# Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

Ralf Wüsthofen

**Abstract.** This paper proves an inconsistency in Peano arithmetic (PA). We express two propositions — a strengthened form of the strong Goldbach conjecture and its negation — using a specific set that varies according to which of the propositions holds. On the other hand, we show that this set remains unchanged under the assumptions of the two statements. This causes a contradiction.

**Notations.** Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

**Theorem.** *PA is contradictory, i.e. the statement FALSE can be derived.*

*Proof.* We define the set $S_g$ := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer n ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers n ≥ 4 appear as m in a middle component mk of $S_g$. So, by the definition of $S_g$ we have

$$\text{SSGB} \iff \forall n \in \mathbb{N}_4 \;\; \exists (pk, mk, qk) \in S_g \quad n = m$$

$$\neg\text{SSGB} \iff \exists n \in \mathbb{N}_4 \;\; \forall (pk, mk, qk) \in S_g \quad n \neq m.$$

The set $S_g$ has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of $S_g$ ("*covering*"), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$. So we have

**(C)** $\forall x \in \mathbb{N}_3 \;\; \exists (pk, mk, qk) \in S_g \quad x = pk \;\vee\; x = mk.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 27: (**3·9**, 7·9, 11·9)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 4096: (3·1024, **4·1024**, 5·1024)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set $S_g$ ("*maximality*"). So we have

**(M)** $\forall$ p, q $\in \mathbb{P}_3$, p < q $\forall$ k $\in \mathbb{N}$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The property (C) excludes the possibility that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is different from all $S_g$ triple components pk and mk.

The property (M) excludes the possibility that there is an n ≥ 4 different from all m, i.e. ¬SSGB, for the reason that n is the arithmetic mean of a pair of primes not used in $S_g$. Thus (M) excludes the possibility that ¬SSGB applies due to a missing prime number pair. The proof would no longer be possible if we left out any prime number pair in the formulation of SSGB and $S_g$.

The basic idea is now the following.

*There are two possibilities for $S_g$, exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in $S_g$ or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.*

*Since, due to (M), an n ≥ 4 different from all m cannot be the arithmetic mean of a pair of primes not used in $S_g$ and since, due to (C), this n equals a component of some $S_g$ triple that exists by definition, the covering of $\mathbb{N}_3$ by the $S_g$ triples in the case n exists (¬SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in $S_g$ take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.*

*We illustrate this by representing the covering of $\mathbb{N}_3$ by an infinite matrix where the k-th row, k ≥ 1, is determined by all triple components pk, mk, qk :*

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 4 5 | 3 5 7 | ... | 5 6 7 | 5 8 11 | ... | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ● ∘ ∘ | ... | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| 6 8 10 | ∘ ∘ ∘ | ... | ∘ ∘ ∘ | ∘ ∘ ∘ | ... | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |
| ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ● ∘ ∘ | ... | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |
| ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| ∘ ● ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | | ∘ ∘ ∘ | ∘ ∘ ∘ | ∘ ∘ ∘ | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |
| ... | ... | | ... | ... | | ... | ... | ... | ... | | ... | ... | ... | ... |

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*The marked matrix elements ● represent an n ≥ 4 given in the case ¬SSGB, depending on which of the three types the number n is: prime, composite and not a power of 2, power of 2. There are infinitely many other matrix elements (unmarked) that are also equal to n or a multiple of n. In the case SSGB where there is no such n, the matrix is the same. On the other hand, there is a column mk = nk, k ≥ 1, in the case SSGB that does not exist in the case ¬SSGB. So, the matrix is different in the two cases. This is a contradiction.*

After having excluded the possibility that ¬SSGB applies due to ¬(C) or ¬(M), we will now show that ( (C) $\wedge$ (M) ) leads to a contradiction.

The following steps are independent of the choice of n if, in the case of ¬SSGB, there is more than one that is different from all m. For example, the minimal such n works.

We split $S_g$ into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \ \vee\ mk = yk' \ \vee\ qk = yk' \}$

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \ \wedge\ mk \neq yk' \ \wedge\ qk \neq yk' \}$.

