OPEN ACCESS

J. Innovation Sciences and Sustainable Technologies, 3(1)(2023), 23 - 40.
https://doie.org/10.0421/JISST.2023834560

Original Research

# Lie Symmetry Analysis for Soliton Solutions of Generalised Kadomtsev-Petviashvili-Boussinesq Equation in $(3+1)$-dimensions

Vishakha Jadaun* . Navnit Jha†

Received in final form on December 18, 2022

**Abstract**

The Lie group of infinitesimal transformations technique and similarity reduction is performed for obtaining an exact invariant solution to generalized Kadomstev-Petviashvili-Boussinesq (gKPB) equation in (3+1)-dimensions. We obtain generators of infinitesimal transformations, which provide us a set of Lie algebras. In addition, we get geometric vector fields, a commutator table of Lie algebra, and a group of symmetries. A detailed geometrical framework related to the nature of the solutions possessing traveling wave, bright and dark soliton, standing wave with multiple breathers, and one-dimensional kink, for the appropriate values of the parameters involved. It is observed that there is a qualitative difference between dark soliton and bright soliton. Multiple breathers are detected, a breather is a nonlinear wave wherein energy concentrates in a local and oscillatory manner. A partially standing and partially traveling wave with changing amplitude is also observed and it is seen that breathers are localized solutions with varying amplitude.

**Keywords:** $(3+1)$-dimensional Generalised Kadomtsev-Petviashvili-Boussinesq equation, Lie symmetries, Similarity transformations method, Infinitesimal generator, Soliton solutions.

## 1 Introduction

Many nonlinear partial differential equations (PDEs) have been proposed to model various complex physical, chemical, and biological phenomena mathematically throughout the last several decades. For example, diverse theoretical advancements in mathematics such as fluid mechanics (including the interaction of waves, solitary waves, traveling waves, shallow water waves, and rogue waves), and the theory of turbulence (including analysis of chaos) have been perused in a variety of application to model nonlinear phenomena [1, 2]. These nonlinear phenomena are mathematically modeled as a nonlinear system based on the nonlinear system of equations, which is a set of simultaneous differential equations [3, 4]. Therefore,

*Department of Mathematics, Malla Reddy University, Hyderabad, India, E-mail: vishakhasjadaun@gmail.com
†Department of Mathematics, South Asian University, New Delhi, India, E-mail: navnitjha@sau.ac.in

the qualitative analysis of various solutions to nonlinear evolution PDEs plays a significant role in such studies. It is known that the nonlinear PDEs are not straightforward to solve. A variety of techniques including the multiple exp-function method [5], generalized symmetry method [6, 7, 9, 8, 10], the Bäcklund transformation method, Hirota's bilinear method [11], Pfaffian technique, Darboux transformation [12], the Painlevé analysis, the inverse scattering method [13], Wronskian and Grammian solutions have been widely employed to understand characteristics of nonlinear evolution equations.

A soliton is a solitary wave that arises from a delicate balance between nonlinear and dispersive effects. It maintains its shape while moving at a constant speed and its pulse width depends on the amplitude. For a non-dissipative system, a soliton is a solitary wave whose amplitude, shape, and velocity are conserved. A soliton collides with another soliton, their fundamental parameters remain conserved, except the phase shift. The term "Bright Soliton" refers to a soliton whose intensity is larger than the background, whereas "Dark Soliton" refers to a soliton whose intensity is smaller than the background. Dark solitons can be generated by the introduction of short dark pulses in an existing prolonged standard pulse. They are robust to losses and more stable than bright solitons. Breather is a soliton with energy concentrated in an oscillatory and localized manner. Topological solitons emerge due to topological constraints. They represent a twist in the value of the soliton and cause the transition from one value to another value. In the current work, we aim to analyze the following gKPB equation in (3+1)-dimensions [14],

$$\Delta := w_{xxxy} + 3w_x w_{xy} + 3w_y w_{xx} + w_{tt} + w_{xt} + w_{yt} - w_{zz} = 0, \tag{1}$$

where $w(x, y, z, t)$ represents the height of the wave at a spatial point $(x, y, z)$ in time $t$ on a three dimensional real space. Removal of the term $w_{tt}$ from the equation (1), gives us the widely popular Kadomtsev-Petviashvili (KP) equation [15]. It is observed that the KP equation is an integrable model and can be represented by a first-order PDE in time, while the Kadomstev-Petviashvili-Boussinesq equation can take the form of a second-order nonlinear PDE in a temporal direction. Moreover, it models both left and right-moving waves. Yu and Sun [16], constructed a direct bilinear Bäcklund transformation and obtained rational and exponential traveling wave solutions with various wavenumbers. The exact lump solutions through the perturbation expansion technique combined with Hirota's bilinear transformation are described in [17]. Lü [18] formulated the lump, breather-wave, and interaction solution with few restrictions considering Hirota transformation and symbolic tools of Mathematica. Recently, Wang *et al.*[19] applied the bifurcation theory of the dynamical system to it and presented traveling wave solutions of the KPB equation.

The present work is arranged in the following manner. In Section 2, the Lie symmetry analysis of the gKPB equation in (3+1)-dimensions is obtained. In Section 3, we perform symmetry reduction and determine the analytic solution for the gKPB equation. In Section 4, we have given a numerical algorithm. Section 5 contains a geometrical representation of extracted solutions by providing a 3D solution surface for various values of parameters. This section also includes brief analytical discussions about the nature of obtained solutions. Finally, Section 6 is concluded with remarks and findings.

## 2 Lie symmetry analysis for $(3+1)-$dimensional gKPB equation

Using Lie symmetry analysis similarity reductions of the gKPB equation is derived as given in [6, 7]. Let us construct the one-parameter Lie group of transformation with $(u_1 = w, x_1 = x, x_2 = y, x_3 = z, x_4 = t)$,

$$\begin{bmatrix} \mathring{\mathrm{x}} \\ \mathring{\mathrm{y}} \\ \mathring{\mathrm{z}} \\ \mathring{\mathrm{t}} \\ \mathring{\mathrm{w}} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} + \varsigma \begin{bmatrix} \xi^1(x,y,z,t,w) \\ \xi^2(x,y,z,t,w) \\ \xi^3(x,y,z,t,w) \\ \tau(x,y,z,t,w) \\ \eta(x,y,z,t,w) \end{bmatrix} + O(\varsigma^2). \tag{2}$$

