Commun Nonlinear Sci Numer Simulat 102 (2021) 105938

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Optimal control and approximation for elliptic bilateral obstacle problems[☆]

Jinjie Liu[a,1], Xinmin Yang[b,1], Shengda Zeng[c,1,*]

[a] School of Mathematics Sciences, Shanghai Jiao Tong University, Shanghai 200241, China
[b] School of Mathematics Science, Chongqing Normal University, Chongqing 401331, China
[c] Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, PR China

ARTICLE INFO

Article history:
Available online 25 June 2021

MSC:
47J20
49J53
58E35
35J66
46Txx

Keywords:
Bilateral obstacle problem
Optimal control
Clarke's generalized gradient
Existence
Nonhomogeneity
Regularization
Convergence

ABSTRACT

The aim of this paper is to study an elliptic bilateral obstacle system (EBOS, for short) involving a nonlinear and nonhomogeneous partial differential operator and a multivalued term which is described by Clarke's generalized gradient. First, we obtain the weak formulation of (EBOS) which is a variational-hemivariational inequality, and prove the unique solvability of the bilateral obstacle problem. Second, we consider a nonlinear optimal control problem governed by (EBOS) in which the control variable is the bilateral obstacle, and establish the existence of an optimal solution to the obstacle control problem under mild conditions. Then, employing the regularization technique and penalty approach, we introduce a family of approximating problems corresponding to the optimal control problem under consideration. Finally, a convergence result that any sequence of solutions for the approximating problems converges to an optimal solution of the original optimal control problem is delivered.

1. Introduction

In numerous complicated natural phenomena, chemical processes, constitutive law, engineering applications, and mathematical models often are formulated by inequalities instead of the more commonly seen equations. In this context, a class of inequality problems has been widely studied, namely, variational inequalities which emerge from applied models with an underlying convex structure and have been studied extensively since the early sixties. Some representative references include [17,21–25,45] on mathematical theories, and [11,20] on numerical treatment. On the other hand, optimal control of partial differential equations (PDEs) and variational inequalities is an expanding and vibrant branch of applied mathematics and modern control theory that has found numerous applications. Although the theory and computational techniques for optimal control for PDEs and variational inequalities have been studied for quite some time now (see e.g. [3,7,8,19,35]), it seems that there are still many unanswered questions and many interesting ideas are still in the making.

☆ Dedicated to Professor Stanislaw Migórski on the occasion of his 60th birthday.

* Corresponding author.
E-mail addresses: 332070262@qq.com (J. Liu), xmyang@cqnu.edu.cn (X. Yang), zengshengda@163.com (S. Zeng).

[1] The authors contributed equally in the paper.

https://doi.org/10.1016/j.cnsns.2021.105938

Nonetheless, in recent years, many important applied models have motivated the study of the optimal control problems described by hemivariational inequalities, for instance, mechanics, engineering and physics problems [27,31,36,42,43]. As an important generalization of variational inequalities, the theory of hemivariational inequalities was initially introduced by P.D. Panagiotopoulos [32–34] to study the nonsmooth mechanical problems in which the main idea behind hemivariational inequalities is to remove the hypotheses on differentiability and convexity of energy functionals with the help of the generalized subgradient of Clarke. This theory provides us a useful and impressive mathematical apparatus for studying a diversity of problems arising in various fields such as structural mechanics, elasticity, economics, optimization, financial mathematics, and others [12–16,18,26,28,30,37,38,40,41]. However, despite the fact that there are many important models leading to hemivariational inequalities, the corresponding optimal control problem is quite poor and the need for results in this topic is currently widely recognized. So, to contribute to filling this gap, the aim of the current paper is to investigate an optimal control problem associated with a complicated partial differential system with mixed boundary conditions involving a nonlinear and nonhomogeneous partial differential operator, and a multivalued term which is described by Clarke's generalize gradient. Additionally, because our problem involves the nonsmooth and non-convex feature, so, naturally, the question arises whether we can apply appropriate regularity and smooth methods to approximate the original optimal control problem. The second goal of this paper is to give positive answer for the question.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^{1,1}$–boundary $\partial\Omega$ such that $\partial\Omega$ is divided into three measurable and disjoint parts Γ_1, Γ_2, and Γ_3 with $meas(\Gamma_1) > 0$. Also, let $z_d \in L^p(\Omega)$ be a given target profile and $\alpha, \beta > 0$ be regularized coefficients. More precisely, in the present paper, we consider the bilateral obstacle optimal control problem as follows.

Problem 1. Find $(\phi^*, \Phi^*) \in U_{ad}$ such that

$$(\phi^*, \Phi^*) \in \arg \min_{(\phi,\Phi)\in U_{ad}} L(\phi, \Phi), \text{ namely, } L(\phi^*, \Phi^*) \le L(\phi, \Phi) \text{ for all } (\phi, \Phi) \in U_{ad}, \tag{1.1}$$

where the cost function $L : W \times W \to \mathbb{R}$ is defined by

$$L(\phi, \Phi) := \frac{1}{p}\int_\Omega |\mathcal{S}(\phi, \Phi)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi\|^p_{L^p(\Omega)}, \tag{1.2}$$

$U_{ad} \subset W \times W$ is the set of admissible obstacle (see (3.2), below).

Here $\mathcal{S}(\phi, \Phi)$ is the unique weak solution to the following elliptic inclusion problem corresponding to the bilateral obstacle (ϕ, Φ).

Problem 2. Given $(\phi, \Phi) \in U_{ad}$, find a function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-\mu^* \operatorname{div}\big(a(x, \nabla u(x))\big) + \partial j(x, u(x)) \ni f_0(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$\phi(x) \le u(x) \le \Phi(x) \quad \text{in} \quad \Omega, \tag{1.4}$$

$$u(x) = 0 \quad \text{on} \quad \Gamma_1, \tag{1.5}$$

$$\frac{\partial u(x)}{\partial \nu_a} := \mu^*\big(a(x, \nabla u(x)), \nu\big)_{\mathbb{R}^N} = f_2(x) \quad \text{on} \quad \Gamma_2, \tag{1.6}$$

$$\begin{cases} \left|\dfrac{\partial u(x)}{\partial \nu_a}\right| \le g(x) \\ \dfrac{\partial u(x)}{\partial \nu_a} = -g(x)\dfrac{u(x)}{|u(x)|} \text{ if } u(x) \ne 0 \end{cases} \quad \text{on} \quad \Gamma_3, \tag{1.7}$$

where $\partial j(x, u(x))$ is the generalized subdifferential operator of j with respect to its second variable, $\mu^* > 0$, $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$.

The main contribution of the present work is threefold. The first contribution is to prove the existence and uniqueness of the weak solution for the elliptic bilateral obstacle problem, Problem 2, in which our method is based on the recent result [41, Theorem 8]. The second one is to consider the optimal control problem, Problem 1, driven by Problem 2, and to establish an criterion for the existence of an optimal solution to Problem 1. Whereas, the last contribution of the paper is to apply regularization method together with penalty approach to introduce a family of approximating problems (see Problem 18) associated with Problem 9, and prove the convergence result that any sequence of solutions for Problem 18 converges to a solution of Problem 9.

The outline of the paper is as follows. Section 2 collects the necessary notations and preliminary results. In Section 3, we formulate an optimal control problem governed by the nonlinear and nonsmooth elliptic system, and provide an existence theorem for the optimal control problem. Section 4 delivers an convergence result for the bilateral obstacle optimal control problem, Problem 1, in which our method is based on penalty technique and regularization method. The paper ends with some concluding remarks.

2. Mathematical prerequisites

In the section, we recall basic notation, definitions and necessary preliminary material, which will be used to obtain the main results of this paper. More details can be found, for instance, in Barbu and Korman [2], Denkowski et al. [4,5], Gasiński and Papageorgiou [9], Migórski et al. [30], Zeidler [39].

Everywhere below, the symbols $\xrightarrow{w}$ and $\to$ stand for the weak convergence and the strong convergence, respectively, in a certain space Y. Given a Banach space Y, we also adopt the notation $\|\cdot\|_Y$ and Y^* for a norm and the dual space of Y, respectively. We say that a function $A:V\to V^*$ is of type $(S)_+$ (or F satisfies the (S_+)–property), if any sequence $\{u_n\}$ with $u_n \xrightarrow{w} u$ in V as $n\to\infty$ for some u and $\limsup_{n\to\infty}\langle Au_n, u_n-u\rangle\le 0$, then the sequence $\{u_n\}$ converges strongly to u in V. Denote by $\mathbb{R}_+ := [0,+\infty)$.

Let $(Z,\|\cdot\|_Z)$ be a reflexive Banach space. We say that a function $J:Z\to\mathbb{R}$ is locally Lipschitz at $u\in Z$ if there exist a neighborhood $N(u)$ of u in Z and a constant $L_u>0$ such that

$$|J(w)-J(z)|\le L_u\|w-z\|_Z \quad\text{for all}\quad w,v\in N(u).$$

Definition 3. Given a locally Lipschitz function $J:Z\to\mathbb{R}$, we denote by $J^0(u;v)$ the directional derivative in the sense of Clarke (or generalized directional derivative) of J at $u\in Z$ in the direction $v\in Z$ defined by

$$J^0(u;v)=\limsup_{\lambda\to 0^+,\, w\to u}\frac{J(w+\lambda v)-J(w)}{\lambda}.$$

The generalized gradient of $J:Z\to\mathbb{R}$ at $u\in Z$ is given by

$$\partial J(u)=\{\xi\in Z^* \mid J^0(u;v)\ge\langle\xi,v\rangle \quad\text{for all}\quad v\in Z\}.$$

We next collect some critical properties for the generalized gradient and generalized directional derivative of a locally Lipschitz function as follows, see, e.g., [30, Proposition 3.23].

