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Research paper

Numerical analysis for a new kind of obstacle problem

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ABSTRACT

In this paper, we consider a new kind of obstacle problem for the elastic membrane. The obstacle is made of a rigid body covered by a soft layer that is deformable and allows penetration. It assigns a reactive normal pressure, which depends on the interpenetration of the membrane and the obstacle, during the contact process. Three equivalent descriptions of the new obstacle problem are derived, namely the energy form, the variational inequality form and the differential equation form. The existence and uniqueness of the solution are proved. Based on the variational inequality form, we derive an optimal order error estimate for the finite element approximate solution under appropriate solution regularity assumptions. We also introduce a series of penalized problems and prove its convergence result as the penalty parameter converges to infinity. Numerical examples are reported on using linear elements to solve the new obstacle problem, and the simulation results are in good agreement with the theoretical analysis.

1. Introduction

The obstacle problem is a typical variational inequality of the first kind [9,15]. It includes the obstacle problem of elastic membrane or plate, elastoplastic torsion of a cylindrical bar, a cavitation problem in hydrodynamic lubrication, the minimal surface problem and so on [7]. Here we only focus on the obstacle problem for an elastic membrane.

The way in which the obstacle problem is constructed has been rigorously described. Assume that the elastic membrane (1) passes through the boundary of a bounded domain Ω; (2) lies above an obstacle of height $\psi \in H^1(\Omega)$ with $\psi \le 0$ on $\partial\Omega$; and (3) is subject to the action of a vertical force which is proportional to $f \in L^2(\Omega)$ [15]. Let u be the vertical displacement component of the membrane. Under certain assumptions, the following three equivalent forms that the displacement u satisfy can be obtained: the energy form

$$u \in K,\ \ E(u) = \inf_{v\in K} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2\, dx - \int_\Omega f v dx \right\}, \tag{1}$$

the variational inequality form

$$u \in K,\ \ \int_\Omega \nabla u \cdot \nabla(v-u) dx \ge \int_\Omega f(v-u) dx, \quad \forall v \in K, \tag{2}$$

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and the differential equation form

$$u-\psi \geq 0, \ -\Delta u - f \geq 0, \ (u-\psi)(-\Delta u - f) = 0 \ \text{ a.e. in } \ \Omega. \tag{3}$$

Here $K = \left\{u \in H_0^1(\Omega) \mid u \geq \psi \ \text{ a.e. in } \Omega\right\}$ is the set of admissible displacements. In the past decades, mathematical theories and numerical analysis of this model, including existence, uniqueness, regularity, numerical methods and convergence results, have been established extensively and thoroughly. Details can be found, e.g., in [2–17,19,25,27].

To our best knowledge, the obstacle in most related references is perfectly rigid. In practice, there are no perfectly rigid bodies. Small penetration could occur, due to the body's and foundation's asperities [28]. Moreover, contact processes between deformable bodies abound in industry and everyday life, and many important results have been obtained in the modeling and analysis of the corresponding contact problems. In general, the contact problem describes the equilibrium state of a deformable body that (1) is subjected to the action of body forces and surface tractions; and (2) is clamped on part of its surface and in contact with an elastic or elastic-rigid foundation on another part of its surface. We refer to Sofonea and Matei [1], Han and Sofonea [21] for more details. The different contact conditions are where our main interest lies. Different from the contact problem, we consider the contact between the body and the deformable obstacle.

In this paper, we consider elastic-rigid foundation in the obstacle problem of the elastic membrane. More precisely, the elastic membrane is constrained to lie above an obstacle which is made of a rigid body covered by a layer made of soft material. The soft layer is deformable and allows penetration, so it assigns a reactive normal pressure depending on the interpenetration of the elastic membrane and the foundation. Similar to the classical obstacle problem, we give three equivalent descriptions of this kind of obstacle problem, and its well-posedness is discussed. Based on the variational inequality form, we study the numerical solution of the new obstacle problem by the finite element method and the penalty technique. The optimal order error estimate of the linear elements is obtained. We also provide numerical examples to visualize the theoretical analysis.

The rest of the paper is organized as follows. In Section 2, we present three equivalent descriptions for the new kind of obstacle problem and prove the existence and uniqueness of the solution to this problem. We consider numerical approximations in Section 3, including convergence analysis and error estimates for numerical solutions. In Section 4, we give numerical experiments to illustrate the rationality of our theoretical analysis. Finally, a summary is given in the last section.