Lie group transformations' generators for the independent and dependent variables are $\xi^1, \xi^2$, $\xi^3$, $\tau$, and $\eta$, respectively, together with continuous group parameter $\varsigma$. The aforementioned transformations' vector field is represented as $X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial w}$, where $\xi^i = \xi^i(x,y,z,t,w)$ $\eta = \eta(x,y,z,t,w), \tau = \tau(x,y,z,t,w)$. By employing the invariance condition: $Pr^{(4)}X(\Delta) = 0$, wnenever $\Delta = 0$, the infinitesimal invariance criteria for Eq. (1) is obtained as

$$\eta_{xxxy} + 3\eta_x w_{xy} + 3w_x\eta_{xy} + 3\eta_y w_{xx} + 3\eta_{xx}w_y + \eta_{tt} + \eta_{xt} + d\eta_{yt} - \eta_{zz} = 0. \tag{3}$$

Now, implementing the fourth prolongation $Pr^{(4)}X$ of $X$ to Eq. (3), one can receive the following determining equations

$$\begin{aligned} \xi^1_t &= \xi^1_x = \frac{\xi^3_z}{3},\ \xi^1_w = 0,\ \xi^1_y = 0,\ \xi^1_z = \frac{\xi^3_t}{2} = 0, \xi^2_t = \xi^2_z = \xi^2_w = 0,\ \xi^2_y = \xi^3_z,\ \xi^2_z = \frac{\xi^3_t}{2}, \\ \xi^3_w &= \xi^3_x = \xi^3_y = \xi^3_{tt} = \xi^3_{tz} = \xi^3_{zz} = 0, \tau_t = \xi^3_z,\ \ \tau_x = \tau_w = \tau_y = 0,\ \ \tau_z = \xi^3_t, \\ \eta_w &= -\frac{\xi^3_z}{3},\ \ \eta_x = \frac{\xi^3_z}{9} = \eta_y,\ \ \eta_{tt} = \eta_{zz}. \end{aligned}$$

Upon simplifying the above determining equations, the infinitesimal generators are

$$\begin{aligned} \xi^1 &= \frac{c_1}{2}z + \frac{c_2}{3}(x+t) + c_6, \quad \xi^2 = 2c_4y + \frac{a}{c}c_3t + c_6, \quad \xi^3 = \frac{c_1}{2}z + c_2y + c3, \\ \tau &= c_1t + c_2z + c_3, \quad \eta = \varphi(t-z) + \psi(t+z) + \frac{c_2)}{9}(x+y-3w), \end{aligned} \tag{4}$$

where $c_i, i = 1(1)6$ and $\varphi(t-z)$ & $\psi(t+z)$ are arbitrary constants and arbitrary functions, respectively. Assume $\varphi(t-z) = c_7$, the Lie algebra of infinitesimal symmetries of Eq. (1) is spanned by following vector fields

$$\begin{aligned} X_1 = &\ \frac{1}{2}(z+2t)\frac{\partial}{\partial x} + \frac{1}{2}z\frac{\partial}{\partial y} + z\frac{\partial}{\partial t} \\ X_2 = &\ \frac{1}{3}(x+t)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{1}{9}(x+y-3w)\frac{\partial}{\partial w} \\ X_3 = &\frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial y}, \quad X_6 = \frac{\partial}{\partial x}, \quad X_7 = \frac{\partial}{\partial w}. \end{aligned} \tag{5}$$

### 2.1 Computation of Lie-Brackets

The Lie algebra commutation relation, through the Lie bracket table, appears to be antisymmetric with $(i,j)^{th}$ entry as $[X_i, X_j] = X_i * X_j = X_i \cdot X_j - X_j \cdot X_i$. Moreover, the diagonal elements of the commutator table are all zero, since $[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]$. Further, the commutator table offers each structure constant in a simplified manner. The following commutation table is constructed using the generators of infinitesimal transformation (5),

| $[\cdot]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |
|---|---|---|---|---|---|---|---|
| $X_1$ | $0$ | $0$ | $\frac{X_5-X_6}{2}+X_4$ | $-X_3$ | $0$ | $0$ | $0$ |
| $X_2$ | $0$ | $0$ | $-X_3$ | $\frac{-X_6}{3}-X_4$ | $-X_5-\frac{X_7}{9}$ | $-\frac{X_6}{3}-\frac{X_7}{9}$ | $\frac{X_7}{3}$ |
| $X_3$ | $\frac{-X_5+X_6}{2}-X_4$ | $X_3$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $X_4$ | $X_3$ | $\frac{X_6}{3}+X_4$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $X_5$ | $0$ | $X_5+\frac{X_7}{9}$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $X_6$ | $0$ | $\frac{X_6}{3}+\frac{X_7}{9}$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $X_7$ | $0$ | $\frac{-X_7}{3}$ | $0$ | $0$ | $0$ | $0$ | $0$ |

Table 1: Lie Brackets for the gKPB equation

As depicted in Table 1, a continuous group of transformations for the (3+1)-dimensional gKPB is found by vector fields that span an infinite-dimensional Lie algebra. The linear combinations of generators $X_i, i = 1, 2, \ldots, 7$ yield infinite subalgebras for the above Lie algebra.

## 3 Symmetry Reduction and Closed-form Solutions

In this section, we seek to derive a group invariant solution for Eq. (1) from the reduced equations. Note that reduced equations are, in turn, derived from invariant functions. It is easy to simplify the characteristic of Lagrange's system to determine invariant functions

$$\frac{dx}{\xi^1(x,y,z,t,w)} = \frac{dy}{\xi^2(x,y,z,t,w)} = \frac{dz}{\xi^3(x,y,z,t,w)} = \frac{dt}{\tau(x,y,z,t,w)} = \frac{dw}{\eta(x,y,z,t,w)}. \tag{6}$$

Invariance is a remarkable property of the Lie group of transformations method. The solutions obtained under a one-parameter Lie group of transformations are invariant. The Lagrange system of characteristic equations allows group invariant solutions to construct differential equations with the one-less independent variable, resulting in an ordinary differential equation (ODE). The resolution of ODE is back substituted to yield a solution of the primary differential equation.

### 3.1 Vector field $\mathbf{X}_1$

The Eq. (6) and the vector field $\mathbf{X}_1$ in (5) are being used to find the following invariant solution for Eq. (1)

$$w(x,y,z,t) = f(X,Y,Z), \quad \text{with} \quad Y = 2y - t, \quad X = 2x - t, \quad Z = z^2 - t^2. \tag{7}$$

From Eqs. (7) and (1), we get following PDE

$$-f_Z - 4Zf_{ZZ} - f_{YY} - 2f_{XY} - f_{XX} + 24f_X f_{XY} + 24f_Y f_{XX} + 16f_{XXXY} = 0. \tag{8}$$

