma^2\mu},$$

where the precession coefficient, $\gamma = \sqrt{1-6\alpha^2}$. Over a period of the motion the true anomaly is increased by an amount $6\pi\alpha^2$.

Although it has gone unnoticed, the division, concomitantly, leads to an *increase* in the semi-latus rectum by an amount $3\mu/c^2$. *The process of linearization has effectively reduced the quadrupole moment to a dipole moment*. And if Einstein result holds true, there must also be a change in the semi-latus rectum. This is a way of testing Einstein's result against Gerber's equation, which also predicts the same precession, but no change in the semi-latus rectum. Gerber obtained his result from the dubious expansion of a retarded potential, but with the square of the retardation instead of the retardation itself.

From (19) we find the value of the dipole interaction to be dependent on the principal quantum number, according to:

$$J_1 = 24\pi\frac{|E|}{c^2}n^2 = 6\pi\frac{\mu}{ac^2}n^2. \tag{27}$$

The greater the orbit, the more pronounced becomes the dipole interaction. Introducing (27) into (23) gives

$$\gamma \approx \sqrt{1 - \frac{6\pi\mu}{ac^2(1-\varepsilon^2)}}, \tag{28}$$

and the true anomaly increases by

$$2\pi/\gamma = 2\pi\left(1 + \frac{3\mu}{pc^2}\right). \tag{29}$$

The second term in (29) is the same expression found by Gerber for the increase in the true anomaly that was confirmed by Einstein almost a quarter of century later.

Now, introducing (27) into (25) leads to a quadratic expression for the ratio of quantum numbers:

$$\left(\frac{k}{n}\right)^2 - \left(\frac{h^2}{\mu a}\right)\frac{k}{n} + 12\pi\left(\frac{h}{ac}\right)^2 = 0.$$

It has two real roots which are:

$$\left(\frac{v_-}{c}\right)^2 = 1 - 6\pi\frac{\mu}{ac^2} \tag{30}$$

$$\left(\frac{v_+}{c}\right)^2 = 1 - \frac{p}{a} + 6\pi\frac{\mu}{ac^2}. \tag{31}$$

The larger root, (30), is associated with the speed at the perihelion, while the smaller root, (31), with the slower speed at the perihelion. The velocity imbalance rotates the ellipse forward, and the precession of the perihelion is simply a manifestation of this imbalance.

Consequently, Newcomb's[7] original suggestion, and Dicke's[5] follow-up, that the quadrupole moment should have an effect upon the precession rate of the orbit, is unfounded. It is even more surprising that additional contributions to the precession which are even smaller than the solar quadrupole moment keep popping up in the literature.[8] These contribution may possibly contribute to the total energy, but certainly not to the shape of the orbital motion.

The relativistic treatment follows the same general trend with the exception that the mass undergoes an apparent decrease by an amount

$$B = \mu\left(1 - \frac{\alpha^2}{2n^2}\right).$$

The apparent decrease in the mass of the sun causes a decrease in the speed of the planet at the aphelion. Other than this, there is no necessity of introducing dimensionless parameters, J_1 and J_2, but, otherwise, the relativistic treatment confirms the general result of the nonrelativistic model so that the advance of the perihelion can be treated solely within a nonrelativistic framework.

2. Uniformly Rotating Discs: Keplerian Ellipses Centered at the Origin

We now turn to a hyperbolic model of constant curvature of a uniformly rotating disc in a harmonic oscillator potential that yields Keplerian ellipses centered at the origin rather than at a focus of the ellipse.

The form factor is

$$B(r) = 1 - \frac{(\omega r)^2}{c^2}, \tag{32}$$

where ω is the angular speed of the rotating disc. The Beltrami metric is now given by

$$ds^2 = \frac{dr^2}{(1-(\omega r/c)^2)^2} + \frac{r^2 d\varphi^2}{(1-(\omega r/c)^2)}. \tag{33}$$

That we are in a non-euclidean setting is made apparent by considering what happens at a constant radius, r. The periphery of the disc is seen to be greater than the circumference of the circle by an amount:

$$\int_0^{2\pi} \frac{r d\varphi}{\sqrt{1-(\omega r/c)^2}} = \frac{2\pi r}{\sqrt{1-(\omega r/c)^2}} > 2\pi r.$$

Additionally we can consider what happens at $\varphi = \text{const.}$, as we vary the radial distance. As the inhabitants (Poincarites) of this two-dimensional world approach the rim of the disc, their rulers shrink in proportion to their body dimensions so that the periphery seems to them to be at an infinite distance away. That is, at constant, φ,

$$s(r) = \int_0^r \frac{dr}{1-(\omega r/c)^2} = \frac{c}{\omega} \tanh^{-1}\left(\frac{\omega r}{c}\right) = \frac{c}{2\omega} \ln\left(\frac{c+\omega r}{c-\omega r}\right).$$