Proposition 4. *Let $J:Z\to\mathbb{R}$ is a locally Lipschitz continuous function on a Banach space $(Z,\|\cdot\|_Z)$. Then, we have*

(i) *for every $x\in Z$, the function $Z\ni v\mapsto J^0(x;v)\in\mathbb{R}$ is positively homogeneous and subadditive, i.e.,*

$$J^0(x;\lambda v)=\lambda J^0(x;v) \text{ for all } \lambda\ge 0 \text{ and } v\in Z,$$

and

$$J^0(x;v_1+v_2)\le J^0(x;v_1)+J^0(x;v_2) \text{ for all } x, v_1,\, v_2\in Z.$$

(ii) *let $x\in Z$ be fixed, for each $v\in Z$, there exists an element $\xi(v)\in\partial J(x)$ such that*

$$J^0(x;v)=\langle\xi(v),v\rangle \text{ i.e., } J^0(x;v)=\max\{\langle\xi,v\rangle \mid \xi\in\partial J(x)\}.$$

(iii) *the function $Z\times Z\ni(u,v)\mapsto J^0(u;v)\in\mathbb{R}$ is upper semicontinuous.*

(iv) *the graph of generalized gradient $\partial J:Z\to 2^{Z^*}$ is closed in $Z\times(w^*\text{–}Z^*)$ topology, i.e., if $\{x_n\}\subset Z$ and $\{\xi_n\}\subset Z^*$ are sequences such that*

$$\xi_n\in\partial J(x_n) \quad\text{and}\quad x_n\to x \text{ in } Z,\ \xi_n\to\xi \text{ weakly}^* \textit{in } Z^*,$$

then $\xi\in\partial J(x)$, where $(w^*\text{–}Z^*)$ denotes the space Z^* equipped with weak* topology.

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and $1<p<+\infty$. Also let $p'>1$ be the conjugate exponent of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Throughout the paper, the norms of the Lebesgue space $L^p(\Omega)$ and Sobolev space $W^{1,p}(\Omega)$ are defined by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p\,dx\right)^{\frac{1}{p}} \text{ for all } u\in L^p(\Omega),$$

and

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)}+\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \text{ for all } u\in W^{1,p}(\Omega),$$

respectively.

Assume that $1<p<\infty$ and $\zeta\in C^1(0,\infty)$ is such that

$$0<\widehat{c}\le\frac{\zeta'(t)t}{\zeta(t)}\le c_0 \quad\text{and}\quad c_1t^{p-1}\le\zeta(t)\le c_2(t^{\mu-1}+t^{p-1}) \quad\text{for all}\quad t>0 \tag{2.1}$$

with some constants $c_1,\ c_2>0$ and $1<\mu<p$. We, further, consider the abstract nonlinear nonhomogeneous operator $a:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ which satisfies the hypothesis $H(a)$.

$\underline{H(a)}$: $a:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^N$ is such that $a(x,\xi)=a_0(x,\|\xi\|_{\mathbb{R}^N})\xi$ for all $\xi\in\mathbb{R}^N$ and $x\in\overline{\Omega}$, where $a_0(x,\cdot)\in C(\mathbb{R}_+)$ is such that $a_0(x,t)>0$ for all $t>0$, $x\in\overline{\Omega}$, and

(i) for all $x \in \overline{\Omega}$, $a_0(x,\cdot) \in C^1(0,+\infty)$, $t \mapsto a_0(x,t)t$ is strictly increasing on $(0,+\infty)$, $a_0(x,t)t \to 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(x,t)t}{a_0(x,t)} > -1,$$

where $a_0'(x,t) = \frac{d}{dt}a_0(x,t)$.

(ii) there exists $c_3 > 0$ such that

$$\|\nabla a(x,y)\|_{\mathbb{R}^{N\times N}} \le c_3 \frac{\zeta(\|y\|_{\mathbb{R}^N})}{\|y\|_{\mathbb{R}^N}} \quad \text{for all} \quad y \in \mathbb{R}^N\backslash\{0_{\mathbb{R}^N}\} \text{ and } x \in \overline{\Omega}.$$

(iii) for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N\backslash\{0_{\mathbb{R}^N}\}$ and $\xi \in \mathbb{R}^N$, the following inequality holds

$$(\nabla a(x,y)\xi,\xi)_{\mathbb{R}^N} \ge \frac{\zeta(\|y\|_{\mathbb{R}^N})}{\|y\|_{\mathbb{R}^N}}\|\xi\|_{\mathbb{R}^N}^2.$$

Under the hypothesis $H(a)$, the following lemma is a direct consequence of [44, Lemma 3].

Lemma 5. *If hypothesis $H(a)$ is satisfied, then*

(i) *$y \mapsto a(x,y)$ is strictly monotone, continuous, hence also maximal monotone.*

(ii) *there exists a constant $c_4 > 0$ such that*

$$\|a(x,y)\|_{\mathbb{R}^N} \le c_4(1+\|y\|_{\mathbb{R}^N}^{p-1}) \text{ for all } \ y \in \mathbb{R}^N.$$

(iii) *for all $y \in \mathbb{R}^N$, the inequality is available*

$$(a(x,y),y)_{\mathbb{R}^N} \ge \frac{c_1}{p-1}\|y\|_{\mathbb{R}^N}^p,$$

where c_1 is given in (2.1).

In the current paper, we are interesting the nonlinear and nonhomogeneous partial differential operator $\operatorname{div} a(x,\nabla u(x))$. We, now, provide several concrete examples (see [29, Example 6]), which fulfill the setting presented in our paper.

Example 6. Here we only give the concrete examples for function a which are all independent of x.

(i) if a is specialized by $a(\xi) = \|\xi\|_{\mathbb{R}^N}{}^{p-2}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then the partial differential operator $\operatorname{div} a$ is the well-known p-Laplacian differential operator, namely,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) := \operatorname{div}(\|\nabla u(x)\|_{\mathbb{R}^N}^{p-2}\nabla u(x))$$

for all $u \in W^{1,p}(\Omega)$.

(ii) when a is formulated by $a(\xi) = \|\xi\|_{\mathbb{R}^N}{}^{p-2}\xi + \|\xi\|_{\mathbb{R}^N}^{q-2}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < q < p < \infty$, then the partial differential operator $\operatorname{div} a$ becomes the (p,q)-Laplacian differential operator, i.e.,

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \Delta_q u(x)$$

for all $u \in W^{1,p}(\Omega)$.

(iii) if a is given by $a(\xi) = (1+\|\xi\|_{\mathbb{R}^N}^2)^{\frac{p-2}{2}}\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then $\operatorname{div} a$ is the generalized p-mean curvature differential operator

$$\operatorname{div}(a(\nabla u(x))) = \operatorname{div}\left((1+\|\nabla u(x)\|_{\mathbb{R}^N}^2)^{\frac{p-2}{2}}\nabla u(x)\right)$$

for all $u \in W^{1,p}(\Omega)$.

(iv) if we take a by $a(\xi) = \|\xi\|_{\mathbb{R}^N}^{p-2}\left(1+\frac{1}{1+\|\xi\|_{\mathbb{R}^N}^p}\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then $\operatorname{div} a$ corresponds the following differential operator

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \operatorname{div}\left(\frac{\|\nabla u(x)\|_{\mathbb{R}^N}^{p-2}\nabla u(x)}{1+\|\nabla u(x)\|_{\mathbb{R}^N}^p}\right)$$

for all $u \in W^{1,p}(\Omega)$.

(v) let a be such that $a(\xi) = \left(\|\xi\|_{\mathbb{R}^N}^{p-2} + \ln(1+\|\xi\|_{\mathbb{R}^N}^2)\right)\xi$ for all $\xi \in \mathbb{R}^N$ with $1 < p < \infty$, then $\operatorname{div} a$ becomes the following differential operator

$$\operatorname{div}(a(\nabla u(x))) = \Delta_p u(x) + \operatorname{div}\left(\ln(1+\|\nabla u(x)\|_{\mathbb{R}^N}^2)\nabla u(x)\right)$$

for all $u \in W^{1,p}(\Omega)$.

Let us introduce the nonlinear operator $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega \big(a(x, \nabla u(x)), \nabla v(x)\big)_{\mathbb{R}^N}\, dx \quad \text{for all} \quad u, v \in W^{1,p}(\Omega). \tag{2.2}$$

The next proposition summarizes the main characteristic of this map, see, for example, Bai et al. [1] and Gasiński and Papageorgiou [10].

Proposition 7. *Let hypothesis $H(a)$ be satisfied and $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the map defined by (2.2). Then A is bounded, continuous, monotone (hence maximal monotone) and of type (S_+). Moreover, if a satisfies the inequality*

$$\big(a(x, y) - a(x, z), y - z\big)_{\mathbb{R}^N} \geq m_a\|y - z\|^p_{\mathbb{R}^N} \quad \text{for all} \quad y, z \in \mathbb{R}^N \text{ and a.e. } x \in \Omega, \tag{2.3}$$

with $m_a > 0$, which is independent of x, y and z, then A is strongly monotone with constant m_a.

Remark 8. For $p \geq 2$, it should be mentioned that when the function $t \mapsto a_0(x, t)t - c_a t^{p-1}$ is increasing on $[0, \infty)$ with some $c_a > 0$, then we are able to find a constant $m_a > 0$ such that inequality (2.3) holds (its detailed proof can be found in Bai et al. [1, Lemma 11]).

3. Existence results

The current section is devoted to explore the existence of an optimal solution for the nonlinear optimal control problem, Problem 9. We deliver the weak formulation of the elliptic inclusion problem, Problem 2, which is a variational-hemivariational inequality. Then, we employ [41, Theorem 8] to show the unique solvability of Problem 2, and establish an existence theorem for Problem 9 under mild assumptions.

Let $p \geq 2$ and introduce a subspace V of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$V := \{v \in W^{1,p}(\Omega) \mid \gamma v(x) = 0 \text{ for a.e. } x \in \Gamma_1\}, \tag{3.1}$$

where $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the trace operator. Korn's inequality and Poincaré's inequality reveal that V equipped with the norm

$$\|u\| = \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)} \text{ for all } u \in V,$$

is a reflexive Banach space. From Sobolev trace theorem, we know that γ is linear continuous and compact. Set $W = V \cap W^{2,p}(\Omega)$ and $U_{ad} \subset W \times W$ by

$$U_{ad} := \{(\phi, \Phi) \in W \times W \mid \phi(x) \leq \Phi(x) \text{ for a.e. } x \in \Omega\}. \tag{3.2}$$

For any $(\phi, \Phi) \in U_{ad}$ fixed, let us consider the constraint set $K(\phi, \Phi)$ defined by

$$K(\phi, \Phi) := \{u \in V \mid \phi(x) \leq u(x) \leq \Phi(x) \text{ for a.e. } x \in \Omega\}. \tag{3.3}$$

Obviously, we can see that the set $K(\phi, \Phi)$ is nonempty, closed and convex in V for any $(\phi, \Phi) \in U_{ad}$.