We define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \ \mathrm{SSGB} \ \wedge\ ( (C) \wedge (M) ) \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg\mathrm{SSGB} \ \wedge\ ( (C) \wedge (M) ) \}$.

Under the assumption SSGB there is no n in addition to all m defined in $S_g$. Therefore, under this assumption, we can choose any $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So,

**(1.1)** we have a proof of $( \forall\, y \in \mathbb{N}_3 \qquad \mathrm{SSGB} \Rightarrow S_1 = S_g+(y) \cup S_g-(y) )$.

Under the assumption $\neg$SSGB there is an $n \in \mathbb{N}_4$ as described above. Then, since the possibilities $\neg$(C) and $\neg$(M) for the existence of n are ruled out,

**(1.2)** we have a proof of $( \neg\mathrm{SSGB} \Rightarrow S_2 = S_g+(n) \cup S_g-(n) )$.

$S_g+(n) \cup S_g-(n)$ is independent of n, since for every n it equals $S_g$. So we have a proof that $S_g+(n) \cup S_g-(n)$ equals $S_g+(y) \cup S_g-(y)$ for every $y \in \mathbb{N}_3$. Hence, from (1.2) we get

**(1.2')** we have a proof of $( \forall\, y \in \mathbb{N}_3 \quad \neg\mathrm{SSGB} \Rightarrow S_2 = S_g+(y) \cup S_g-(y) )$.

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), ( (1.1) $\wedge$ (1.2') ) implies

**(2.1)** we have a proof of

$$( \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad SSGB \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_g{+}(y) \cup S_g{-}(y) \} )$$

$\wedge$

**(2.2)** we have a proof of

$$( \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad \neg SSGB \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_g{+}(y) \cup S_g{-}(y) \} ).$$

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

**(2.1')** we have a proof of $( \forall y \in \mathbb{N}_3 \quad SSGB \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_g{+}(y) \cup S_g{-}(y) \} )$

$\wedge$

**(2.2')** we have a proof of $( \forall y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_g{+}(y) \cup S_g{-}(y) \} )$.

The splitting of $S_g$ into the complementary subsets $S_g{+}(y)$ and $S_g{-}(y)$ is independent of our information about $S_g$, in particular it is independent of the existence of an $n \geq 4$ and the property behind n.

So, since by definition $S_g+(y) \cup S_g-(y)$ equals $S_g$ for every $y \in \mathbb{N}_3$ regardless of whether or not SSGB holds, for every $y \in \mathbb{N}_3$ the set $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ is a fixed set that is either equal to $\mathbb{N}_4$ or to some non-empty proper subset Y of $\mathbb{N}_4$.

Therefore, ( (2.1') $\wedge$ (2.2') ) splits as follows.

( **(3.11)** we have a proof of ( SSGB => $M_1 = \mathbb{N}_4$ )

$\wedge$

**(3.21)** we have a proof of ( $\neg$SSGB => $M_2 = \mathbb{N}_4$ ) )

$\vee$

( **(3.12)** we have a proof of ( SSGB => $M_1 = Y$ )

$\wedge$

**(3.22)** we have a proof of ( $\neg$SSGB => $M_2 = Y$ ) ).

Now, we will establish a contradiction to ( ((3.11) $\wedge$ (3.21)) $\vee$ ((3.12) $\wedge$ (3.22)) ).

Under the assumption SSGB the numbers m defined in $S_g$ take all integer values $x \geq 4$ whereas under $\neg$SSGB they don't. Therefore,

**(4.1)** we have a proof of ( SSGB => $M_1 = \mathbb{N}_4$ )

$\wedge$

**(4.2)** we have a proof of ( $\neg$SSGB => $M_2 = Y$, for some non-empty proper subset Y of $\mathbb{N}_4$ ).

Then, due to (4.1), we obtain that (3.12) is false, and, due to (4.2), we obtain that (3.21) is false.

Thus, ( ((4.1) $\wedge$ (4.2)) $\wedge$ ( ((3.11) $\wedge$ (3.21)) $\vee$ ((3.12) $\wedge$ (3.22)) ) ) yields the statement FALSE.

□