

Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). The contradiction we derive is based on two properties of a specific set which we use to reformulate a strengthened form of the strong Goldbach conjecture. We show that the conjecture and its negation, if we express them as assumptions using that set, on the one hand make no difference, but on the other hand they do.

Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_n denote the natural numbers starting from $n > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g . So, by the definition of S_g we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg \text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of \mathbb{N}_3 can be expressed by the triple components of S_g ("covering"), because every integer $x \geq 3$ can be written as some pk with $k = 1$ when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3, k \in \mathbb{N}$. So we have

$$(C) \quad \forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \quad \vee \quad x = mk = 4k.$$

A few examples of the covering:

$x = 19$: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

$x = 36$: (**3·12**, 7·12, 11·12)

$x = 38$: (**19·2**, 21·2, 23·2)

$x = 42$: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

$x = 64$: (3·16, **4·16**, 5·16)

$x = 10000$: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g (“maximality”). So we have

(M) $\forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g$, where $m = (p + q) / 2$.

The proof is motivated by the following view.

There are two possibilities for S_g , exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to \neg SSGB.

Since, due to (C), every n given by \neg SSGB as well as every multiple $nk, k \in \mathbb{N}$, equals a component of some S_g triple that exists by definition, the covering of \mathbb{N}_3 by the S_g triples if n exists (\neg SSGB) is equal to that if n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case \neg SSGB they don't.

First of all, we note that each of the two properties (C) and (M) is a condition sine qua non for the proof, for the following reasons.

\neg (C) immediately implies \neg SSGB, since an $n \geq 4$ different from all S_g triple components pk, mk, qk is in particular different from all m in S_g .

The proof would no longer be possible if, for example, we omitted the factor k in the definition of S_g , because then the corresponding (C) could no longer be guaranteed.

Similarly, the property (M) rules out the possibility that there is an $n \geq 4$ different from all m (i.e. $\neg\text{SSGB}$) and n is the arithmetic mean of a pair of primes not used in S_g . Thus (M) excludes the possibility that $\neg\text{SSGB}$ applies due to a missing prime number pair. This means that the proof would no longer be possible here either if we left out any prime number pair in the formulation of SSGB and S_g .

We will now show that $((C) \wedge (M))$ leads to a contradiction.

The following proof is independent of the choice of n if there is more than one in the case of $\neg\text{SSGB}$. For example, the minimal such n works.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_{g+}(y) \cup S_{g-}(y)$, with

$S_{g+}(y) := \{ (pk, mk, qk) \in S_g \mid \exists k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$ and

$S_{g-}(y) := \{ (pk, mk, qk) \in S_g \mid \forall k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by $\neg\text{SSGB}$ as described above. Then, we have

(*) $\neg\text{SSGB} \Rightarrow S_g = S_{g+}(n) \cup S_{g-}(n)$.

More precisely, under the assumption $\neg\text{SSGB}$ with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form $(pk = nk', mk, qk)$ with $k = k'$ in case n is prime, due to (C)

(ii) S_g triples of the form $(pk = nk', mk, qk)$ with $k \neq k'$ in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form $(3k, 4k = nk', 5k)$ in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form $(pk = nk', mk, qk)$, $(pk, mk = nk', qk)$ or $(pk, mk, qk = nk')$

and

(v) S_g triples of the form $(pk \neq nk', mk \neq nk', qk \neq nk')$, i.e. those S_g triples where none of the nk' equals a component.

So, $S_{g+}(n)$ is the union of the triples of the above types (i) to (iv) and $S_{g-}(n)$ is the union of the triples of type (v).

Now, we define

$$S_1 := \{ (pk, mk, qk) \in S_g \mid \neg \text{SSGB holds} \}$$

$$S_2 := \{ (pk, mk, qk) \in S_g \mid \text{SSGB holds} \}.$$

So, by (*) we obtain

$$(1) \neg \text{SSGB} \Rightarrow S_1 = S_g = S_{g^+}(n) \cup S_{g^-}(n).$$

Since $S_{g^+}(n) \cup S_{g^-}(n)$ is independent of n , we can write

$$(1') \forall y \in \mathbb{N}_3 \quad \neg \text{SSGB} \Rightarrow S_1 = S_g = S_{g^+}(y) \cup S_{g^-}(y).$$

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_{g^+}(y) \cup S_{g^-}(y)$. So, we obtain

$$(2) \forall y \in \mathbb{N}_3 \quad \text{SSGB} \Rightarrow S_2 = S_g = S_{g^+}(y) \cup S_{g^-}(y).$$

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x -tuples are equal, then the sets of their corresponding i -th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$$M(k) := \{ mk \mid (pk, mk, qk) \in S_g \}$$

$$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$$

$$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}.$$

Then, applying the principle above to the middle component of the triples (pk, mk, qk) , (1') and (2) imply

$$(3) \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3$$

$$(\neg \text{SSGB} \Rightarrow M_1(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \})$$

\wedge

$$(4) \forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3$$

$$(\text{SSGB} \Rightarrow M_2(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_{g^+}(y) \cup S_{g^-}(y) \}).$$

We set $M := M(1)$, $M_1 := M_1(1)$ and $M_2 := M_2(1)$. Since

$$\neg \text{SSGB} \Rightarrow M_1 = M \quad \text{and} \quad \text{SSGB} \Rightarrow M_1 = \{ \} \neq M$$

and

$$\text{SSGB} \Rightarrow M_2 = M \quad \text{and} \quad \neg \text{SSGB} \Rightarrow M_2 = \{ \} \neq M,$$

the implications in (3) and (4) are in fact equivalences.