2. Description of the new kind of obstacle problem

Let $\Omega \subset \mathbb{R}^d (d = 1, 2)$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. $H^1(\Omega)$ denotes the Sobolev space $W^{1,2}(\Omega)$ with the usual norm, $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, and $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. For a normed space X, we denote by $\|\cdot\|_X$ its norm. For a Hilbert space Y, we denote by $(\cdot,\cdot)_Y$ the inner product of two elements in Y and simply write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_Y$ when there is no confusion. Δ refers to the Laplace operator, and ∇ the gradient operator.

Given $f \in L^2(\Omega)$, non-negative function $g \in H^1(\Omega)$, and $\psi \in H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. We consider the following problem: Find the equilibrium position of an elastic membrane constrained to lie above an obstacle, with a height of ψ, made of a rigid body covered by a soft layer with thickness g. The soft layer of the obstacle is deformable and allows penetration, so it exerts a reactive normal pressure on the elastic membrane during the contact process, which depends on the penetration.

In order to derive the mathematical model, we firstly discuss the expression of the reactive normal pressure assigned by the deformed soft layer. Here we consider a commonly used normal compliance function, which can be found in many publications (see e.g. [1,22–24]), to present the reactive pressure. Define $p : \mathbb{R} \to \mathbb{R}$ by

$$p(r) = \begin{cases} c_\nu [r]_+, & \text{if } r < \eta, \\ c_\nu \eta, & \text{if } r \geq \eta, \end{cases} \tag{4}$$

where the constant $c_\nu > 0$ is stiffness coefficient and $[a]_+ = \max\{a, 0\}$. $\eta > 0$ is a constant determined by the specific problem. Obviously, the function p satisfies:

$$\begin{cases} (a) \ p : \Omega \times \mathbb{R} \to R_+. \\ (b) \ |p(x, r_1) - p(x, r_2)| \leq c_\nu |r_1 - r_2|, \ \forall r_1, r_2 \in \mathbb{R}, \ \text{ a.e. } x \in \Omega. \\ (c) \ ((p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0, \ \forall r_1, r_2 \in \mathbb{R}, \ \text{ a.e. } x \in \Omega. \\ (d) \ \text{The mapping } x \to p(x, r) \text{ is measurable on } \Omega, \text{for any } r \in \mathbb{R}. \\ (e) \ p \in C(\mathbb{R}). \end{cases} \tag{5}$$

Denote by u the vertical displacement component of the membrane, then our expression for the normal reactive pressure is

$$p(\psi - u) = \begin{cases} c_\nu [\psi - u]_+, & \text{if } \psi - u < g, \\ c_\nu g, & \text{if } \psi - u \geq g. \end{cases}$$

Indeed, at each point x, if $\psi(x) - u(x) < 0$, there is no contact and the normal pressure vanishes. When the penetration exceeds $g(x)$, the soft layer of the obstacle offers no additional resistance to penetration. Using the Riesz's representation theorem, we define the operator $G : H_0^1(\Omega) \to H_0^1(\Omega)$ by the equality

$$(Gu, v) = -\int_\Omega p(\psi - u) v dx, \ \forall u, v \in H_0^1(\Omega). \tag{6}$$

According to the property of p shown in (5), it is not hard to get that G is a monotone Lipschitz continuous operator.

Assume that the elastic membrane is clamped on the boundary of Ω and is subject to the action of a vertical force which is proportional to f. Similar to the basic assumption and discussion in [7], the total potential energy of the elastic membrane can be defined as

$$E(u) = \frac{1}{2}a(u,u) + j(u) - (f,u). \tag{7}$$

Here $j(u) = \int_\Omega \int_u^\psi p(\psi - \tau)d\tau dx$ denotes the work done by the normal reactive pressure during the actual displacement, and $a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$ is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$, for which there exist constants β_1, $\beta_2 > 0$ such that

$$|a(u,v)| \le \beta_1 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \ \forall u, v \in H_0^1(\Omega)$$

and

$$a(u,u) \ge \beta_2 \|u\|^2_{H^1(\Omega)}, \ \forall u \in H_0^1(\Omega).$$

The principle of minimal energy from mechanics asserts that the equilibrium position u is a minimizer of the total energy, i.e.

$$u \in K, \ E(u) = \inf_{v \in K} E(v) \tag{8}$$

where $K = \left\{ v \in H_0^1(\Omega) | v \ge \psi - g \text{ a.e. in } \Omega \right\}$ is the admissible displacement. This is the mathematical model to the underlying obstacle problem. Before analyzing the properties of (8), let us recall the following two results.