Using the similarity transformation method (STM), the infinitesimal generators for Eq. (8) are

$$\xi_X = \frac{a_1}{4}X + a_3,\ \xi_Y = \frac{a_1}{2}Y + a_2,\ \xi_Z = a_1 Z,\ \eta_f = a_5 \log Z + \frac{a_1}{48}(X+Y) + a_4, \tag{9}$$

where $\xi_X, \xi_Y, \xi_Z$ are the generators of infinitesimal transformations for independent variables $X,\ Y,\ Z$, respectively while $\eta_f$ for dependent variable $f$; $a_i$, $i = 1(1)5$, are constants. A set of vector fields for generators of infinitesimal transformation (9) is given by

$$\begin{aligned} \pi_1 = & \quad \frac{X}{4}\frac{\partial}{\partial X} + \frac{Y}{2}\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z} + \frac{X+Y}{48}\frac{\partial}{\partial f}, \\ \pi_2 = & \quad \frac{\partial}{\partial Y}, \quad \pi_3 = \frac{\partial}{\partial X}, \quad \pi_4 = \frac{\partial}{\partial f}, \quad \pi_5 = \log Z \frac{\partial}{\partial f}. \end{aligned} \tag{10}$$

### 3.1.1 Vector field $\pi_2$

Lagrange's characteristic equations for vector field $\pi_2$ in (10) provide the function $f$ in the following invariant form

$$f(X,Y,Z) = H(r,q), \quad r = X, \quad q = Z.$$

Now, we reduce Eq. (8) as follows

$$H_q + 4H_{qq} + H_{rr} = 0. \tag{11}$$

Using back substitution, solution of Eq. (11) provides solution of primary equation (1) as

$$\begin{aligned} w(x,y,z,t) \quad &= \frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2x-t)^2)+K_2\right)\left(\mathrm{BesselJ}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)})K_3\right)}{\exp(\sqrt{K_1}(2x-t))} \\ &+ \frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2x-t)^2)+K_2\right)\left(\mathrm{BesselY}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)})K_4\right)}{\exp(\sqrt{K_1}(2x-t))}, \end{aligned} \tag{12}$$

where $K_i$, $i = 1(1)4$ are constants.

### 3.1.2 Vector field $\pi_3$

Lagrange's characteristic equations for vector field $\pi_3$ in (10) provide the function $f$ in the invariant form shown below

$$f(X,Y,Z) = H(r,q) \quad \text{and} \quad r = Y, \quad q = Z.$$

The above equation reduces Eq. (8) into following PDE

$$H_q + 4H_{qq} + H_{rr} = 0. \tag{13}$$

Using back substitution, the solution of the Eq. (13) provides solution of primary equation (1)

$$\begin{aligned} w(x,y,z,t) &= \frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2y-t)^2)+K_2\right)\left(\mathrm{BesselJ}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)}K_3\right)}{\exp(\sqrt{K_1}(2y-t))} \\ &+\frac{(z^2-t^2)^{\frac{3}{8}}\left(K_1^2\exp(K_1(2y-t)^2)+K_2\right)\left(\mathrm{BesselY}(\frac{3}{4},\sqrt{-K_1(z^2-t^2)}K_4\right)}{\exp(\sqrt{K_1}(2y-t))}, \end{aligned} \tag{14}$$

where $K_i$, $i = 1(1)4$ are constants.

## 3.2 Vector field $\mathbf{X}_3$

The equation (1) is converted into the given invariant form with the aid of Eq. (6) and vector field $\mathbf{X}_3$ in (5)

$$w(x,y,z,t) = f(X,Y,T), \quad Y = y \quad X = x, \quad T = t,$$

Using above equation and Eq. (1), we get the following PDE

$$f_{XXXY} + 3f_X f_{XY} + 3f_Y f_{XX} + f_{XT} + f_{YT} + f_{TT} = 0. \tag{15}$$

The traveling wave solutions of Eq. (15) provide the following traveling wave solutions for Eq. (1)

$$w(x,y,z,t) = 2K_2 \tanh\left(K_2 x + K_3 y + (-\alpha_1 - \alpha_2)t + K_1\right) + K_5, \tag{16}$$

$$w(x,y,z,t) = 2K_2 \tanh\left(K_2 x + K_3 y + (-\alpha_1 - \alpha_2)t + K_1\right) + K_5, \tag{17}$$

where $\alpha_1 = \frac{K_2+K_3}{2}$, $\alpha_2 = \frac{1}{2}\sqrt{-16K_2^3K_3 + K_2^2 + 2K_2K_3 + K_3^2}$ and $K_i$, $i = 1(1)5$ are constants. Now, we apply the similarity transformation approach on Eq.(15) to produce the infinitesimal generators shown below

$$\begin{aligned} \xi_X &= \frac{a_1}{3}(X+T) + a_4, \quad \xi_Y = a_1 Y + a_3, \quad \tau_T = a_1 T + a_2, \\ \eta_f &= \frac{a_1}{9}(X+Y-3f) + a_5 T + a_6, \end{aligned}$$

where $a_i$, $i = 1(1)6$ are arbitrary constants. Vector fields associated with the aforementioned infinitesimal generators are given by

$$\begin{aligned} \pi_1 = &\ \frac{1}{3}(T+X)\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + T\frac{\partial}{\partial T} + \frac{1}{9}(X-3f+Y)\frac{\partial}{\partial f}, \\ \pi_2 = &\ \frac{\partial}{\partial T}, \quad \pi_3 = \frac{\partial}{\partial Y}, \quad \pi_4 = \frac{\partial}{\partial X}, \quad \pi_5 = T\frac{\partial}{\partial f}, \quad \pi_6 = \frac{\partial}{\partial f}. \end{aligned} \tag{18}$$

### 3.2.1 Vector field $\pi_2$

An invariant function $H(r,q)$ and invariant variable $r$, $q$ for Eq. (15) are produced by using Eq.(6) and $\pi_2$ in (18)

$$f(X,Y,T) = H(r,q), \quad r = X, \quad q = Y.$$

Further, the reduction of Eq. (15) is

$$3H_rH_{rq} + 3H_qH_{rr} + H_{rrrq} = 0. \tag{19}$$

Apply the Lie group of transformations method on Eq. (19) to obtain infinitesimal generators

$$\xi_r = b_1r + b_2, \quad \xi_q = P(q), \quad \eta_H = -b_1H + b_3, \quad \text{where } b_1, b_2, b_3 \text{ are constants.}$$

Assuming $P(q) = 0$, the invariant solution $H(r, q)$ is written as $H(r, q) = \frac{G(\zeta)}{r+b_2}$, with similarity variable $\zeta = q$. Using above invariant function, Eq. (19) is reduced into following ODE

$$+3G'^2(\zeta) + 6G'(\zeta)G(\zeta) - 6G'(\zeta) = 0. \tag{20}$$

We back substitute the solutions of Eq. (20) to obtain solutions of Eq. (1)

$$w(x, y, z, t) = K_3 + \frac{1 + K_1 \exp(-2y)}{x + K_2}, \tag{21}$$

$$w(x, y, z, t) = K_3 + \frac{K_1}{x + K_2}, \tag{22}$$

where $K_i$, $i = 1, 2, 3$ are constants.