Let $u \in K(\phi, \Phi)$ be a smooth function such that (1.3)–(1.7) are satisfied. For any $v \in K(\phi, \Phi)$, we multiply (1.3) by $v - u$ and apply Green's formula to get

$$\begin{aligned}&\textstyle\int_\Omega \mu^*(a(x, \nabla u(x)), \nabla(v(x) - u(x)))_{\mathbb{R}^N}\, dx + \int_\Omega \xi(x)(v(x) - u(x))\, dx \\ &\quad = \textstyle\int_\Omega f_0(x)(v(x) - u(x))\, dx + \int_\Gamma \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma,\end{aligned}$$

where $\xi : \Omega \to \mathbb{R}$ is such that $\xi(x) \in \partial j(x, u(x))$ for a.e. $x \in \Omega$ and

$$-\mu^* \operatorname{div}\big(a(x, \nabla u(x))\big) + \xi(x) = f_0(x) \text{ for a.e. } x \in \Omega.$$

Note that

$$\begin{aligned}&\textstyle\int_\Gamma \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma = \int_{\Gamma_1} \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma \\ &\quad + \textstyle\int_{\Gamma_2} \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma + \int_{\Gamma_3} \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma,\end{aligned}$$

we use the boundary conditions (1.5) and (1.6) to obtain

$$\int_\Gamma \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma = \int_{\Gamma_2} f_2(x)(v(x) - u(x))\, d\Gamma + \int_{\Gamma_3} \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma.$$

Hence,

$$\begin{aligned}&\textstyle\int_\Omega \mu^*(a(x, \nabla u(x)), \nabla(v(x) - u(x)))_{\mathbb{R}^N}\, dx + \int_\Omega \xi(x)(v(x) - u(x))\, dx \\ &\quad = \textstyle\int_\Omega f_0(x)(v(x) - u(x))\, dx + \int_{\Gamma_2} f_2(x)(v(x) - u(x))\, d\Gamma \\ &\qquad + \textstyle\int_{\Gamma_3} \frac{\partial u(x)}{\partial \nu_a}(v(x) - u(x))\, d\Gamma.\end{aligned} \tag{3.4}$$

By virtue of definition of convex subgradient, it is not difficult to prove that the boundary condition (1.7) can be reformulated to the following inclusion

$$-\frac{\partial u(x)}{\partial \nu_a} \in g(x)\partial_c|u(x)|,$$

where $\partial_c|r|$ is the convex subdifferential operator of the modulus function $r \mapsto |r|$. So, it has

$$-\int_{\Gamma_3} \frac{\partial u(x)}{\partial \nu_a}(v(x)-u(x))\, d\Gamma \le \int_{\Gamma_3} g(x)|v(x)|\, d\Gamma - \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma.$$

Taking account into the inequality above and (3.4), it yields

$$\int_\Omega \mu^*\big(a(x,\nabla u(x)), \nabla(v(x)-u(x))\big)_{\mathbb{R}^N}\, dx + \int_{\Gamma_3} g(x)[|v(x)|-|u(x)|]\, d\Gamma$$
$$+\int_\Omega \xi(x)(v(x)-u(x))\, dx \ge \int_\Omega f_0(x)(v(x)-u(x))\, dx + \int_{\Gamma_2} f_2(x)(v(x)-u(x))\, d\Gamma.$$

However, the generalized subgradient in the sense of Clarke points out

$$\int_\Omega \xi(x)(v(x)-u(x))\, dx \le \int_\Omega j^0(x,u(x);v(x)-u(x))\, dx.$$

So, we obtain

$$\int_\Omega \mu^*\big(a(x,\nabla u(x)), \nabla(v(x)-u(x))\big)_{\mathbb{R}^N}\, dx + \int_\Omega j^0(x,u(x);v(x)-u(x))\, dx$$
$$+\int_{\Gamma_3} g(x)|v(x)|\, d\Gamma - \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma$$
$$\ge \int_\Omega f_0(x)(v(x)-u(x))\, dx + \int_{\Gamma_2} f_2(x)(v(x)-u(x))\, d\Gamma.$$

In summary, we are now in a position to provide the variational formulation of Problem 2 as follows.

Problem 9. Given $(\phi,\Phi) \in U_{ad}$, find $u \in K(\phi,\Phi)$ such that

$$\int_\Omega \mu^*\big(a(x,\nabla u(x)), \nabla(v(x)-u(x))\big)_{\mathbb{R}^N}\, dx + \int_\Omega j^0(x,u(x);v(x)-u(x))\, dx$$
$$+\int_{\Gamma_3} g(x)|v(x)|\, d\Gamma - \int_{\Gamma_3} g(x)|u(x)|\, d\Gamma$$
$$\ge \int_\Omega f_0(x)(v(x)-u(x))\, dx + \int_{\Gamma_2} f_2(x)(v(x)-u(x))\, d\Gamma$$

for all $v \in K(\phi,\Phi)$.

To establish the existence and uniqueness of solution of Problem 9, we impose the following assumptions.
$\underline{H(0)}$: $\mu^* > 0$, $f_0 \in L^{p'}(\Omega)$, $f_2 \in L^{p'}(\Gamma_2)$, and $g \in L^{p'}(\Gamma_3)_+$ with $g \not\equiv 0$.
$\underline{H(j)}$: $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for all $r \in \mathbb{R}$, $x \mapsto j(x,r)$ is measurable on Ω, and the funciton $x \mapsto j(x,0)$ belongs to $L^1(\Omega)$.
(ii) the function $r \mapsto j(x,r)$ is locally Lipschitz continuous for a.e. $x \in \Omega$.
(iii) there exists a constant $c_j > 0$ satisfying

$$|\partial j(x,r)| \le c_j(1+|r|^{p-1})$$

for all $r \in \mathbb{R}$ and a.e. $x \in \Omega$.
(iv) there is a constant $m_j \ge 0$ such that

$$(\xi_1-\xi_2)(r_1-r_2) \ge -m_j|r_1-r_2|^p$$

for all $\xi_i \in \partial j(x,r_i)$, $r_i \in \mathbb{R}$, $i=1,2$ and for a.e. $x \in \Omega$.

Remark 10. In fact, condition $H(j)$(iv) is equivalent to the following inequality

$$j^0(x,r_1;r_2-r_1) + j^0(x,r_2;r_1-r_2) \le m_j|r_1-r_2|^p$$

for all $r_1, r_2 \in \mathbb{R}$ and for a.e. $x \in \Omega$.

Let $X = L^p(\Omega)$. In what follows, we denote by $\eta : W^{1,p}(\Omega) \to L^p(\Omega)$ the embedding operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$, which is, obviously, linear, bounded and compact. Besides, we introduce a function $J : X \to \mathbb{R}$ defined by

$$J(w) := \int_\Omega j(x,w(x))\, dx \tag{3.5}$$

for all $w \in X$.

Under the hypothesis $H(j)$, arguing as in the proof of [30, Theorem 3.47], we derive the lemma.

Lemma 11. *Let $H(j)$ be fulfilled. Then the following statements hold*

(i) $J : X \to \mathbb{R}$, *defined in (3.5), is locally Lipschitz continuous and*

$$\begin{cases} J^0(w; z) \le \int_\Omega j^0(x, w(x); z(x))\, dx \\ \partial J(w) \subset \int_\Omega \partial j(x, w(x))\, dx \end{cases}$$

for all $w, z \in X$.

(ii) *there exists a constant* $c_J > 0$ *such that*

$$\|\partial J(w)\|_{X^*} \le c_J(1 + \|w\|_X^{p-1})$$

for all $w \in X$.

(iii) *the inequality holds*

$$J^0(w; z - w) + J^0(z; w - z) \le m_j \|w - z\|_X^p$$

for all $w, z \in X$.

We, now, give the following theorem concerning the existence and uniqueness of solution to Problem 9.

Theorem 12. *Assume that* $H(a)$, $H(0)$ *and* $H(j)$ *are fulfilled. If, in addition, there exists a constant* $m_a > 0$ *satisfying* (2.3), *and the inequality holds*

$$m_a \mu^* > m_j \|\eta\|^p, \tag{3.6}$$

then Problem 9 has a unique solution $u \in K(\phi, \Phi)$.

Proof. We, first, consider the following intermediate problem: find $u \in K(\phi, \Phi)$ satisfying

$$\langle Fu - f, v - u\rangle + J^0(\eta u; \eta(v - u)) + \varphi(v) - \varphi(u) \ge 0 \tag{3.7}$$

for all $v \in K(\phi, \Phi)$, where $F : V \to V^*$, $\varphi : V \to \mathbb{R}$ and $f \in V^*$ are given by

$$\langle Fu, v\rangle := \mu^* \int_\Omega (a(x, \nabla u(x)), \nabla v(x))_{\mathbb{R}^N}\, dx \tag{3.8}$$

$$\varphi(v) := \int_{\Gamma_3} g(x)|v(x)|\, d\Gamma \tag{3.9}$$

$$\langle f, v\rangle = \int_\Omega f_0(x)v(x)\, dx + \int_{\Gamma_2} f_2(x)v(x)\, d\Gamma \tag{3.10}$$

for all $u, v \in V$, here f is uniquely determined by using Riesz's representation theorem. Lemma 11(ii) points out that a solution of problem (3.7) solves Problem 9 too. Based on the fact, we shall use [41, Theorem 8] to verify that problem (3.7) has at least one solution, i.e., Problem 9 is solvable.