Theorem 2.1 (The Weierstrass theorem)**.** *Let X be a Hilbert space and K a nonempty closed convex subset of X. Let $J : K \to \mathbb{R}$ be a convex lower semicontinuous function. Then J is bounded from below and attains its infimum on K whenever one of the following two conditions hold:*

(i) K is bounded;

(ii) J is coercive.

Moreover, if J is a strictly convex function, then J attains its infimum on K at only one point.

Proposition 2.2 (Proposition 1.29 in [1])**.** *Let $(X, \|\cdot\|_X)$ be a normed space, K a nonempty closed convex subset of X and $\varphi : K \to \mathbb{R}$ a convex lower semicontinuous function. Then φ is bounded from below by an affine function, i.e. there exist $l \in X^*$ and $\alpha \in \mathbb{R}$ such that $\varphi(v) \ge l(v) + \alpha$ for all $v \in K$.*

By Theorem 2.1 and Proposition 2.2, we have the following existence and uniqueness result.

Theorem 2.3. *There exists a unique solution to (8).*

Proof. It is not hard to get that E is strictly convex and lower semicontinuous. Therefore, if K is bounded, we can immediately get the conclusion according to the Weierstrass theorem.

Assume now that K is not bounded. Since a is coercive, it follows that

$$E(v) \ge \frac{\beta_2}{2} \|v\|^2_{H^1(\Omega)} + j(v) - (f,v), \forall v \in K. \tag{9}$$

Since p is nondecreasing and continuous, we have j is convex, proper, and lower semicontinuous over $H_0^1(\Omega)$ [18]. Then Proposition 2.2 and Riesz's representation theorem guarantee that there exist $l \in H_0^1(\Omega)$ and $\alpha \in R$ such that

$$j(v) \ge (l,v) + \alpha, \ \forall v \in K. \tag{10}$$

Combining (9) and (10) and using the Cauchy-Schwarz inequality, we obtain that E is coercive. Therefore, according to Theorem 2.1, (8) has a unique solution. □

Next, we show that the solution of (8) can be characterized by a variational inequality.

Theorem 2.4. *An element u is a solution to (8) iff it satisfies*

$$u \in K, \ a(u, v-u) + (Gu, v-u) \ge (f, v-u), \quad \forall v \in K. \tag{11}$$

Proof. Assume that u is a solution to (8) and let $v \in K$ and $t \in (0,1)$. It follows that

$$E(u) \le E(u + t(v-u)),$$

then the definition of E yields

$$ta(u, v-u) + \frac{t^2}{2}a(u-v, u-v) - t(f, v-u) + j(u + t(v-u)) - j(u) \ge 0.$$

Divide the above inequality by $t > 0$ and let $t \to 0^+$, we get that

$$a(u, v-u) + (Gu, v-u) \ge (f, v-u), \ \forall v \in K.$$

Conversely, suppose that u is a solution of the variational inequality (11). That is

$$a(u, v-u) + (Gu, v-u) - (f, v-u) \geq 0, \ \forall v \in K.$$

Since j is convex, it follows that $j(v) - j(u) \geq (Gu, v-u)$. According to the definition of E and the above two inequalities, we obtain that any $v \in K$ satisfies

$$\begin{aligned} E(v) &= E(u + (v-u)) \\ &= E(u) + a(u, v-u) - (f, v-u) + j(u + (v-u)) - j(u) + \frac{1}{2}a(v-u, v-u) \\ &\geq E(u) + a(u, v-u) - (f, v-u) + (Gu, v-u) + \frac{1}{2}a(v-u, v-u) \\ &\geq E(u). \end{aligned}$$

Therefore, u is the solution of (8). □

Now we have obtained the energy form and the variational inequality form of the new kind of obstacle problem and showed their equivalence. It is possible to derive a differential equation form based on the variational inequality (11). Here we use a similar discussing procedure as in [15]. Assume that f, ψ, g all belong to $C(\Omega)$, then we get the following equivalent result.