#### 3.2.2 Vector field $\pi_3$

By solving Lagrange's characteristic equations for vector field $\pi_3$ in (18), one can express $f$ in terms of $H(r, q)$ as

$$f(X, Y, T) = H(r, q), \qquad r = X, \quad q = T.$$

Further, the Eq. (15) simplifies to

$$H_{qq} + H_{rq} = 0. \tag{23}$$

By back substituting the solutions of Eq. (23), we find solutions of the Eq. (1) as

$$w(x, y, z, t) = K_7 \tanh^3(K_3(t - x) - K_1) + K_5 \tanh(K_3(t - x) - K_1) + K_4, \tag{24}$$

$$\begin{aligned} w(x, y, z, t) = {} & K_7 \tanh^3(K_3(t - x) - K_1) + K_6 \tanh^2(K_3(t - x) - K_1) \\ & + K_5 \tanh(K_3(t - x) - K_1) + K_4, \end{aligned} \tag{25}$$

where $K_i$, $i = 1(1)7$, are constants.

#### 3.2.3 Vector field $\pi_4$

Lagrange's characteristic equations are calculated as follows Solution of the characteristic equations provides the function $f$ in the terms of an invariant function $H(r, q)$

$$f(X, Y, T) = H(r, q), \quad r = Y, \quad q = T. \tag{26}$$

Using the above invariant function, the reduced PDE from the Eq. (15) is

$$H_{ss} + H_{rs} = 0. \tag{27}$$

The solutions of Eq. (27) produce the following solutions of Eq. (1)

$$w(x,y,z,t) = K_7 \tanh^3(K_3(t-y) - K_1) + K_5 \tanh(K_3(t-y) - K_1) + K_4, \tag{28}$$

$$\begin{aligned} w(x,y,z,t) = {} & K_7 \tanh^3(K_3(t-y) - K_1) + K_6 \tanh^2(K_3(t-y) - K_1) \\ & + K_5 \tanh(K_3(t-y) - K_1) + K_4, \end{aligned} \tag{29}$$

where $K_i$, $i = 1(1)7$, are constants.

## 3.3 Vector field $\mathbf{X}_4$

Using characteristic equations of Lagrange (6) for vector field $\mathbf{X}_4$, the Eq. (1) is written in the form of invariant function $f(X,Y,Z)$

$$w(x,y,z,t) = f(X,Y,Z), \qquad Y = y, \quad X = x, \quad Z = z. \tag{30}$$

Plugging Eq.(30) into Eq. (1) yields the following PDE

$$3f_X f_{XY} + 3f_Y f_{XX} + f_{XXXY} - f_{ZZ} = 0. \tag{31}$$

The traveling wave solutions of (31) give the following solutions for main Eq. (1)

$$w(x,y,z,t) = 2K_2 \tanh(K_2 x + K_3 y - 2\sqrt{K_2^3 K_3}\, z + K_1) + K_5, \tag{32}$$

$$w(x,y,z,t) = 2K_2 \tanh(K_2 x + K_3 y + 2\sqrt{K_2^3 K_3}\, z + K_1) + K_5, \tag{33}$$

where $K_i$, $i = 1(1)5$, are constants. Apply the similarity transformation approach to Eq.(31) to produce the infinitesimal generators shown below

$$\begin{aligned} \xi_X &= \frac{1}{3}(2a_1 - a_3)X + a_5, \;\; \xi_Y = a_3 Y + a_4, \;\; \xi_Z = a_1 Z + a_2, \\ \eta_f &= -\frac{1}{3}(2a_1 - a_3)f + a_6 Z + a_7, \end{aligned}$$

where $a_i$, $i = 1(1)7$, are arbitrary constants. The vector fields associated to these infinitesimal generators are

$$\begin{aligned} \pi_1 = {} & \frac{2X}{3}\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Z} - \frac{2f}{3}\frac{\partial}{\partial f}, \quad \pi_2 = \frac{\partial}{\partial Z}, \\ \pi_3 = {} & -\frac{1}{3}X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + \frac{1}{3}f\frac{\partial}{\partial f}, \quad \pi_4 = \frac{\partial}{\partial Y}, \\ \pi_5 = {} & \frac{\partial}{\partial X}, \quad \pi_6 = Z\frac{\partial}{\partial f}, \quad \pi_7 = \frac{\partial}{\partial f}. \end{aligned} \tag{34}$$

#### 3.3.1 Vector field $\pi_1$

The vector field $\pi_1$ in (34) and Lagrange system is solved that lead to the following invariant functions

$$f(X,Y,Z) = \frac{H(r,q)}{Z^{\frac{2}{3}}}, \quad r = \frac{X^3}{Z^2}, \quad q = Y. \tag{35}$$

We produce the following PDE with one less independent variable using Eqs. (35) and (1)

$$\begin{aligned} - \quad & \tfrac{10}{9}H - \tfrac{26}{3}rH_r + 6H_{rq} + 27r^{\frac{4}{3}}H_rH_{rq} - 4r^2H_{rr} + 18r^{\frac{1}{3}}H_rH_q \\ & +27r^{\frac{4}{3}}H_qH_{rr} + 54rH_{rrq} + 27r^2H_{rrrq} = 0. \end{aligned} \tag{36}$$

Apply the similarity transformation approach to Eq.(36) to produce the infinitesimal generators shown below

$$\xi_r = -b_1 r, \quad \xi_s = b_1 q + b_2, \quad \eta_f = \frac{b_1}{3}H, \quad \text{where } b_1, b_2 \text{ are constants.}$$

Invariant function and invariant variable are produced by the characteristic equations of Lagrange's system for Eq. (36)

$$H(r,q) = (q+b_2)^{\frac{1}{3}}G(\zeta), \quad \zeta = r(q+b_2). \tag{37}$$

The Eq. (37) reduces Eq. (36) into a fourth-order nonlinear ODE

$$\begin{aligned} &-\tfrac{10}{9}G(\zeta) + (8 - \tfrac{26}{3}\zeta)G'(\zeta) + 6\zeta^{\frac{1}{3}}G(\zeta)G\prime(\zeta) + 54\zeta^{\frac{4}{3}}G'^2(132\zeta - 4\zeta^2)G''(\zeta) \\ &+9\zeta^{\frac{4}{3}}G(\zeta)G''(\zeta) + 54\zeta^{\frac{7}{3}}G'(\zeta)G''(\zeta) + 63\zeta^2G'''(\zeta) + 2\zeta^3G''''(\zeta) = 0. \end{aligned} \tag{38}$$

In this case, we did not find an exact solution, though one can solve it numerically using discretization method.