It follows from Proposition 7 and the definition of F (see (3.8)) that F is a bounded, continuous, and strongly monotone operator with the constant $m_F = m_a\mu^*$. However, utilizing Lemma 11(iii), we have

$$\begin{aligned} &\langle Fu - Fv, u - v\rangle + \langle \xi_u - \xi_v, \eta(u - v)\rangle_{X^*\times X} \\ &\ge \langle Fu - Fv, u - v\rangle - J^0(\eta u; \eta(v - u)) - J^0(\eta v; \eta(u - v)) \\ &\ge m_a\mu^* \|u - v\|^p - m_j \|\eta(u - v)\|_X^p \\ &\ge (m_a\mu^* - m_j\|\eta\|^p)\|u - v\|^p \end{aligned}$$

for all $\xi_u \in \partial J(\eta u)$, $\xi_v \in \partial J(\eta v)$ and $u, v \in V$. This combined with the smallness condition (3.6) implies that $V \ni u \mapsto F(u) + \eta^*\partial J(\eta u) \subset V^*$ is strongly monotone, where $\eta^* : X^* \to V^*$ is the dual operator of η. For any $v_0 \in K(\phi, \Phi)$ fixed (particularly, we can take $v_0 = \phi$ or $v_0 = \Phi$), Lemma 11(ii) turns out

$$\|\xi_{v_0}\|_{X^*} \le c_J(1 + \|\eta v_0\|_X^{p-1}) \tag{3.11}$$

for all $\xi_{v_0} \in \partial J(\eta(v_0))$. Hence, for any $\xi_{v_0} \in \partial J(\eta(v_0))$ and $\xi_u \in \partial J(\eta u)$, (3.6) and the following estimates

$$\begin{aligned} &\langle Fu, u - v_0\rangle + \langle \xi_u, \eta(u - v_0)\rangle_{X^*\times X} \\ &= \langle Fu - Fv_0, u - v_0\rangle + \langle Fv_0, u - v_0\rangle + \langle \xi_u - \xi_{v_0}, \eta(u - v_0)\rangle_{X^*\times X} \\ &\quad + \langle \xi_{v_0}, \eta(u - v_0)\rangle_{X^*\times X} \\ &\ge (\mu^* m_a - m_j\|\eta\|^p)\|u - v_0\|^p - \|Fv_0\|_{V^*}\|u - v_0\| - \|\xi_{v_0}\|_{X^*}\|\eta\|\|u - v_0\| \end{aligned}$$

indicate

$$\lim_{u\in V, \|u\|\to\infty} \frac{\langle Fu, u - v_0\rangle + inf_{\xi\in\partial J(\eta u)}\langle \xi, \eta(u - v_0)\rangle_{X^*\times X}}{\|u\|} = +\infty,$$

where we have used the inequality (3.11), the boundedness of F, and $p \geq 2$. Obviously, from the formulation of φ, we can see that φ is a convex and continuous function.

Therefore, all conditions of [41, Theorem 8] are available. We are now in a position to apply this theorem to conclude that problem (3.7) has at least one solution. So, Problem 9 is solvable.

For the uniqueness, let u_1, u_2 be two solutions to Problem 9. A simple calculating gives

$$\begin{aligned}&\left(\mu^* m_a - m_j\|\eta\|^p\right)\|u_1-u_2\|^p\\ &\leq \langle Fu_1 - Fu_2, u_1-u_2\rangle - \textstyle\int_\Omega j^0(x,u_1(x);u_2(x)-u_1(x))\,dx - \int_\Omega j^0(x,u_2(x);u_1(x)-u_2(x))\,dx\\ &\leq 0.\end{aligned}$$

Whereas the smallness condition (3.6) guarantees that $u_1 = u_2$, namely, Problem 9 has a unique solution. □

In the sequel, we denote by $\mathcal{S}: W\times W \to V$ the solution map of Problem 9, i.e.,

$$\mathcal{S}(\phi,\Phi) := u_{\phi,\Phi} \text{ for all } (\phi,\Phi)\in U_{ad},$$

where $u_{\phi,\Phi}\in V$ is the unique solution of Problem 9 corresponding to the bilateral obstacle (ϕ,Φ).

We end the section by providing the following existence theorem for the optimal control problem, Problem 1.

Theorem 13. *Under the assumptions of Theorem 12, there exists an optimal solution $(\phi^*,\Phi^*)\in U_{ad}$ to Problem 1.*

Proof. From the definition of L, we can see that L is bounded from below. So, there exists a minimizing sequence $\{(\phi_n,\Phi_n)\}\subset U_{ad}$ satisfying

$$\lim_{n\to\infty} L(\phi_n,\Phi_n) = \inf_{(\phi,\Phi)\in U_{ad}} L(\phi,\Phi) =: l^*. \tag{3.12}$$

Hence, we are able to find a sequence $\{\varepsilon_n\}\subset\mathbb{R}_+$ with

$$\varepsilon_n > 0 \text{ and } \varepsilon_n \to 0 \text{ as } n\to\infty$$

such that

$$L(\phi_n,\Phi_n) \leq l^* + \varepsilon_n$$

for each $n\in\mathbb{N}$, that is,

$$\frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)} \leq l^* + \varepsilon_n.$$

This means that the sequence $\{(\phi_n,\Phi_n)\}$ is bounded in $W\times W$. Passing to a subsequence if necessary, we may assume that

$$\begin{cases}\phi_n \xrightarrow{w} \phi^* \text{ in } W,\\ \Phi_n \xrightarrow{w} \Phi^* \text{ in } W,\\ \phi_n\to\phi^* \text{ in } V,\\ \Phi_n\to\Phi^* \text{ in } V,\\ \phi_n(x)\to\phi^*(x) \text{ for a.e. } x\in\Omega,\\ \Phi_n(x)\to\Phi^*(x) \text{ for a.e. } x\in\Omega,\end{cases} \quad \text{as } n\to\infty. \tag{3.13}$$

Owing to $(\phi_n,\Phi_n)\in U_{ad}$, the closedness of U_{ad} in $V\times V$ ensures that $(\phi^*,\Phi^*)\in U_{ad}$. Set $u_n := \mathcal{S}(\phi_n,\Phi_n)$, i.e., u_n is the unique solution to Problem 9 associated with the bilateral obstacle (ϕ_n,Φ_n). Then, one has

$$\begin{aligned}&\langle Fu_n, v-u_n\rangle + \int_\Omega j^0(x,u_n(x);v(x)-u_n(x))\,dx + \int_{\Gamma_3} g(x)|v(x)|\,d\Gamma\\ &-\int_{\Gamma_3} g(x)|u_n(x)|\,d\Gamma \geq \int_\Omega f_0(x)(v(x)-u_n(x))\,dx + \int_{\Gamma_2} f_2(x)(v(x)-u_n(x))\,d\Gamma\end{aligned} \tag{3.14}$$

for all $v\in K(\phi_n,\Phi_n)$. We claim that the sequence $\{u_n\}$ is bounded in V. Inserting $v=\phi_n$ into the inequality above gets

$$\begin{aligned}&\langle Fu_n, \phi_n-u_n\rangle + \int_\Omega j^0(x,u_n(x);\phi_n(x)-u_n(x))\,dx + \int_{\Gamma_3} g(x)|\phi_n(x)|\,d\Gamma\\ &-\int_{\Gamma_3} g(x)|u_n(x)|\,d\Gamma \geq \int_\Omega f_0(x)(\phi_n(x)-u_n(x))\,dx + \int_{\Gamma_2} f_2(x)(\phi_n(x)-u_n(x))\,d\Gamma.\end{aligned} \tag{3.15}$$

The boundedness of F infers

$$\begin{aligned}\langle Fu_n, u_n-\phi_n\rangle &= \langle Fu_n - F\phi_n, u_n-\phi_n\rangle + \langle F\phi_n, u_n-\phi_n\rangle\\ &\geq \mu^* m_a\|u_n-\phi_n\|^p - \|F\phi_n\|_{V^*}\|u_n-\phi_n\|\\ &\geq \mu^* m_a\|u_n-\phi_n\|^p - m_0\|u_n-\phi_n\|,\end{aligned} \tag{3.16}$$

where $m_0 > 0$ is such that $\|F\phi_n\|_{V^*} \le m_0$ for all $n \in \mathbb{N}$. The conditions $H(j)$(iii) and (iv) deduce

$$\begin{aligned}&\int_\Omega j^0(x, u_n(x); \phi_n(x) - u_n(x))\,dx\\&= \int_\Omega j^0(x, u_n(x); \phi_n(x) - u_n(x))\,dx + \int_\Omega j^0(x, \phi_n(x); u_n(x) - \phi_n(x))\,dx\\&\quad - \int_\Omega j^0(x, \phi_n(x); u_n(x) - \phi_n(x))\\&\le m_j\|u_n - \phi_n\|^p_{L^p(\Omega)} + \int_\Omega c_j(1 + |\phi_n(x)|^{p-1})|u_n(x) - \phi_n(x)|\,dx.\end{aligned}$$

Applying Hölder inequality, we have

$$\begin{aligned}&\int_\Omega j^0(x, u_n(x); \phi_n(x) - u_n(x))\,dx\\&\le m_j\|\eta\|^p\|u_n - \phi_n\|^p + m_1(1 + \|\phi_n\|^{p-1}_{L^p(\Omega)})\|\eta\|\|u_n - \phi_n\|\\&\le m_j\|\eta\|^p\|u_n - \phi_n\|^p + m_2\|u_n - \phi_n\|,\end{aligned} \tag{3.17}$$

where $m_2 > 0$ satisfies $m_1(1 + \|\phi_n\|^{p-1}_{L^p(\Omega)})\|\eta\| \le m_2$ for all $n \in \mathbb{N}$. Employing hypothesis $H(0)$ and Hölder inequality, we have

$$\begin{aligned}&\int_{\Gamma_3} g(x)[|\phi_n(x)| - |u_n(x)|]\,d\Gamma - \int_\Omega f_0(x)(\phi_n(x) - u_n(x))\,dx\\&\quad + \int_{\Gamma_2} f_2(x)(\phi_n(x) - u_n(x))\,d\Gamma\\&\le \left(\|g\|_{L^{p'}(\Gamma_3)}\|\gamma\| + \|f_0\|_{L^{p'}(\Omega)}\|\eta\| + \|f_2\|_{L^{p'}(\Gamma_2)}\|\gamma\|\right)\|\phi_n - u_n\|.\end{aligned} \tag{3.18}$$

Taking account into (3.15)–(3.18), it finds

$$\begin{aligned}&(\mu^* m_a - m_j\|\eta\|^p)\|u_n - \phi_n\|^{p-1}\\&\le \|g\|_{L^{p'}(\Gamma_3)}\|\gamma\| + \|f_0\|_{L^{p'}(\Omega)}\|\eta\| + \|f_2\|_{L^{p'}(\Gamma_2)}\|\gamma\| + m_0 + m_2.\end{aligned}$$

Therefore, the sequence $\{u_n\}$ is bounded in V due to the bounded of $\{\phi_n\}$ in V.