Theorem 2.5. *If the solution u of (11) satisfies $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then u satisfies the following complementary form a.e. in Ω :*

$$\begin{cases} u \geq \psi - g, \\ -\Delta u - p(\psi - u) - f \geq 0, \\ (u - (\psi - g))(-\Delta u - p(\psi - u) - f) = 0. \end{cases} \tag{12}$$

On the contrary, if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (12) a.e. in Ω, then u must be a solution of problem (11).

Proof. Assume that u is the solution of (11) and satisfies $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Choosing $v = u + \phi$ in (11), with $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$, we obtain

$$\int_\Omega \nabla u \cdot \nabla \phi dx - \int_\Omega p(\psi - u)\phi dx - \int_\Omega f\phi dx \geq 0.$$

Hence, we can get

$$\int_\Omega (-\Delta u - p(\psi - u) - f)\phi dx \geq 0, \ \forall \phi \in C_0^\infty(\Omega), \phi \geq 0,$$

by performing integration by parts. Therefore, u must satisfy

$$-\Delta u - p(\psi - u) - f \geq 0 \text{ in } \Omega.$$

Suppose that for some $x_0 \in \Omega$ satisfies $u(x_0) > \psi(x_0) - g(x_0)$. Then there exists a neighborhood $U(x_0) \in \Omega$ of x_0 and a positive number δ such that

$$u(x) > \psi(x) - g(x) + \delta, \forall x \in U(x_0).$$

For any $\phi \in C_0^\infty(U(x_0))$ satisfying $\|\phi\|_\infty \leq 1$, applying (11) with $v = u \pm \delta\phi$ and performing integration by parts yield

$$\pm \int_\Omega (-\Delta u - p(\psi - u) - f)\phi dx \geq 0, \ \forall \phi \in C_0^\infty(U(x_0)), \|\phi\|_\infty \leq 1.$$

That is

$$\int_\Omega (-\Delta u - p(\psi - u) - f)\phi dx = 0, \ \forall \phi \in C_0^\infty(U(x_0)), \|\phi\|_\infty \leq 1.$$

Then we further get that

$$\int_\Omega (-\Delta u - p(\psi - u) - f)\phi dx = 0, \ \forall \phi \in C_0^\infty(U(x_0)).$$

Therefore, if $u(x_0) > \psi(x_0) - g(x_0), x_0 \in \Omega$, then we have

$$(-\Delta u - p(\psi - u) - f)(x_0) = 0.$$

In summary, if the solution of (11) has the regularity $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then the following complementary form holds a.e. in Ω:

$$\begin{cases} u \geq \psi - g, \\ -\Delta u - p(\psi - u) - f \geq 0, \\ (u - (\psi - g))(-\Delta u - p(\psi - u) - f) = 0. \end{cases}$$

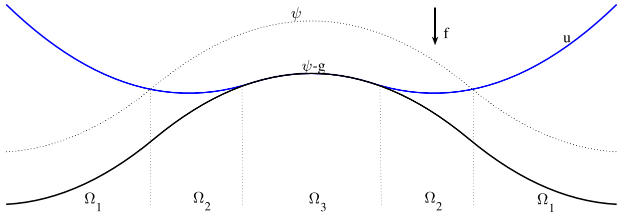


Fig. 1. A one-dimensional obstacle problem.

On the contrary, assume that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (12) a.e. in Ω. By the second inequality in (12), we have

$$\int_\Omega (-\Delta u - p(\psi - u) - f)(v - (\psi - g))dx \geq 0, \ \ \forall v \in K.$$

Then combined with

$$\int_\Omega (-\Delta u - p(\psi - u) - f)(u - (\psi - g))dx = 0,$$

we obtain

$$\int_\Omega (-\Delta u - p(\psi - u) - f)(v - u)dx \geq 0, \ \ \forall v \in K.$$

Performing integration by parts yields

$$\int_\Omega \nabla u \cdot \nabla (v - u) - \int_\Omega p(\psi - u)(v - u)dx - \int_\Omega f(v - u)dx \geq 0, \ \ \forall v \in K.$$

Therefore, u is the solution of problem (11). □

According to the complementary form (12), the domain Ω is decomposed into three parts, denoted by Ω_1, Ω_2 and Ω_3 respectively, where

$u - \psi > 0$ and $-\Delta u - f = 0$ in Ω_1,

$\psi - g < u \leq \psi$ and $-\Delta u - p(\psi - u) - f = 0$ in Ω_2,

$u = \psi - g$ and $-\Delta u - c_\nu g - f \geq 0$ in Ω_3.