#### 3.3.2 Vector field $\pi_2$

Invariant functions are formed by reducing Eq. (31) to its similarity

$$f(X,Y,Z) = H(r,q), \quad r = X, \quad q = Y.$$

The equation (31), with the help of above invariant variables, is reduced in a PDE with two independent variables

$$3H_rH_{rq} + 3H_qH_{rr} + H_{rrrq} = 0. \tag{39}$$

Eq. (39), by applying Lie group analysis, yields the infinitesimal generators

$$\xi_r = b_1 r + b_2, \quad \xi_q = P(q), \quad \eta_H \quad = -b_1 H + b_3, \quad \text{where } b_1, b_2, b_3 \text{ are constants.}$$

Assuming $P(q) = 0$, the function $H(r,q)$ is written in the form of invariant function $G(\zeta)$ $H(r,s) = \frac{G(\zeta)}{r+b_2}$, where $\zeta = s$. Upon simplifying the above invariant function, the reduction of Eq. (39) into a first-order ODE is

$$6G(\zeta)G'(\zeta) - 6G'(\zeta) + 3G'^2(\zeta) = 0 \tag{40}$$

To find solutions of the Eq. (1), we back substitute the solutions of Eq. (40)

$$w(x, y, z, t) = K_3 + \frac{1 + K_1 \exp(-2y)}{x + K_2}, \tag{41}$$

$$w(x, y, z, t) = K_3 + \frac{K_1}{x + K_2}, \tag{42}$$

where $K_i$, $i = 1, 2, 3$ are constants.

#### 3.3.3 Vector field $\pi_3$

The Eq. (31) is converted into the following invariant form using Lagrange Characteristic equations for vector field $\pi_3$ $f(X, Y, Z) = Y^{\frac{1}{3}} H(r, q), \quad r = X^3 Y, \quad q = Z$. Using the above invariant functions, the reduction of Eq. (31) is

$$\begin{aligned} &-H_{qq} + 8H_r + 132rH_{rr} + 54r^{\frac{4}{3}} H_r^2 + 54r^{\frac{7}{3}} H_r H_{rr} \\ &+6r^{\frac{1}{3}} H H_r + 9r^{\frac{4}{3}} H H_{rr} + 144r^2 H_{rrr} + 27r^3 H_{rrrr} = 0. \end{aligned} \tag{43}$$

When applied to Eq. (43), the Lie group analysis method produces the infinitesimal generators $\xi_r = 2b_1 r, \quad \xi_q = b_1 q + b_2, \quad \eta_H = -\frac{2}{3} b_1 H$, where $b_1, b_2$ are constants. The characteristic equation (6) for these infinitesimal generators reduces Eq. (43) into the similarity form $\zeta = \frac{r}{\overset{*}{q + b_2}}, \quad H(r, q) = \frac{G(\zeta)}{r^{\frac{1}{3}}}$. The above equation reduces Eq. (43) into a fourth-order nonlinear ODE

$$\begin{aligned} &-\tfrac{10}{9} G(\zeta) - 26\zeta G'(\zeta) + 8G\prime(\zeta) + 6\zeta^{\frac{1}{3}} G' G(\zeta) + 54\zeta^{\frac{4}{3}} G'(\zeta) \\ &-4\zeta^2 G''(\zeta) G(\zeta) + 132\zeta G''(\zeta) + 9\zeta^{\frac{4}{3}} G(\zeta) G''(\zeta) \\ &+54\zeta^{\frac{7}{3}} G''(\zeta) G(\zeta) + 144\zeta^2 G'''(\zeta) + 27\zeta^3 G''''(\zeta) = 0. \end{aligned} \tag{44}$$

In this case, we did not find an exact solution, though it is easy to solve them by available discretization techniques.

#### 3.3.4 Vector field $\pi_4$

The vector field $\pi_4$ in (34) provides similarity reduction of Eq. (31) as $f(X, Y, Z) = H(r, q)$, where $r = X, \ \ q = Z$. By using above expression, the reduction of Eq. (31) is

$$H_{ss} = 0. \tag{45}$$

Hence, Eq. (1) has following solution for this case

$$w(x, y, z, t) = \alpha(x) z + \beta(x). \tag{46}$$

#### 3.3.5 Vector field $\pi_5$

The vector field $\pi_5$ in (34) provides similarity reduction of the Eq. (31) as $f(X, Y, Z) = H(r, q), r = Y, \ \ q = Z$. Equation (31) can be simplified as follows by using above invariant function

$$H_{qq} = 0. \tag{47}$$

Hence, we obtain the following solution for Eq. (1).

$$w(x,y,z,t) = \alpha(y)z + \beta(y) \tag{48}$$

### 3.4 Vector field $\mathbf{X}_5$

On solving Lagrange characteristic equation for vector field $\mathbf{X}_5$, invariant function $f(X,Z,T)$ formed by reducing Eq. (1)

$$w(x,y,z,t) = f(X,Z,T), \quad X = x, \quad T = t, \quad Z = z. \tag{49}$$

By putting Eq.(49) into Eq. (1), the following PDE is generated

$$f_{XT} + f_{TT} - f_{ZZ} = 0. \tag{50}$$

By solving Eq. (50), we back substitute its solutions to obtain the following traveling wave type solutions

$$\begin{aligned} w(x,y,z,t) &= K_8 \tanh^3\left(K_3 t - \frac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) \\ &+ K_6 \tanh\left(K_3 t - \frac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_5, \qquad (51) \\ w(x,y,z,t) &= K_8 \tanh^3\left(K_3 t - \frac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) \\ &+ K_7 \tanh^2\left(K_3 t - \frac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) \\ &+ K_6 \tanh\left(K_3 t - \frac{K_3^2 - K_4^2}{K_3}x + K_4 z + K_1\right) + K_4, \qquad (52) \end{aligned}$$

where $K_i$, $i = 1(1)8$, are constants.

### 3.5 Vector field $\mathbf{X}_6$

Using vector field $\mathbf{X}_6$ and Lagrange characteristic equations, the Eq. (1) is converted to the following invariant form when new similarity variables are introduced $w(x,y,z,t) = f(Y,Z,T), \quad Y = y, \quad Z = z, \quad T = t$. From above expression and Eq. (1), the following PDE is obtained

$$f_{YT} + f_{TT} - f_{ZZ} = 0. \tag{53}$$

This equation (53) is similar to Eq. (50) so it will have similar kind of traveling wave solutions.