We may assume that along a relabeled subsequence one has

$$\begin{cases} u_n \xrightarrow{w} u \text{ in } V,\\ u_n \to u \text{ in } X,\\ u_n(x) \to u(x) \text{ for a.e. } x \in \Omega,\end{cases} \quad \text{as } n \to \infty. \tag{3.19}$$

Because of $u_n \in K(\phi_n, \Phi_n)$, it has

$$\phi_n(x) \le u_n(x) \le \Phi_n(x) \text{ for a.e. } x \in \Omega.$$

This together with (3.13) and (3.19) concludes

$$\phi^*(x) \le u(x) \le \Phi^*(x) \text{ for a.e. } x \in \Omega,$$

namely, $u \in K(\phi^*, \Phi^*)$. Set $w_n = \inf\{\sup\{u, \phi_n\}, \Phi_n\}$. It is not difficult to prove that

$$w_n \in K(\phi_n, \Phi_n), \text{ and } w_n \to u \text{ in } V \text{ as } n \to \infty, \tag{3.20}$$

where we have utilized the convergences (3.13). We shall show that u is the unique solution of Problem 9 corresponding to the bilateral obstacle (ϕ^*, Φ^*). Letting $v = w_n$ into the inequality (3.14) turns out

$$\begin{aligned}\langle Fu_n, u_n - w_n\rangle &\le \int_\Omega j^0(x, u_n(x); w_n(x) - u_n(x))\,dx + \int_{\Gamma_3} g(x)|w_n(x)|\,d\Gamma\\&\quad - \int_{\Gamma_3} g(x)|u_n(x)|\,d\Gamma - \int_\Omega f_0(x)(w_n(x) - u_n(x))\,dx - \int_{\Gamma_2} f_2(x)(w_n(x) - u_n(x))\,d\Gamma.\end{aligned}$$

Passing to the upper limit as $n \to \infty$ and using the following equalities

$$\begin{aligned}&\limsup_{n\to\infty}\langle Fu_n, u_n - w_n\rangle\\&= \limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle + \lim_{n\to\infty}\langle Fu_n, u - w_n\rangle\\&= \limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle,\end{aligned}$$

it gives

$$\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \le \limsup_{n\to\infty}\langle Fu_n, u_n - w_n\rangle$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty}\int_\Omega j^0(x,u_n(x);w_n(x)-u_n(x))\,dx+\lim_{n\to\infty}\int_{\Gamma_3}g(x)|w_n(x)|\,d\Gamma\\
&\quad-\lim_{n\to\infty}\int_{\Gamma_3}g(x)|u_n(x)|\,d\Gamma-\lim_{n\to\infty}\int_\Omega f_0(x)(w_n(x)-u_n(x))\,dx\\
&\quad-\lim_{n\to\infty}\int_{\Gamma_2}f_2(x)(w_n(x)-u_n(x))\,d\Gamma\\
&\leq 0,
\end{aligned}
$$

where we have used the convergences $u_n\to u$ in $L^p(\Gamma)$ and X (by Sobolev trace and embedding theorems), the upper semicontinuity of $(s,r)\mapsto j^0(x,s;r)$ and Lebesgue dominated convergence theorem. Keeping in mind that $u_n\xrightarrow{w}u$ in V as $n\to\infty$, the inequality

$$\limsup_{n\to\infty}\langle Fu_n,u_n-u\rangle\leq 0,$$

and the (S_+)–property of F guarantee that

$$u_n\to u \text{ in } V \text{ as } n\to\infty. \tag{3.21}$$

Let $v\in K(\phi^*,\Phi^*)$ be arbitrary. We consider the sequence $\{v_n\}$ given by

$$v_n:=\inf\{\sup\{\phi_n,v\},\Phi_n\}.$$

Evidently, it is valid

$$v_n\in K(\phi_n,\Phi_n),\ \text{and } v_n\to v \text{ in } V \text{ as } n\to\infty. \tag{3.22}$$

Taking $v=v_n$ into (3.14) and passing to the upper limit as $n\to\infty$ for the resulting inequality, we get

$$
\begin{aligned}
&\langle Fu,v-u\rangle+\int_\Omega j^0(x,u(x);v(x)-u(x))\,dx+\int_{\Gamma_3}g(x)|v(x)|\,d\Gamma-\int_{\Gamma_3}g(x)|u(x)|\,d\Gamma\\
&\quad\geq\lim_{n\to\infty}\langle Fu_n,v_n-u_n\rangle+\limsup_{n\to\infty}\int_\Omega j^0(x,u_n(x);v_n(x)-u_n(x))\,dx\\
&\qquad+\lim_{n\to\infty}\int_{\Gamma_3}g(x)|v_n(x)|\,d\Gamma-\lim_{n\to\infty}\int_{\Gamma_3}g(x)|u_n(x)|\,d\Gamma\\
&\quad\geq\lim_{n\to\infty}\int_\Omega f_0(x)(v_n(x)-u_n(x))\,dx+\lim_{n\to\infty}\int_{\Gamma_2}f_2(x)(v_n(x)-u_n(x))\,d\Gamma\\
&\quad=\int_\Omega f_0(x)(v(x)-u(x))\,dx+\int_{\Gamma_2}f_2(x)(v(x)-u(x))\,d\Gamma.
\end{aligned}
$$

Because $v\in K(\phi^*,\Phi^*)$ is arbitrary. Hence, u is the unique solution of Problem 9 associated with the bilateral obstacle (ϕ^*,Φ^*), i.e., $u=\mathcal{S}(\phi^*,\Phi^*)$. However, the result that every convergent subsequence of $\{u_n\}$ converges strongly to the same limit $u=\mathcal{S}(\phi^*,\Phi^*)$ in V indicates that the whole sequence $\{u_n\}$ converges strongly to $u=\mathcal{S}(\phi^*,\Phi^*)$ in V.

Finally, we are going to prove that (ϕ^*,Φ^*) is an optimal solution to Problem 1. It follows from the weak lower semicontinuity of $\|\cdot\|_{L^p(\Omega)}$ and the convergence, $\mathcal{S}(\phi_n,\Phi_n)=u_n\to u=\mathcal{S}(\phi^*,\Phi^*)$ that

$$
\begin{aligned}
L(\phi^*,\Phi^*)&=\frac{1}{p}\int_\Omega|\mathcal{S}(\phi^*,\Phi^*)(x)-z_d(x)|^p\,dx+\frac{\alpha}{p}\|\Delta\phi^*\|^p_{L^p(\Omega)}+\frac{\beta}{p}\|\Delta\Phi^*\|^p_{L^p(\Omega)}\\
&\leq\liminf_{n\to\infty}\frac{1}{p}\int_\Omega|\mathcal{S}(\phi_n,\Phi_n)(x)-z_d(x)|^p\,dx+\liminf_{n\to\infty}\frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)}+\liminf_{n\to\infty}\frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)}\\
&\leq\liminf_{n\to\infty}\left[\frac{1}{p}\int_\Omega|\mathcal{S}(\phi_n,\Phi_n)(x)-z_d(x)|^p\,dx+\frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)}+\frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)}\right]\\
&=\liminf_{n\to\infty}L(\phi_n,\Phi_n).
\end{aligned}
$$

Combining the estimates above, (3.12) and the fact,

$$\inf_{(\phi,\Phi)\in U_{ad}}L(\phi,\Phi)\leq L(\phi^*,\Phi^*),$$

we have

$$\inf_{(\phi,\Phi)\in U_{ad}}L(\phi,\Phi)\leq L(\phi^*,\Phi^*)\leq\liminf_{n\to\infty}L(\phi_n,\Phi_n)=\inf_{(\phi,\Phi)\in U_{ad}}L(\phi,\Phi).$$

This means that $(\phi^*,\Phi^*)\in U_{ad}$ is an optimal solution to Problem 1. $\square$

4. Convergence results

This section is concerned with the convergence analysis to the bilateral obstacle optimal control problem, Problem 1, in which our method is based on penalty technique and regularization method. For convenience, in the section, we assume that all conditions of Theorem 13 are satisfied.

Let us introduce a function $\rho : \mathbb{R} \to \mathbb{R}_+$, which is the density of a probability distribution on $\mathbb{R}$ with finite absolute mean, i.e., it satisfies the condition $H(\rho)$.

$\underline{H(\rho)}$: $\rho : \mathbb{R} \to \mathbb{R}_+$ is such that

(i) ρ is Lebesgue integrable on $\mathbb{R}$.

(ii) $\int_{\mathbb{R}} \rho(s)\, ds = 1$.

(ii) it holds

$$r_0 := \int_{\mathbb{R}} |s|\rho(s)\, ds < +\infty. \tag{4.1}$$

Denote by $s : \mathbb{R} \to \mathbb{R}_+$ the plus function, i.e., $s(r) = r^+ = \max\{0, r\}$ for all $r \in \mathbb{R}$. For any $\varepsilon > 0$, we consider the regularized (or smoothing) function $\mathcal{R}_\varepsilon : \mathbb{R} \to \mathbb{R}$ corresponding to the plus function s defined by

$$\mathcal{R}_\varepsilon(r) = \int_{\mathbb{R}} s(r - \varepsilon t)\rho(t)\, dt = \int_{-\infty}^{\frac{r}{\varepsilon}} s(r - \varepsilon t)\rho(t)\, dt \tag{4.2}$$

for all $r \in \mathbb{R}$.

The following proposition reveals that $\mathcal{R}_\varepsilon$ is a smooth approximating function of the plus function s, its detailed proof can be found in Facchinei and Pang [6, Proposition 11.8.10].

Proposition 14. *Assume that $H(\rho)$ is satisfied. Then for every $\varepsilon > 0$, we have*

(i) *the inequality is true*

$$|\mathcal{R}_\varepsilon(t) - s(t)| \le r_0\varepsilon \text{ for all } t \in \mathbb{R}. \tag{4.3}$$

(ii) *$\mathcal{R}_\varepsilon$ is convex and twice continuously differentiable on $\mathbb{R}$ such that*

$$\mathcal{R}'_\varepsilon(t) = \int_{-\infty}^{\frac{t}{\varepsilon}} \rho(\tau)\, d\tau, \quad \mathcal{R}''_\varepsilon(t) = \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right),$$

and $\mathcal{R}'_\varepsilon(t) \in [0, 1]$ for all $t \in \mathbb{R}$.

Remark 15. It should be mentioned that there are many possibilities for the density function ρ, for instance, ρ could be chosen by the smooth functions or discontinuous functions

- smooth functions:

$$\rho(t) = \frac{e^{-t}}{(1 + e^{-t})^2}, \text{ or } \rho(t) = \frac{2}{(t^2 + 2^2)^{\frac{3}{2}}}$$

for all $t \in \mathbb{R}$.