Thus, as $r \to c/\omega$, the arc length, $s(r)$, tends to infinity. This seeming finite segment has an infinite hyperbolic length. The hyperbolic plane, rather than the pseudo-sphere, is the true analog of the sphere for constant negative curvature.[9] The right-hand side is the logarithm of the longitudinal Doppler effect, and the logarithm of the cross-ratio relates it to the Lobachevskian 'straight' line in the hyperbolic plane.[10]

The conservation of angular momentum is expressed as

$$\frac{r^2 \dot{\varphi}}{1-(\omega r/c)^2} = h, \tag{34}$$

and the conservation of energy reads

$$\frac{1}{2}\left\{\frac{\dot{r}^2}{(1-(\omega r/c)^2)^2} + \frac{r^2\dot{\varphi}^2}{1-(\omega r/c)}\right\} = E + V(r). \tag{35}$$

Introducing (34) into (35) give the orbital equation as

$$\frac{h^2}{2}\left[\frac{1}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2}\right] + V(r) = E + \frac{1}{2}\left(\frac{h\omega}{c}\right)^2.$$

Hence, the curvature appears only as a constant in the last term on the right-hand side, so that projected orbits from the hyperboloid are the same as in the Euclidean plane.

Since we know from Bertrand's theorem that only *circular*, closed orbits exist only for the Newtonian potential, (6), and the harmonic oscillator, (3), we choose either one and use it in Clairaut's equation. A first integral exists

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{\Omega^2}{h^2u^2} = 2\frac{E}{h^2} + \left(\frac{\omega}{c}\right)^2,$$

where we chose the harmonic oscillator potential, $V(r) = \Omega^2 r^2/2$. The equation may easily be integrated, and the result is

$$r^2 = \frac{\sqrt{A}^{-1}}{1+\varepsilon\sin[2\Omega/h\cdot(\varphi-\varphi_0)]}, \tag{36}$$

where $A = E/h^2 + (\omega/c)^2/2$. For non-negative values of Ω^2, the orbits will be ellipses centered at the origin. The larger the value of Ω, the smaller will be the eccentricity,

$$\varepsilon = \sqrt{1-(\Omega/hA)^2},$$

and the more circular will be the orbit.

If the projection were from a sphere, the curvature would appear in the combination, $2E - (h\omega/c)^2$, which would limit the motion to those trajectories for which $2E > (h\omega/c)^2$, or $2E > h^2/R^2$, where R is the radius of the sphere. Only stable elliptical orbits exist when the total energy is greater than the centrifugal energy. No such condition exists in the hyperbolic plane, and negative energies are admissible so long as $2E + h^2/R^2 > 0$.

3. Rotational Deformation

The foregoing metric of constant negative curvature, characterizing a uniform rotating disc, can be used to describe the rotational flattening of a star. A spheroid is modelled using the Legendre polynomial of degree 2,

$$P_2(\cos\vartheta) = \frac{1}{2}\left(3\cos^2\vartheta - 1\right),$$

where ϑ is the angle measured from the axis of symmetry, i.e., colatitude. A rapidly rotating solar core, in contrast to a slowly rotating photosphere, could offer an explanation for solar oblateness.[11] Letting a denote the radius of the photosphere, the distortion coefficient, (32), becomes

$$R(a) = a \cdot B(a) = a\left[1 - \left(\frac{\omega a}{c}\right)^2 P_2(\cos\vartheta)\right], \tag{37}$$

becomes an equipotential surface at a radius a. The second term is the angle dependent centrifugal potential.

Introducing Kepler's III law, $\omega^2 = \mu/a^3$,*[2] into (37) the deformed shape of the sun's photosphere will be described by equipotential surface

$$R(a) = a\left[1 - \frac{\mu}{ac^2}P_2(\cos\theta)\right]. \tag{38}$$

The coefficient of the second term is one-third the coefficient in the modified Einstein equation of the orbit.

[2]*Kepler's law states that the energy flow is equal the $\frac{3}{2}$-power of twice the total energy,

$$\mu\omega = (-2E)^{3/2}.$$

Rotational flattening, leading to tidal bulging, is treated as an oblate spheroid with two long axes, and one short one. These occur at $\vartheta = 0$, and $\vartheta = \pi/2$, respectively. The values at the surface are

$$R_L(a) = a\left(1 + \frac{\mu}{2ac^2}\right) \qquad \text{and} \qquad R_S = a\left(1 - \frac{\mu}{ac^2}\right).$$

These are the principal axes of an oblate ellipsoid. Dynamic, rotational, equilibrium, sometimes referred to as hydrostatic equilibrium, requires the deformations to sum to zero, i.e., $2R_L + R_S = 3a$. The square of the eccentricity is given in terms of the squares of the shortest and longest principal axes, i.e.,

$$\varepsilon^2 = 1 - \frac{R_S^2(a)}{R_L^2(a)} = 3\frac{\mu}{ac^2} + \cdots \tag{39}$$

Thus, to lowest order, the square of the eccentricity is precisely the Einstein expression for his modified orbital equation, (26).