Note that Ω_1 denotes the non-coincidence set where the reaction of the obstacle vanishes, and Ω_3 denotes the coincidence set of the elastic membrane and the rigid part of the obstacle, i.e. the penetration bound g is reached. In Ω_2, the membrane is in contact with the soft layer, but the penetration is less than g where the reaction of the obstacle is uniquely determined by u. As with the classical obstacle problem, we cannot predetermine the position of Ω_2 and Ω_3. In Fig. 1, a one-dimensional obstacle problem is visualized.

Remark. It can be seen that the classical obstacle problem is a special case of the problem considered in this article. When $g = 0$, no interpenetration is allowed. It is the same as the classical obstacle problem.

3. Numerical analysis

In this section, we consider numerical schemes for variational inequality (11). We first introduce the finite element approximation and derive a priori error estimates. Then, since the penalty method is an effective approach in the numerical solution of problems with constraints, we apply it to the underlying approximation problem and prove its convergence.

3.1. Finite element approximation of the variational inequality

Let $\mathcal{T}_h$ be a "classical" triangulation [18] of Ω with $h > 0$ denoting a spatial discretization parameter, and $\mathcal{N}_h$ the set of nodes of $\mathcal{T}_h$. We approximate $H_0^1(\Omega)$ by

$$V_h = \left\{ v_h \in C(\bar{\Omega}), v_h|_{\partial\Omega} = 0, v_h|_T \in P_1, \ \ \forall T \in \mathcal{T}_h \right\},$$

where P_1 denotes the space of polynomials of degree no more than 1. Obviously, V_h is a finite dimensional subspace of $H_0^1(\Omega)$. It is then quite natural to approximate K by

$$K_h = \{ v_h \in V_h | v_h(P) \geq (\psi - g)(P), \ \ \forall P \in \mathcal{N}_h \}.$$

This leads us to the finite element approximation of (11): Find $u_h \in K_h$ such that

$$a(u_h, v_h - u_h) + (Gu_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in K_h, \tag{13}$$

where

$$(Gu_h, v_h - u_h) = -\int_\Omega p(\psi - u_h) v_h dx$$

and

$$p(\psi - u_h) = \begin{cases} c_\nu[\psi - u_h]_+, & \text{if } \psi - u_h < g, \\ c_\nu g, & \text{otherwise.} \end{cases}$$

Now, we consider the operator $A_1 : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $A_1 = -\Delta + G$. It is clear that A_1 is a strongly monotone Lipschitz continuous operator. By standard arguments (see, e.g., [1]), one deduces that there exists a unique solution $u_h \in K_h$ to problem (13). In the following, we assume that $\psi - g$ is a convex function so that $K_h \subset K$.

Theorem 3.1. *Let u and u_h be the solutions of (11) and (13) respectively. Assuming $u \in H^2(\Omega)$, then*

$$\|u - u_h\|_{H^1(\Omega)} \le ch$$

where the constant $c > 0$ depends on u.

Proof. By the bilinearity and coercivity of a, we obtain

$$\begin{aligned} \beta_2 \|u - u_h\|^2_{H^1(\Omega)} &\le a(u - u_h, u - u_h) \\ &= a(u, u) + a(u_h, u_h) - a(u, u_h) - a(u_h, u). \end{aligned} \tag{14}$$

Applying (11) and $K_h \subset K$,

$$a(u, u) \le a(u, u_h) + (Gu, u_h - u) + (f, u - u_h).$$

Let $v_h \in K_h$ be arbitrary. Then, according to (13), we obtain

$$a(u_h, u_h) \le a(u_h, v_h) + (Gu_h, v_h - u_h) + (f, u_h - v_h).$$

Substituting the above two inequalities into (14) yields

$$\beta_2 \|u - u_h\|^2_{H^1(\Omega)} \le a(u_h, v_h - u) + (Gu, u_h - u) + (Gu_h, v_h - u_h) + (f, u - v_h). \tag{15}$$

By (6) and the monotonicity of G, we obtain

$$(Gu, u_h - u) + (Gu_h, v_h - u_h) = (Gu - Gu_h, u_h - u) + (Gu_h, v_h - u) \le (Gu_h, v_h - u).$$