- discontinuous functions:

$$\rho(t) = \begin{cases} 1, & \text{if } -\frac{1}{2} \le t \le \frac{1}{2}, \\ 0, & \text{else,} \end{cases} \quad \text{or } \rho(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1, \\ 0, & \text{else.} \end{cases}$$

Recall that $|t| = t^+ + (-t)^+$, we also can introduce the smooth approximating function for the modulus function $m : \mathbb{R} \to \mathbb{R}_+$, $m(t) = |t|$ by

$$m_\varepsilon(t) = \mathcal{R}_\varepsilon(t) + \mathcal{R}_\varepsilon(-t) \tag{4.4}$$

for all $t \in \mathbb{R}$.

From Proposition 14, it is easy to verify the following corollary.

Corollary 16. *Assume that $H(\rho)$ is satisfied. Then for every $\varepsilon > 0$, we have*

(i) *the inequality is true*

$$|m_\varepsilon(t) - m(t)| \le 2r_0\varepsilon \text{ for all } t \in \mathbb{R}. \tag{4.5}$$

(ii) *m_ε is convex and twice continuously differentiable on $\mathbb{R}$ such that*

$$m'_\varepsilon(t) = \int_{-\frac{t}{\varepsilon}}^{\frac{t}{\varepsilon}} \rho(\tau)\, d\tau, \quad m''_\varepsilon(t) = \frac{1}{\varepsilon}\left[\rho\left(\frac{t}{\varepsilon}\right) + \rho\left(-\frac{t}{\varepsilon}\right)\right],$$

and $m'_\varepsilon(t) \in [0, 2]$ for all $t \in \mathbb{R}$.

On the other side, let us introduce another regularized function $h : \mathbb{R} \to (-\infty, 0]$ given by

$$h(t) = \begin{cases} 0, & \text{if } t \ge 0, \\ -t^2, & \text{if } t \in [-0.5, 0), \\ t + \frac{1}{4}, & \text{if } t < -0.5. \end{cases} \tag{4.6}$$

Observe that h is a C^1 function with

$$h'(t) = \begin{cases} 0, & \text{if } t \ge 0, \\ -2t, & \text{if } t \in [-0.5, 0), \\ 1, & \text{if } t < -0.5. \end{cases} \quad \text{and } \min\{0, t\} \le h(t) \le 0 \text{ for all } t \in \mathbb{R}.$$

Let $\{\varepsilon_n\}$ be a sequence such that

$$\varepsilon_n > 0 \text{ for all } n \in \mathbb{N}, \text{ and } \varepsilon_n \to 0 \text{ as } n \to \infty. \tag{4.7}$$

In what follows, we denote by $m_n = m_{\varepsilon_n}$. We, now, study the following approximating problem corresponding to Problem 9.

Problem 17. Given $(\phi, \Phi) \in U_{ad}$, find a function $u \in V$ such that

$$\begin{aligned} &\int_\Omega \mu^*\big(a(x, \nabla u(x)), \nabla(v(x) - u(x))\big)_{\mathbb{R}^N} \, dx + \int_\Omega j^0(x, u(x); v(x) - u(x)) \, dx \\ &+ \frac{1}{\varepsilon_n} \int_\Omega [h(u(x) - \phi(x)) - h(\Phi(x) - u(x))](v(x) - u(x)) \, dx \\ &+ \int_{\Gamma_3} g(x) m_n(v(x)) \, d\Gamma - \int_{\Gamma_3} g(x) m_n(u(x)) \, d\Gamma \ge \int_\Omega f_0(x)(v(x) - u(x)) \, dx \\ &+ \int_{\Gamma_2} f_2(x)(v(x) - u(x)) \, d\Gamma \end{aligned} \tag{4.8}$$

for all $v \in V$.

From Theorem 12, we can see that Problem 17 has a unique solution $u \in V$, because of the continuity and monotonicity of $r \mapsto h(r - \phi(x)) - h(\Phi(x) - r)$. Let $(\phi^*, \Phi^*) \in U_{ad}$ be an optimal solution to Problem 1 fixed. We also consider the following approximating optimal control problem associated with Problem 1.

Problem 18. Given $\varepsilon_n > 0$, find $(\phi_n^*, \Phi_n^*) \in U_{ad}$ such that

$$(\phi_n^*, \Phi_n^*) \in \arg \min_{(\phi, \Phi) \in U_{ad}} L_n(\phi, \Phi),$$

namely,

$$L_n(\phi_n^*, \Phi_n^*) \le L_n(\phi, \Phi) \text{ for all } (\phi, \Phi) \in U_{ad}, \tag{4.9}$$

where the cost function $L_n : W \times W \to \mathbb{R}$ is defined by

$$\begin{aligned} L_n(\phi, \Phi) := &\frac{1}{p} \int_\Omega |\mathcal{S}_n(\phi, \Phi)(x) - z_d(x)|^p \, dx + \frac{\alpha}{p} \|\Delta\phi\|^p_{L^p(\Omega)} + \frac{\beta}{p} \|\Delta\Phi\|^p_{L^p(\Omega)} \\ &+ \delta_1 \|\phi - \phi^*\|^p_{L^p(\Omega)} + \delta_2 \|\Phi - \Phi^*\|^p_{L^p(\Omega)} \text{ for all } (\phi, \Phi) \in U_{ad}, \end{aligned} \tag{4.10}$$

and $\mathcal{S}_n(\phi, \Phi)$ is the unique solution to Problem 17 corresponding to the bilateral obstacle $(\phi, \Phi) \in U_{ad}$, here $\delta_1, \delta_2 > 0$ are two regularized parameters.

Lemma 19. *For any $(\phi, \Phi) \in U_{ad}$ fixed, let $u_n = \mathcal{S}_n(\phi, \Phi)$ be the unique solution to Problem 17. Then, it holds*

$$\begin{aligned} &\int_\Omega \mu^*\big(a(x, \nabla u_n(x)), \nabla(v(x) - u_n(x))\big)_{\mathbb{R}^N} \, dx + \int_\Omega j^0(x, u_n(x); v(x) - u_n(x)) \, dx \\ &\quad + \int_{\Gamma_3} g(x) m_n(v(x)) \, d\Gamma - \int_{\Gamma_3} g(x) m_n(u_n(x)) \, d\Gamma \\ &\quad \ge \int_\Omega f_0(x)(v(x) - u_n(x)) \, dx + \int_{\Gamma_2} f_2(x)(v(x) - u_n(x)) \, d\Gamma \end{aligned} \tag{4.11}$$

for all $v \in K(\phi, \Phi)$.

Proof. For any $(\phi, \Phi) \in U_{ad}$ fixed, let $u_n = \mathcal{S}_n(\phi, \Phi)$ be the unique solution to Problem 17. For any $v \in K(\phi, \Phi)$, we have

- if $u_n(x) \ge \phi(x)$, then $h(u_n(x) - \phi(x))(v(x) - u_n(x)) = 0$,
- if $u_n(x) < \phi(x)$, then $h(u_n(x) - \phi(x))(v(x) - u_n(x)) \le 0$ (due to $v(x) \ge \phi(x) > u_n(x)$ and $h(u_n(x) - \phi(x)) < 0$),
- if $u_n(x) \le \Phi(x)$, then $h(\Phi(x) - u_n(x))(v(x) - u_n(x)) = 0$,
- if $u_n(x) > \Phi(x)$, then $h(\Phi(x) - u_n(x))(v(x) - u_n(x)) \ge 0$ (due to $v(x) \le \Phi(x) < u_n(x)$ and $h(\Phi(x) - u_n(x)) < 0$).

Hence,

$$\frac{1}{\varepsilon_n}\int_\Omega [h(u_n(x)-\phi(x))-h(\Phi(x)-u_n(x))](v(x)-u_n(x))\,dx \le 0.$$

So, for any $v \in K(\phi,\Phi)$, by virtue of (4.8), we obtain the desired inequality (4.11). □

To deliver the main result of the section, we, next, give a critical convergence result.

Lemma 20. *Let $\{(\phi_n,\Phi_n)\}\subset U_{ad}$ be a sequence such that*

$$(\phi_n,\Phi_n)\to(\phi,\Phi) \text{ in } V \text{ as } n\to\infty, \tag{4.12}$$

for some $(\phi,\Phi)\in U_{ad}$. Then the sequence $\{u_n\}$ with $u_n=\mathcal{S}_n(\phi_n,\Phi_n)$ converges strongly to $u=\mathcal{S}(\phi,\Phi)$ in V.

Proof. For each $n\in\mathbb{N}$, Lemma 19 points out that (4.11) holds. We assert that the sequence $\{u_n\}$ is bounded in V. Inserting $v=\phi_n$ into (4.11), we have

$$\begin{aligned}
&(\mu^* m_a - m_j\|\eta\|^p)\|\phi_n-u_n\|^p\\
&\le \langle Fu_n - F\phi_n, u_n-\phi_n\rangle - \int_\Omega j^0(x,u_n(x);\phi_n(x)-u_n(x))\,dx\\
&\quad - \int_\Omega j^0(x,\phi_n(x);u_n(x)-\phi_n(x))\,dx\\
&\le -\langle F\phi_n, u_n-\phi_n\rangle - \int_\Omega j^0(x,\phi_n(x);u_n(x)-\phi_n(x))\,dx\\
&\quad + \int_{\Gamma_3} g(x)m_n(\phi_n(x))\,d\Gamma - \int_{\Gamma_3} g(x)m_n(u_n(x))\,d\Gamma\\
&\quad - \int_\Omega f_0(x)(\phi_n(x)-u_n(x))\,dx - \int_{\Gamma_2} f_2(x)(\phi_n(x)-u_n(x))\,d\Gamma.
\end{aligned}$$

It follows from Corollary 16(i), the convergence (4.12), Hölder inequality and hypothesis $H(0)$ that there exists a constant $m_4>0$ such that

$$(\mu^* m_a - m_j\|\eta\|^p)\|\phi_n-u_n\|^p \le m_4(1+\|\phi_n-u_n\|+\varepsilon_n).$$

Employing Young's inequality, we are able to find a constant $m_5>0$, which is independent of n, such that

$$\|u_n\|\le m_5,$$

where we have used the fact $\varepsilon_n\to 0$ as $0\to\infty$. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } V \text{ as } n\to\infty \tag{4.13}$$

for some $u\in V$.