So we get

$$(3') \forall y \in \mathbb{N}_3 \quad (\neg \text{SSGB} \Leftrightarrow M_1 = M = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \})$$

\wedge

$$(4') \forall y \in \mathbb{N}_3 \quad (\text{SSGB} \Leftrightarrow M_2 = M = \{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}).$$

Since for every $y \in \mathbb{N}_3$ $S_{g^+}(y) \cup S_{g^-}(y)$ equals S_g , there is a set X such that for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_{g^+}(y) \cup S_{g^-}(y) \}$ equals X . So, from (3') and (4') we obtain

$$(5) (\neg \text{SSGB} \Leftrightarrow M_1 = M = X) \quad \wedge \quad (\text{SSGB} \Leftrightarrow M_2 = M = X).$$

The set X is a free variable in (5) that is either equal to \mathbb{N}_4 or to some non-empty proper subset Y of \mathbb{N}_4 . Therefore, (5) splits as follows.

$$(5.1) \quad ((\neg \text{SSGB} \Leftrightarrow M_1 = M = \mathbb{N}_4) \quad \wedge \quad (\text{SSGB} \Leftrightarrow M_2 = M = \mathbb{N}_4))$$

\vee

$$(5.2) \quad ((\neg \text{SSGB} \Leftrightarrow M_1 = M = Y \neq \mathbb{N}_4) \quad \wedge \quad (\text{SSGB} \Leftrightarrow M_2 = M = Y \neq \mathbb{N}_4)).$$

On the other hand, under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg \text{SSGB}$ they don't. Therefore, we have

$$(6.1) \quad \text{SSGB} \Leftrightarrow M_2 = M = \mathbb{N}_4$$

$$(6.2) \quad \neg \text{SSGB} \Leftrightarrow M_1 = M \neq \mathbb{N}_4.$$

The second conjunct in (5.1) is identical to (6.1) which is a tautology. So we can omit that second conjunct. And since (6.2) is another tautology, we can add it to (5.1). So, (5.1) implies

$$(7.1) \quad ((\neg \text{SSGB} \Leftrightarrow M_1 = M = \mathbb{N}_4) \quad \wedge \quad (\neg \text{SSGB} \Leftrightarrow M_1 = M \neq \mathbb{N}_4)).$$

Similarly, (5.2) implies

$$(7.2) \quad ((\text{SSGB} \Leftrightarrow M_2 = M = \mathbb{N}_4) \quad \wedge \quad (\text{SSGB} \Leftrightarrow M_2 = M \neq \mathbb{N}_4)).$$

Now we make use of this rule: Let P, Q, P', Q' be propositions. Suppose we have $(P \vee Q)$, $(P \Rightarrow P')$ and $(Q \Rightarrow Q')$. Then, we can conclude that $(P' \vee Q')$.

We apply the rule with

$P := (5.1)$

$Q := (5.2)$

$P' := (7.1)$

$Q' := (7.2)$

and obtain

$$(7.1) \ ((\neg \text{SSGB} \Leftrightarrow M_1 = M = \mathbb{N}_4) \ \wedge \ (\neg \text{SSGB} \Leftrightarrow M_1 = M \neq \mathbb{N}_4))$$

(7) \vee

$$(7.2) \ ((\text{SSGB} \Leftrightarrow M_2 = M = \mathbb{N}_4) \ \wedge \ (\text{SSGB} \Leftrightarrow M_2 = M \neq \mathbb{N}_4)).$$

By transitivity of equivalence, from (7.1) we get that $M_1 = M = \mathbb{N}_4$ is equivalent to $M_1 = M \neq \mathbb{N}_4$, which is false. Similarly, from (7.2) we get that $M_2 = M = \mathbb{N}_4$ is equivalent to $M_2 = M \neq \mathbb{N}_4$, which is false too.

Therefore, we obtain

(7.1) \Rightarrow FALSE

and

(7.2) \Rightarrow FALSE.

So, (7) implies (FALSE \vee FALSE), which is equivalent to FALSE.

□

Remark. We can generalize the above argument which shows that (7) is false in such a way that an inconsistency in the set theory ZFC can be proved.

Let P be a proposition and let A and B be non-empty sets. We define

$A_1 := \{ a \in A \mid P \text{ holds} \}$ and $A_2 := \{ a \in A \mid \neg P \text{ holds} \}$.

We consider the statement

$$(F1) \quad ((P \Rightarrow A_1 = A = B) \wedge (P \Rightarrow A_1 = A \neq B))$$

(F) \vee

$$(F2) \quad ((\neg P \Rightarrow A_2 = A = B) \wedge (\neg P \Rightarrow A_2 = A \neq B)).$$

Then, the contradiction $((F) \wedge \neg(F))$ can be derived.

Proof. Since either P or $\neg P$ is false, either (F1) or (F2) is trivially true. So, (F) is true.

On the other hand, since

$$P \Rightarrow A_1 = A \text{ and } \neg P \Rightarrow A_1 = \{ \} \neq A$$

and

$$\neg P \Rightarrow A_2 = A \text{ and } P \Rightarrow A_2 = \{ \} \neq A,$$

the implications in (F1) and (F2) are in fact equivalences.

Then, by transitivity of equivalence, from (F1) we get that $A_1 = A = B$ is equivalent to $A_1 = A \neq B$, which is false. Similarly, from (F2) we get that $A_2 = A = B$ is equivalent to $A_2 = A \neq B$, which is false too.

Therefore, we obtain

$$(F1) \Rightarrow \text{FALSE}$$

and

$(F2) \Rightarrow \text{FALSE}$.

So, (F) implies $(\text{FALSE} \vee \text{FALSE})$, which is equivalent to FALSE . And so, (F) is false.

□