## 4 Numerical Algorithm

To obtain the approximate solution values, the space, and time partial derivatives are replaced by equivalent finite difference discretization formula. The region $\Omega = \big\{(x,y,z,t) : a \leq$

$x \leq b,\ c \leq y \leq d,\ e \leq z \leq f,\ 0 \leq t \leq T\}$ is covered with the spatial grid points $x_l = a+lh_x, y_m = c+mh_y, z_n = e+nh_z$, and temporal grid points $t_k = kh_t$, with the step-size $h_x = (b-a)/(L+1), h_y = (d-c)/(M+1), h_z = (f-e)/(N+1), h_t = T/K$, where $l = 0(1)(L+1), m = 0(1)(M+1), n = 0(1)(N+1), k = 0(1)K$ and $L, M, N, K \in Z_+$. The solution of $w(x, y, z, t)$ of the gKPB Eq. (1) at the grid point $(x_l, y_m, z_n, t_p)$ is denoted by $w^k_{l,m,n}$. Let

$$w_x = v, \tag{54}$$

then Eq. (1) can be written as

$$v_{xxy} + 3vv_y + 3w_yv_x + w_{tt} + v_t + w_{yt} - w_{zz} = 0. \tag{55}$$

Replacing the spatial partial derivatives in Eqs. (54) and (55) by the central difference formula and time derivative by forward/backward difference formula, we arrive at the following algorithm for estimating approximate solution values.

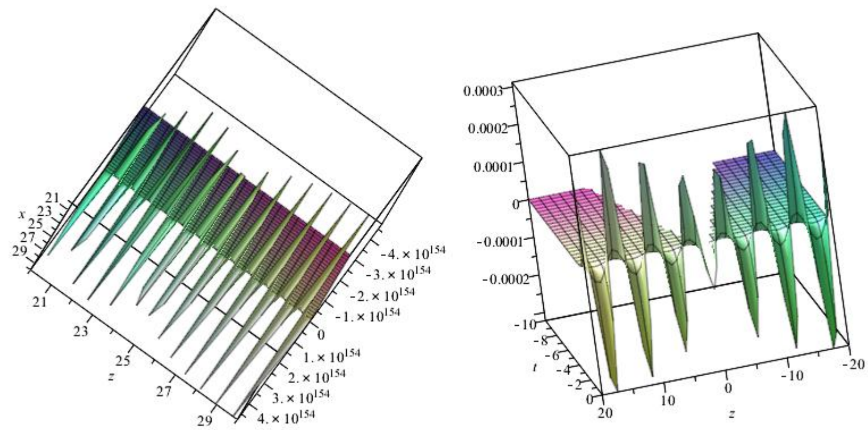
**Algorithm:**

$$\begin{aligned}
[v]_{(x_l,y_m,z_n,t_k)} = v^k_{l,m,n} &:= \frac{(w^k_{l+1,m,n} - w^k_{l-1,m,n})}{2h_x},\\
[v_{xx}]_{(x_l,y_m,z_n,t_k)} &:= \frac{(v^k_{l+1,m,n} - 2v^k_{l,m,n} + v^k_{l-1,m,n})}{h_x^2},\\
[v_{xxy}]_{(x_l,y_m,z_n,t_k)} &:= \frac{\beta_1}{2h_x^2 h_y},\\
[w_y]_{(x_l,y_m,z_n,t_k)} &:= \frac{(w^k_{l,m+1,n} - w^k_{l,m-1,n})}{2h_y},\\
[v_x]_{(x_l,y_m,z_n,t_k)} &:= [w_{xx}]_{(x_l,y_m,z_n,t_k)} = \frac{(w^k_{l+1,m,n} - 2w^k_{l,m,n} + w^k_{l-1,m,n})}{h_x^2},\\
[v_t]_{(x_l,y_m,z_n,t_k)} &:= [w_{xt}]_{(x_l,y_m,z_n,t_k)} := \frac{\beta_2}{2h_x h_t},\\
[w_{yt}]_{(x_l,y_m,z_n,t_k)} &:= \frac{(w^{k+1}_{l,m+1,n} - w^k_{l,m+1,n} - w^{k+1}_{l,m-1,n} + w^k_{l,m-1,n})}{2h_y h_t},\\
[w_{zz}]_{(x_l,y_m,z_n,t_k)} &:= \frac{(w^k_{l,m,n+1} - 2w^k_{l,m,n} + w^k_{l,m,n-1})}{h_z^2},\\
[w_{tt}]_{(x_l,y_m,z_n,t_k)} &:= \frac{(w^{k+1}_{l,m,n} - 2w^k_{l,m,n} - w^{k-1}_{l,m,n})}{h_t^2},
\end{aligned}$$

where $\beta_1 = \big(v^k_{l+1,m+1,n} - v^k_{l+1,m-1,n} - 2(v^k_{l,m+1,n} - v^k_{l,m-1,n}) + v^k_{l-1,m+1,n} - v^k_{l-1,m-1,n}\big)$, $\beta_2 = (w^{k+1}_{l+1,m,n} - w^p_{l+1,m,n} - w^{k+1}_{l-1,m,n} + w^k_{l-1,m,n})$. Substituting the above approximations in Eqs. (54) and (55), the resulting three-level scheme can be solved easily by incorporating appropriate initial and boundary data. An explicit compact discretization of the Kadomtsev-Petviashvili equation in two dimensions is obtained in [20], and the present algorithmic procedure can be similarly interpreted for consistency and stability analysis. The system of nonlinear discrete equations obtained after finite-difference replacement is solved by the Newton-Raphson method with a suitable initial guess, see [21].

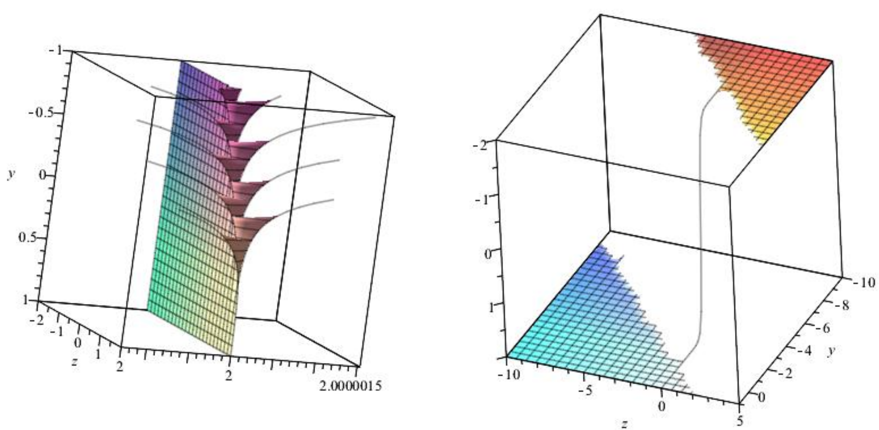
# 5 Graphical Interpretation and Discussion

In this section, a graphical interpretation for the different types of soliton solutions of Eq. (1) is presented.