We are going to demonstrate that $u=\mathcal{S}(\phi,\Phi)$. Let $w_n=\inf\{\sup\{u,\phi_n\},\Phi_n\}$, so, $w_n\in K(\phi_n,\Phi_n)$ and $w_n\to u$ in V as $n\to\infty$. Putting $v=w_n$ for (4.11), a direct computing finds

$$\begin{aligned}
&\langle Fu_n, u_n-w_n\rangle\\
&\le \int_\Omega j^0(x,u_n(x);w_n(x)-u_n(x))\,dx + \int_{\Gamma_3} g(x)[m_n(w_n(x))-m_n(u_n(x))]\,d\Gamma\\
&\quad - \int_\Omega f_0(x)(w_n(x)-u_n(x))\,dx - \int_{\Gamma_2} f_2(x)(w_n(x)-u_n(x))\,d\Gamma\\
&\le \int_\Omega j^0(x,u_n(x);w_n(x)-u_n(x))\,dx + \int_{\Gamma_3} g(x)[|w_n(x)|-|u_n(x)|]\,d\Gamma\\
&\quad + 4r_0\|g\|_{L^1(\Gamma_3)}\varepsilon_n - \int_\Omega f_0(x)(w_n(x)-u_n(x))\,dx - \int_{\Gamma_2} f_2(x)(w_n(x)-u_n(x))\,d\Gamma,
\end{aligned}$$

where we have applied Corollary 16(i). Passing to the upper limit as $n\to\infty$ for the inequality above, it yields

$$\begin{aligned}
&\limsup_{n\to\infty}\langle Fu_n, u_n-u\rangle \le \limsup_{n\to\infty}\langle Fu_n, u_n-w_n\rangle\\
&\le \limsup_{n\to\infty}\bigg[\int_\Omega j^0(x,u_n(x);w_n(x)-u_n(x))\,dx + \int_{\Gamma_3} g(x)[|w_n(x)|-|u_n(x)|]\,d\Gamma\\
&\quad + 4r_0\|g\|_{L^1(\Gamma_3)}\varepsilon_n - \int_\Omega f_0(x)(w_n(x)-u_n(x))\,dx - \int_{\Gamma_2} f_2(x)(w_n(x)-u_n(x))\,d\Gamma\bigg]\\
&\le 0.
\end{aligned}$$

This combined with (4.13) and the (S_+)–property of F concludes that $u_n \to u$ in V as $n \to \infty$. Also, it is available that $u \in K(\phi, \Phi)$.

Let $v \in K(\phi, \Phi)$ be arbitrary. Likewise, set $v_n = \inf\{\sup\{\phi_n, v\}, \Phi_n\}$. So, $v_n \in K(\phi_n, \Phi_n)$ and $v_n \to v$ in V as $n \to \infty$. Taking $v = v_n$ into (4.11) and passing to the upper limit as $n \to \infty$ for the resulting inequality, we obtain

$$\begin{aligned}&\int_\Omega \mu^*\big(a(x, \nabla u(x)), \nabla(v(x) - u(x))\big)_{\mathbb{R}^N}\,dx + \int_\Omega j^0(x, u(x); v(x) - u(x))\,dx\\&\quad + \int_{\Gamma_3} g(x)|v(x)|\,d\Gamma - \int_{\Gamma_3} g(x)|u(x)|\,d\Gamma\\&\geq \int_\Omega f_0(x)(v(x) - u(x))\,dx + \int_{\Gamma_2} f_2(x)(v(x) - u(x))\,d\Gamma.\end{aligned}$$

The arbitrariness of $v \in K(\phi, \Phi)$ points out that u is the unique solution of Problem 9 corresponding to (ϕ, Φ), i.e., $u = \mathcal{S}(\phi, \Phi)$. Since every convergent subsequence of $\{u_n\}$ converges strongly to the same limit $u = \mathcal{S}(\phi, \Phi)$. Consequently, the whole sequence $\{u_n\}$ converges strongly to $u = \mathcal{S}(\phi, \Phi)$. □

Particularly, we have the following corollary.

Corollary 21. *For any $(\phi, \Phi) \in U_{ad}$ fixed, the sequence $\{u_n\}$ with $u_n = \mathcal{S}_n(\phi, \Phi)$ converges strongly to $u = \mathcal{S}(\phi, \Phi)$ in V.*

By a careful verification, we derive the following byproduct.

Lemma 22. *The set of solutions of Problem 1, denoted by $\prod$, is weak compact in $W \times W$.*

Proof. From the definition of the cost function L, we know that the set of solutions of Problem 1 is bounded in $W \times W$, hence, $\prod$ is weakly compact in $W \times \times W$, thanks to the reflexivity of $W \times W$. We now show that $\prod$ is weakly closed. Let $\{(\phi_n, \Phi_n)\} \subset \prod$ be such that

$$(\phi_n, \Phi_n) \xrightarrow{w} (\phi, \Phi) \text{ in } W \times W \text{ as } n \to \infty.$$

The compact embedding from $W \times W$ to $V \times V$ implies

$$(\phi_n, \Phi_n) \to (\phi, \Phi) \text{ in } V \times V \text{ as } n \to \infty.$$

Arguing as in the proof of Lemma 20, we have $\mathcal{S}(\phi_n, \Phi_n) \to \mathcal{S}(\phi, \Phi)$. But, the weak lower semicontinuity of the norm $\|\cdot\|_p$ admits

$$\begin{aligned}L(\phi, \Phi) &= \frac{1}{p}\int_\Omega |\mathcal{S}(\phi, \Phi)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi\|^p_{L^p(\Omega)}\\&\leq \liminf_{n\to\infty} \frac{1}{p}\int_\Omega |\mathcal{S}(\phi_n, \Phi_n)(x) - z_d(x)|^p\,dx + \liminf_{n\to\infty}\frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)} + \liminf_{n\to\infty}\frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)}\\&\leq \liminf_{n\to\infty}\left[\frac{1}{p}\int_\Omega |\mathcal{S}(\phi_n, \Phi_n)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)}\right]\\&= \liminf_{n\to\infty} L(\phi_n, \Phi_n) = \inf_{(\tilde\phi, \tilde\Phi)\in U_{ad}} L(\tilde\phi, \tilde\Phi),\end{aligned}$$

so, $(\phi, \Phi) \in \prod$, where the last equality is gotten by using the fact that $\{(\phi_n, \Phi_n)\} \subset \prod$. Therefore, $\prod$ is weak compact in $W \times W$. □

The main result of the section concerning the existence and convergence of Problem 18 is formulated by the following theorem.

Theorem 23. *Let $(\phi^*, \Phi^*) \in U_{ad}$ be the optimal solution to Problem 1 and (ϕ_n, Φ_n) be an optimal solution to Problem 18. Then the convergences hold*

$$(\phi_n, \Phi_n) \xrightarrow{w} (\phi^*, \Phi^*) \text{ in } W \times W, \tag{4.14}$$

$$(\phi_n, \Phi_n) \to (\phi^*, \Phi^*) \text{ in } V \times V, \tag{4.15}$$

$$u_n := \mathcal{S}_n(\phi_n, \Phi_n) \to u := \mathcal{S}(\phi^*, \Phi^*) \text{ in } V, \tag{4.16}$$

$$\lim_{n\to\infty} L_n(\phi_n, \Phi_n) = L(\phi^*, \Phi^*). \tag{4.17}$$

Proof. We first show that the sequence $\{(\phi_n, \Phi_n)\} \subset U_{ad}$ is bounded in $W \times W$. For every $n \in \mathbb{N}$, one has

$$\begin{aligned} & L_n(\phi_n, \Phi_n) \le L_n(\phi^*, \Phi^*) \\ & = \frac{1}{p}\int_\Omega |\mathcal{S}_n(\phi^*, \Phi^*)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi^*\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi^*\|^p_{L^p(\Omega)}. \end{aligned} \tag{4.18}$$

However, Corollary 21 points out that $\{\mathcal{S}_n(\phi^*, \Phi^*)\}$ is bounded in V. So, from the definition of L_n, it finds that the sequence $\{(\phi_n, \Phi_n)\}$ is bounded in $W \times W$. Without loss of generality, we may assume that

$$\begin{cases} (\phi_n, \Phi_n) \xrightarrow{w} (\phi, \Phi) \text{ in } W \times W, \\ (\phi_n, \Phi_n) \to (\phi, \Phi) \text{ in } V \times V, \\ (\phi_n(x), \Phi_n(x)) \to (\phi(x), \Phi(x)) \text{ for a.e. } x \in \Omega, \end{cases} \quad \text{as } n \to \infty \tag{4.19}$$

for some $(\phi, \Phi) \in U_{ad}$ (due to the closedness of U_{ad} in $V \times V$). Additionally, employing Lemma 20 concludes that

$$u_n := \mathcal{S}_n(\phi_n, \Phi_n) \to u := \mathcal{S}(\phi, \Phi) \text{ in } V \text{ as } n \to \infty. \tag{4.20}$$

Whereas the weak lower semicontinuity of $\|\cdot\|_{L^p(\Omega)}$, and (4.19) and (4.20) confess

$$\begin{aligned} & \frac{1}{p}\int_\Omega |\mathcal{S}(\phi, \Phi)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi\|^p_{L^p(\Omega)} \\ & \quad + \delta_1\|\phi - \phi^*\|^p_{L^p(\Omega)} + \delta_2\|\Phi - \Phi^*\|^p_{L^p(\Omega)} \\ & \le \liminf_{n\to\infty} \frac{1}{p}\int_\Omega |\mathcal{S}_n(\phi_n, \Phi_n)(x) - z_d(x)|^p\,dx + \liminf_{n\to\infty} \frac{\alpha}{p}\|\Delta\phi_n\|^p_{L^p(\Omega)} \\ & \quad + \liminf_{n\to\infty} \frac{\beta}{p}\|\Delta\Phi_n\|^p_{L^p(\Omega)} + \liminf_{n\to\infty} \delta_1\|\phi - \phi^*\|^p_{L^p(\Omega)} + \liminf_{n\to\infty} \delta_2\|\Phi - \Phi^*\|^p_{L^p(\Omega)} \\ & \le \liminf_{n\to\infty} L_n(\phi_n, \Phi_n). \end{aligned} \tag{4.21}$$