Flatness, or ellipticity, of the sun is defined as

$$f = 1 - \frac{R_S}{R_L} = \frac{3}{2}\frac{\mu}{ac^2}.$$

For $a = 1.17 \times 10^5$ km,[11] the value of the ellipticity is $f = 2.84 \times 10^{-5}$, compared to the actual value, 5×10^{-5}.[12]

4. Tidal Deformation

In addition to rotational deformation, there also occurs tidal deformation. Tidal forces can be derived by generalizing the metric associated with (9), viz.,

$$ds^2 = d\vartheta^2 + G^2(\vartheta)\,d\varphi^2, \tag{40}$$

where in the specific case treated above, $G(\vartheta) = \sin\vartheta$. We can express the trigonometric identity as $G' = \sqrt{1 - G^2}$. The sectional curvatures are given by

$$\frac{G''}{G} = -1$$
$$\frac{1 - G'^2}{G^2} = 1$$

The former is radial curvature, while the latter is tangential curvature to a sphere. Both sectional curvatures are constant, as we expected. They run the entire gamut of sectional curvatures.[13]

The sectional curvatures can be generalized to non-constant curvatures. Introducing, Kepler's III law,

$$G^3 = \frac{\mu}{\omega^2}, \tag{41}$$

converts the constant curvature plane, with $G' = \sqrt{1 - (\omega G)^2/c^2}$, into a non-constant curvature one,

$$G' = \sqrt{1 - \frac{\mu}{c^2 G}}.$$

The radial and tangential sections curvatures are now given by

$$\frac{G''}{G} = -\frac{1}{2}\frac{\mu}{c^2G^3}$$
$$\frac{1 - G'^2}{G^2} = \frac{\mu}{c^2G^3},$$

respectively. The right-hand sides are the tidal force components.

Equipotential surfaces resulting from tidal deformation can be represented by an ellipsoid whose increments are given by the sectional curvatures. The greatest principal axis is due to tangential sectional curvature, and the other two principal axes are smaller and equal. The ellipsoid is therefore oblate, in contrast to the prolate ellipsoid for rotational deformation.

In a state of dynamic equilibrium the sum of the increments in the principal axes vanish. This is none other than Einstein's condition of 'emptiness', or the vanishing of the eigenvalue of the Ricci tensor. According to Dirac[14], this

> constitutes his law of gravitation. 'Empty' here means that there is no matter present and no physical fields except the gravitational field. the gravitational field does not disturb the emptyness [*sic*]. Other fields do.

From Kepler's III law, (41), it is difficult to imagine a gravitational field *without* matter. The same condition of 'emptiness' should also apply to rotational deformation, which destroys the uniqueness of the gravitational field as the sole field present.

The metric, (40), is referred to as a 'warped' metric.[15] The Schwarzschild metric is an example of a 'doubly warped' metric. The coefficient of the time-component of the metric, F^2, must necessarily satisfy, $F = G'$, and, like G, be time-independent. Although F only possesses radial sectional curvature, the derivatives of F and G, are of opposite signs, and this leads to a vanishing of their sum so that Einstein's condition of 'emptiness' is formally obeyed. Thus, time and space components of the metric are not on equal footing, and it says nothing about the distortion of a sphere into an oblate spheroid due to the action of tidal forces.

The introduction of a time dimension destroys the fact that the tidal forces can be referred to a triaxial ellipsoid which is entirely spatial. The sectional curvatures in time are neither physically explicable, nor are the cross-terms involving time and space. In fact, the physical meaning of rotational and tidal forces, which are completely static,[16] is less than transparent when a time dimension is introduced.

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(2019). "Planetary orbitsin constant curvature planes," confuses ϑ with r/R in the definition of the elliptic distance, $\hat{r}/R = \tan(r/R)$, where R is constant,

$$\hat{r} = R\tan(r/R) = \frac{p}{1+\varepsilon\cos(\varphi-\varphi_0)}.$$

It is the right-hand side which is r, and not the tangent of r/R. In the hyperbolic case, $\bar{r} = R\tanh(r/R)$, ellipses would not result for all values of the semi-latus rectum, i.e.,

$$r = \tanh^{-1}\left(\frac{p}{1+\varepsilon\cos(\varphi-\varphi)}\right).$$

Ellipses are ellipses not matter whether we are in the hyperbolic, elliptic, or Euclidean planes.

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