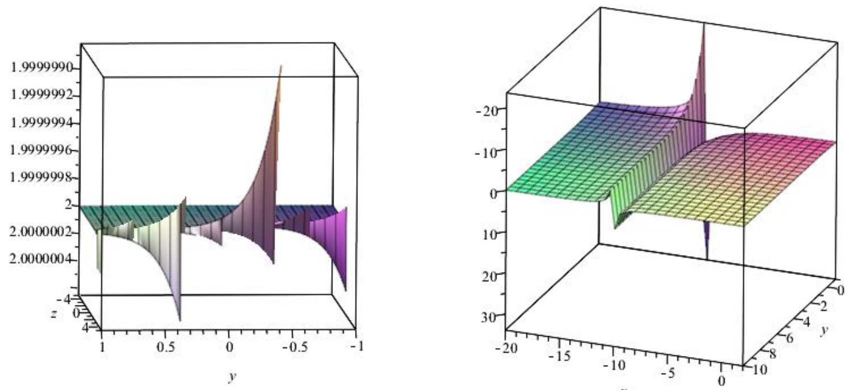

(a) (b)

Figure 1: (A) In Eq. (12), $t = 10.358,\ K_1 = 50,\ K_2 = 2,\ K_3 = 3,\ K_4 = 4$ and $x, z \in [20, 30]$, (B) In Eq. (14), $x = 1.024,\ K_1 = 1.0950,\ K_2 = 2,\ K_3 = 3,\ K_4 = 4$ and $z \in [-20, 20], t \in [-10, 0]$.



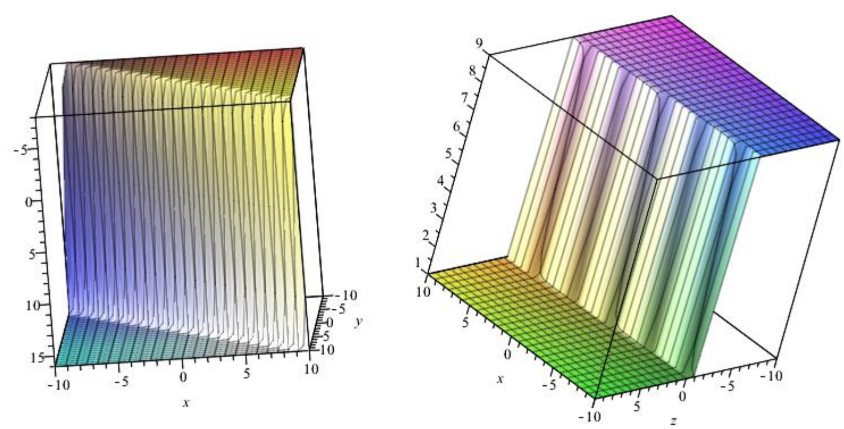

(a) (b)

Figure 2: (A) In Eq. (16), $x = 1.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $z \in [-2, 2],\ y \in [-1, 1]$, (B) In Eq. (16), $x = 0.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $y \in [-10, 1],\ z \in [-10, 5]$.



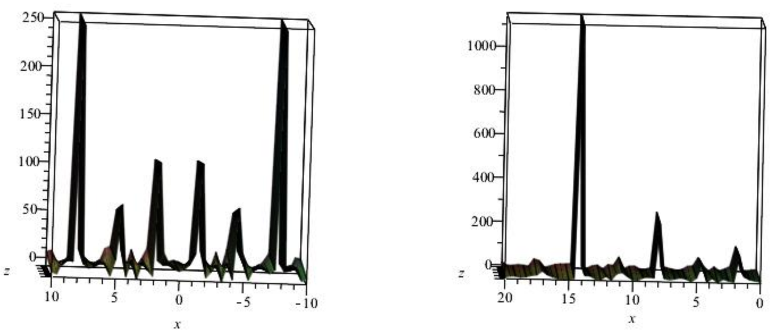

(a) (b)

Figure 3: (A) In Eq. (16), $x = 1.025,\ K_1 = 11,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $y \in [-1, 1],\ z \in [-5, 5]$, (B) In Eq. (21), $K_1 = 1.98,\ K_2 = 10.105,\ K_3 = 1.508$ and $x \in [-20, 2],\ y \in [-1, 10]$.



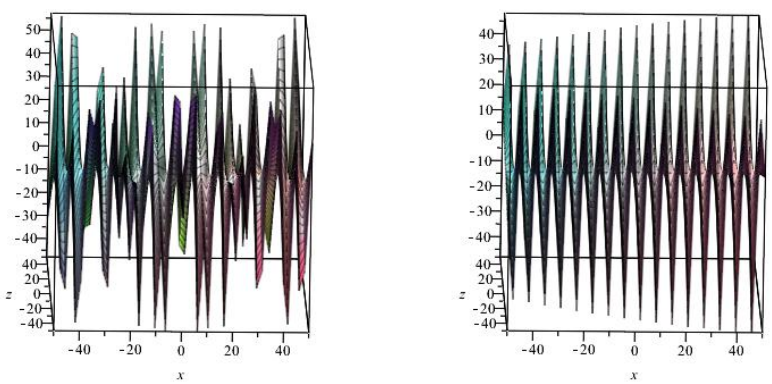

(a) (b)

Figure 4: (A) In Eq. (24), $K_1 = 1, K_3 = 3, K_4 = 4, K_5 = 5, K_7 = 7$ and $x, y \in [-10, 10]$, (B) In Eq. (28), $y = 0.025,\ K_1 = 1,\ K_2 = 2,\ K_3 = 3,\ K_5 = 1$ and $x, z \in [-10, 10]$.




(a) (b)

Figure 5: (A) In Eq. (46), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-10, 10]$, (B) In Eq. (46), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-20, 20]$.




(a) (b)

Figure 6: (A) In Eq. (46), $\alpha(x) = \sin(x^2),\ \beta(x) = \sec^2(x)$ and $x, z \in [-50, 50]$, (B) In Eq. (46), $\alpha(x) = \sin(x),\ \beta(x) = \mathrm{sech}(x^2)$ and $x, z \in [-50, 50]$.