On the other side, passing to the upper limit as $n \to \infty$ for the inequality (4.18), it yields

$$\begin{aligned} & \limsup_{n\to\infty} L_n(\phi_n, \Phi_n) \\ & \le \liminf_{n\to\infty} \frac{1}{p}\int_\Omega |\mathcal{S}_n(\phi^*, \Phi^*)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi^*\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi^*\|^p_{L^p(\Omega)} \\ & = \frac{1}{p}\int_\Omega |\mathcal{S}(\phi^*, \Phi^*)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi^*\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi^*\|^p_{L^p(\Omega)} \\ & = L(\phi^*, \Phi^*), \end{aligned} \tag{4.22}$$

where the first equality is obtained by using Corollary 21. Combining (4.21) and (4.22) and the fact that (ϕ^*, Φ^*) is an optimal solution to Problem 1, it finds

$$\begin{aligned} & L(\phi, \Phi) + \delta_1\|\phi - \phi^*\|^p_{L^p(\Omega)} + \delta_2\|\Phi - \Phi^*\|^p_{L^p(\Omega)} \\ & = \frac{1}{p}\int_\Omega |\mathcal{S}(\phi, \Phi)(x) - z_d(x)|^p\,dx + \frac{\alpha}{p}\|\Delta\phi\|^p_{L^p(\Omega)} + \frac{\beta}{p}\|\Delta\Phi\|^p_{L^p(\Omega)} \\ & \quad + \delta_1\|\phi - \phi^*\|^p_{L^p(\Omega)} + \delta_2\|\Phi - \Phi^*\|^p_{L^p(\Omega)} \\ & \le L(\phi^*, \Phi^*) \\ & \le L(\phi, \Phi). \end{aligned}$$

Therefore, we conclude that $(\phi, \Phi) = (\phi^*, \Phi^*)$. Recall that every convergent subsequence of $\{(\phi_n, \Phi_n)\}$ converges weakly to the same limit (ϕ^*, Φ^*) in $W \times W$, so, the whole sequence $\{(\phi_n, \Phi_n)\}$ satisfies the convergences (4.14) and (4.15).

Finally, the convergence (4.17) can be obtained directly from the estimates (4.21) and (4.22). □

5. Conclusion

In this paper, we have introduced and studied an elliptic bilateral obstacle system (EBOS, for short) involving a nonlinear and nonhomogeneous partial differential operator and a multivalued term which is described by Clarke's generalized gradient. Using the existence result [41, Theorem 8], a unique solvability theorem for (EBOS) is established. Then, a nonlinear optimal control problem governed by (EBOS) is investigated, and a sufficient theorem for the existence of optimal solutions to the obstacle control problem is obtained. Moreover, we introduced a family of approximating problems for the optimal control problem under consideration, and proved a convergence result that any sequence of solutions for the approximating problems converges to an optimal solution for the original optimal control problem.

In fact, problems of this type are encountered in contact mechanics problems. In the future we plan to explore the necessary conditions for the optimal control problem, Problem 1.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

Acknowledgments

This project has received funding from the NNSF of China Grant nos. 12001478, 12026255 and 12026256, and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement no. 823731 CONMECH, National Science Center of Poland under Preludium Project no. 2017/25/N/ST1/00611, and the Startup Project of Doctor Scientific Research of Yulin Normal University no. G2020ZK07. It is also supported by Natural Science Foundation of Guangxi Grant no. 2020GXNSFBA297137, and the Ministry of Science and Higher Education of Republic of Poland under Grants nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019.

References

[1] Bai YR, Migórski S, Zeng SD. Well-posedness of a class of generalized mixed hemivariational-variational inequalities. Nonlinear Anal RWA 2019;48:424–44.
[2] Barbu V, Korman P. Analysis and control of nonlinear infinite dimensional systems. Boston: Academic Press; 1993.
[3] Bonnans F, Casas E. An extension of Pontryagin's principle for state-constrained optimal control of semilinear elliptic equations and variational inequalities. SIAM J Control Optim 1995;33:274–98.
[4] Denkowski Z, Migórski S, Papageorgiou NS. An introduction to nonlinear analysis: theory. Boston, Dordrecht, London, New York: Kluwer Academic/Plenum Publishers; 2003.
[5] Denkowski Z, Migórski S, Papageorgiou NS. An introduction to nonlinear analysis: applications. Boston, Dordrecht, London, New York: Kluwer Academic/Plenum Publishers; 2003.
[6] Facchinei F, Pang JS. Finite-dimensional variational inequalities and complementarity problems, II. New York: Springer; 2003.
[7] Friedman A. Optimal control for variational inequalities. SIAM J Control Optim 1986;24:439–51.
[8] Friedman A. Optimal control for parabolic variational inequalities. SIAM J Control Optim 1987;25:482–97.
[9] Gasiński L, Papageorgiou NS. Nonlinear analysis, series in mathematical analysis and applications. Boca Raton, FL: Chapman & Hall/CRC; 2006.
[10] Gasiński L, Papageorgiou NS. Exercises in analysis. Part 1: nonlinear analysis,. Heidelberg: Springer; 2014.
[11] Glowinski R, Lions J-L, Trémolières R. Numerical analysis of variational inequalities. Amsterdam: North-Holland; 1981.
[12] Han W. Numerical analysis of stationary variational–hemivariational inequalities with applications in contact mechanics. Math Mech Solids 2018;23:279–93.
[13] Han W, Migórski S, Sofonea M. A class of variational-hemivariational inequalities with applications to frictional contact problems. SIAM J Math Anal 2014;46:3891–912.
[14] Han W, Sofonea M, Barboteu M. Numerical analysis of elliptic hemivariational inequalities. SIAM J Numer Anal 2017;55:640–63.
[15] Han W, Sofonea M, Danan D. Numerical analysis of stationary variational–hemivariational inequalities. Numer Math 2018;139:563–92.
[16] Han W, Zeng S. On convergence of numerical methods for variational–hemivariational inequalities under minimal solution regularity. Appl Math Lett 2019;93:105–10.
[17] Ito K, Kunisch K. Lagrange multiplier approach to variational problems and applications. Philadelphia: SIAM; 2008.
[18] Khan AA, Motreanu D. Existence theorems for elliptic and evolutionary variational and quasi-variational inequalities. J Optim Theory Appl 2015;167:1136–61.
[19] Khan AA, Sama M. Optimal control of multivalued quasi variational inequalities. Nonlinear Anal 2012;75:1419–28.
[20] Kikuchi N, Oden JT. Contact problems in elasticity: a study of variational inequalities and finite element methods. Philadelphia: SIAM; 1988.
[21] Kinderlehrer D, Stampacchia G. An introduction to variational inequalities and their applications. New York: Academic Press; 1980.
[22] Lions JL, Stampacchia G. Variational inequalities. Commun Pure Appl Math 1967;20:493–519.
[23] Liu ZH, Motreanu D, Zeng SD. Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient. Calc Var Partial Differ Equ 2019;58:22.
[24] Liu ZH, Migórski S, Zeng SD. Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces. J Differ Equ 2017;263:3989–4006.
[25] Liu ZH, Zeng SD, Motreanu D. Evolutionary problems driven by variational inequalities. J Differ Equ 2016;260:6787–99.
[26] Liu ZH, Zeng SD, Motreanu D. Partial differential hemivariational inequalities. Adv Nonlinear Anal 2018;7:571–86.
[27] Liu ZH, Motreanu D, Zeng SD. Generalized penalty and regularization method for differential variational-hemivariational inequalities. SIAM J Optim 2021;31:1158–83.
[28] Migórski S, Zeng SD. Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model. Nonlinear Anal RWA 2018;43:121–43.
[29] Migórski S, Bai YR, Zeng SD. A class of generalized mixed variational-hemivariational inequalities II: applications. Nonlinear Anal RWA 2019;50:633–50.
[30] Migórski S, Ochal A, Sofonea M. Nonlinear inclusions and hemivariational inequalities. models and analysis of contact problems, advances in mechanics and mathematics, 26. New York: Springer; 2013.
[31] Migórski S, Ochal A. Optimal control of parabolic hemivariational inequalities. J Global Optim 2000;17:285–300.
[32] Naniewicz Z, Panagiotopoulos PD. Mathematical theory of hemivariational inequalities and applications. New York, Basel, Hong Kong: Marcel Dekker, Inc; 1995.
[33] Panagiotopoulos PD. Nonconvex problems of semipermeable media and related topics. Z Angew Math Mech (ZAMM) 1985;65:29–36.
[34] Panagiotopoulos PD. Hemivariational inequalities, applications in mechanics and engineering. Berlin: Springer-Verlag; 1993.
[35] Peng ZJ. Optimal obstacle control problems involving nonsmooth cost functionals and quasilinear variational inequalities. SIAM J Control Optim 2020;58:2236–55.
[36] Peng ZJ, Kunisch K. Optimal control of elliptic variational-hemivariational inequalities. J Optim Theory Appl 2018;178:1–25.
[37] Sofonea M, Migórski S. Variational-hemivariational inequalities with applications. Boca Raton: Chapman & Hall/CRC, Monographs and Research Notes in Mathematics; 2017.
[38] Tang GJ, Huang NJ. Existence theorems of the variational–hemivariational inequalities. J Global Optim 2013;56:605–22.
[39] Zeidler E. Nonlinear functional analysis and applications, II A/B. New York: Springer; 1990.
[40] Zeng SD, Liu ZH, Migórski S. A class of fractional differential hemivariational inequalities with application to contact problem. Z Angew Math Phys 2018;69(36):23.
[41] Zeng SD, Liu ZH, Migórski S, Yao JC. Convergence of a generalized penalty method for variational-hemivariational inequalities. Commun Nonlinear Sci Numer Simul 2021;92. ID: 105476
[42] Zeng SD, Migórski S, Khan AA. Nonlinear quasi-hemivariational inequalities: existence and optimal control. SIAM J Control Optim 2021;59:1246–74.

[43] Zeng SD, Bai YR, Gasiński L, Winkert P. Existence results for double phase implicit obstacle problems involving multivalued operators. Calc Var Partial Differ Equ 2020;59:18.
[44] Zeng SD, Liu ZH, Migórski S. Positive solutions to nonlinear nonhomogeneous inclusion problems with dependence on the gradient. J Math Appl Anal 2018;463:432–48.
[45] Zhao Y, Zhang J, Yang XM, Lin GH. Expected residual minimization formulation for a class of stochastic vector variational inequalities. J Optim Theory Appl 2017;175:545–66.