## 5.1 Discussion:

Fig.(1)(A),(B), and Fig. (2) (A), (B) exhibit a standing wave with multiple breathers. The undulating array of breathers with opposite phases can be regarded as a standing wave. The initial oscillatory dynamics of multi-breathers with undulating amplitude are suggestive of the most unstable interaction among solitons. Gradually, the wave loses coherence and the chaotic regime prevails. Note that such multi-breather interactions are not perfectly elastic. Fig.(3)(A) shows a topological defect with a one-dimensional kink. The presence of at least two discontinuous fibro-fatty plaques may not be entirely discrete from each other because a one-dimensional kink connects these fibro-fatty plaques with distinct spatial localization. In higher dimensions, multiple topological defects can be connected by higher-dimensional kinks. It means that intra-arterial micro- and macro-transport is done through kink solitons. Note that the solitary wave for intra-vascular transport is kink soliton. Fig. (3)(B) shows how the standing wave interacts with bright and dark solitons. The phase transition interacts with an array of multiple breathers, which appears as a standing wave, due to the beaming correspondence among solitons. As a result, nonlinear wave *i.e.* partially standing and partially traveling waves with changing amplitudes, propagates. Such solutions are unstable due to oscillatory instabilities. Fig.(4)(A),(B) exhibit a traveling wave solution that characterizes the flows in the forward direction. Fig.(5)(A),(B) exhibit three bright solitons with progressively decreasing solitary wave amplitude, whereas Fig.(I) has symmetry in three bright solitons at $x = 0$. Cumulative accretion of potential energy contributes to the creation of bright soliton. The temporospatial localization of its energy and narrowing of time duration contributes to an increment in wave speed. As the wave propagates further, its speed is progressively retarded because bright soliton vanishes after the peak. In this case, the slower wave comes in contact with another soliton of almost the same height; again, a bright soliton emerges. Fig.(6)(A),(B) depict multiple breathers. While discussing Fig.(2)(A), we have seen that multiple breathers with varying amplitude are associated with more unstable interactions than other solitons. Fig.(6)(A) exhibits quasi-periodic oscillatory dynamics associated with increasing wave incoherence and spatial inhomogeneity. Fig. (6) (B) exhibits periodic oscillatory dynamics with progressively growing amplitude.

# 6 Conclusions

The (3+1)-dimensional gKPB equation widely appears to model physical phenomena in fluid dynamics. In fact, the gKPB equation provides more accurate approximations of the dynamics of water under a fewer number of constraints than the KP equation. We have investigated various exact solutions of the (3+1)-dimensional gKPB equation using the Lie group of transformations method in the present work. The generators of infinitesimal transformations are obtained by utilizing Lie symmetry group analysis. These generators rely on a number of parameters, and from these generators, we obtained a set of Lie algebras. Finally, using the property of invariance of the Lie group of transformations, we obtained various exact solutions for the gKPB equation in (3+1)-dimensions. The solutions we have derived are represented by the equations (12), (14), (16), (21), (24), (28), (33), (41), (46) etc. A geometrical profile of these solutions in 3-dimensional plots ( by giving appropriate values to arbitrary constants) with an analytical discussion is provided. Traveling wave solutions, bright and dark soliton solutions, kink, and standing waves with multiple breather profiles of solutions are presented. Our results show that the symmetry method is relevant to solve nonlinear evolution equations associated with modeling nonlinear phenomena analytically.

# References


[1] C. Rogers and W. F. Shadwick, " Bäcklund Transformations and Their Applications", Mathematics in Science and Engineering, 161, Academic Press, Inc., New York, 1982.

[2] N. Zhang and T. Xia, A hierarchy of lattice soliton equations associated with a new discrete eigenvalue problem and Darboux transformations, Int. J. Nonlinear Sci. Numer. Simul., 16(2015)(7-8), 301 - 306.

[3] L. Wang et al., Breather-to-soliton transitions, nonlinear wave interactions, and modulational instability in a higher-order generalized nonlinear Schrödinger equation, Phys. Rev. E, 93(2016)(1), 012214, 12 pp.

[4] L.L. Feng and T.T. Zhang, Breather wave, rogue wave and solitary wave solutions of a coupled nonlinear Schrödinger equation, Appl. Math. Lett., 78(2018), 133 - 140.

[5] M. Wang, X. Li and J. Zhang, The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A, 372(2008)(4), 417 - 423.

[6] G. W. Bluman and S. Kumei, " Symmetries and Differential Equations", Applied Mathematical Sciences, 81, Springer-Verlag, New York, 1989.

[7] P. J. Olver, " Applications of Lie Groups to Differential Equations", Second Edition, Graduate Texts in Mathematics, 107, Springer-Verlag, New York, 1993.

[8] V. Jadaun and N. R. Singh, Soliton solutions of generalized $(3 + 1)$-dimensional Yu-Toda-Sasa-Fukuyama equation using Lie symmetry analysis, Anal. Math. Phys., 10(2020)(4), Paper No. 42, 24 pp.

[9] V. Jadaun and S. Kumar, Symmetry analysis and invariant solutions of (3+1)-dimensional Kadomtsev-Petviashvili equation, Int. J. Geom. Method Mod. Phys., 15(2018)(8), 1850125, 19 pp.

[10] V. Jadaun, Pulsatile blood flow in healthy aorta: An application of nonlinear evolution equation, Int. J. Biomath. **15** (2022), no. 7, Paper No. 2250047, 30 pp.

[11] R. Hirota, " The Direct Method in Soliton Theory", Cambridge University Press, Cambridge, 2004.

[12] V. B. Matveev and M. A. Salle, " Darboux Transformation and Solitons", Springer, Berlin, 1991.

[13] M. J. Ablowitz and P. A. Clarkson, " Solitons, Nonlinear Evolution Equations and Inverse Scattering", London Mathematical Society Lecture Note Series, 149, Cambridge University Press, Cambridge, 1991.

[14] A.M. Wazwaz and S. A. El-Tantawy, A new $(3 + 1)$-dimensional generalized Kadomtsev-Petviashvili equation, Nonlinear Dynam., 84(2016)(2), 1107 - 1112.

[15] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl., 15(1970), 539 – 541.

[16] J.P. Yu and Y.-L. Sun, A direct Bäcklund transformation for a $(3 + 1)$-dimensional Kadomtsev-Petviashvili-Boussinesq-like equation, Nonlinear Dynam., 90(2017)(4), 2263 - 2268.

[17] Lakhveer Kaur and A.M. Wazwaz, Dynamical analysis of lump solutions for (3+1) dimensional generalized KP–Boussinesq equation and Its dimensionally reduced equations, Physica Scripta, 93(2018)(7), 075203.

[18] J. Lü and S. Bilige, Diversity of interaction solutions to the $(3 + 1)$-dimensional Kadomtsev-Petviashvili-Boussinesq-like equation, Modern Phys. Lett. B, 32(2018)(26), 1850311, 13 pp.

[19] L. Wang, Y Zhou, Q Liu, and Q Zhang, Traveling waves of the $(3 + 1)$-dimensional Kadomtsev-Petviashvili-Boussinesq equation, Journal of Applied Analysis and Computation, 10(2020)(1), 267 - 281.

[20] A. G. Bratsos and E. H. Twizell, An explicit finite-difference scheme for the solution of the Kadomtsev-Petviashvili equation, International Journal of Computer Mathematics, 68(1998), 175 - 187.

[21] N. Jha P. Lin, Digital simulations for three-dimensional nonlinear advection-diffusion equations using quasi-variable meshes high-resolution implicit compact scheme, Research Reports on Computer Science, 1(2022), 85 - 110.