Plugging this into (15) yields

$$\begin{aligned} \beta_2 \|u - u_h\|^2_{H^1(\Omega)} &\le a(u_h, v_h - u) + (Gu_h, v_h - u) + (f, u - v_h) \\ &= a(u_h - u, v_h - u) + R_{v_h} \end{aligned} \tag{16}$$

where $R_{v_h} := a(u, v_h - u) + (Gu_h, v_h - u) + (f, u - v_h)$. Now let us bound each term of (16). Since

$$a(u_h - u, v_h - u) \le \beta_1 \|u - u_h\|_{H^1(\Omega)} \|u - v_h\|_{H^1(\Omega)}$$

and

$$\|u - u_h\|_{H^1(\Omega)} \|u - v_h\|_{H^1(\Omega)} \le \frac{1}{2}\left(\frac{\beta_2}{\beta_1}\|u - u_h\|^2_{H^1(\Omega)} + \frac{\beta_1}{\beta_2}\|u - v_h\|^2_{H^1(\Omega)}\right),$$

we obtain

$$a(u_h - u, v_h - u) \le \frac{\beta_2}{2}\|u - u_h\|^2_{H^1(\Omega)} + \frac{\beta_1^2}{2\beta_2}\|u - v_h\|^2_{H^1(\Omega)}. \tag{17}$$

According to the assumption of $u \in H^2(\Omega)$, we have

$$a(u, v_h - u) = \int_\Omega \nabla u \cdot \nabla(v_h - u) dx = -\int_\Omega \Delta u (v_h - u) dx \le \|\Delta u\|_{L^2(\Omega)} \|v_h - u\|_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned} R_{v_h} &= a(u, v_h - u) + (Gu_h, v_h - u) + (f, u - v_h) \\ &\le (\|\Delta u\|_{L^2(\Omega)} + \|c_\nu g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \|v_h - u\|_{L^2(\Omega)}. \end{aligned} \tag{18}$$

Combining the above results (17) and (18) with (16), we get that

$$\frac{\beta_2}{2}\|u - u_h\|^2_{H^1(\Omega)} \le c \inf_{v_h \in K_h} \left\{ \|u - v_h\|^2_{H^1(\Omega)} + \|u - v_h\|_{L^2(\Omega)} \right\}.$$

Hence, we get the following Céa-type inequality

$$\|u - u_h\|_{H^1(\Omega)} \le c \inf_{v_h \in K_h} \left\{ \|u - v_h\|_{H^1(\Omega)} + \|u - v_h\|^{\frac{1}{2}}_{L^2(\Omega)} \right\}.$$

Finally, under the assumption of $u \in H^2(\Omega)$, we obtain the following optimal order error bound

$$\|u - u_h\|_{H^1(\Omega)} \le ch$$

by applying standard finite element interpolation theory (see, e.g., [20]). □

3.2. Approximation of the finite element problem by a penalty method

For simplicity, we denote $\bar{\psi} \triangleq \psi - g$ in the following. By employing a penalty method, the constrained problem (13) can be approximated as follows: Find $u_h^\gamma \in V_h$ such that

$$a(u_h^\gamma, v_h) - (p(\psi - u_h^\gamma), v_h) + (r(\gamma; u_h^\gamma), v_h) = (f, v_h), \ \ \forall v_h \in V_h, \tag{19}$$

where $\gamma > 0$ is a constant and $r(\gamma; u_h^\gamma) := -\max\left\{\gamma(\bar{\psi} - u_h^\gamma), 0\right\}$. Since r is monotonic and Lipschitz continues, the Minty-Browder theorem [26] yields the existence and uniqueness of a solution to (19) for every $\gamma > 0$.

Theorem 3.2. *The solution u_h^γ of the penalty form (19) is uniformly bounded with respect to γ, and converges to the solution u_h of (13) as $\gamma \to \infty$.*

Proof. Fix an element v_h from K_h, i.e. $v_h \geq \bar{\psi}$. According to (19), we have

$$a(u_h^\gamma, v_h - u_h^\gamma) - (p(\psi - u_h^\gamma), v_h - u_h^\gamma) + (r(\gamma; u_h^\gamma), v_h - u_h^\gamma) = (f, v_h - u_h^\gamma).$$

Since $r(\gamma; v_h) = 0$ and r is monotonous, we obtain

$$(r(\gamma; u_h^\gamma), v_h - u_h^\gamma) = -(r(\gamma; v_h) - r(\gamma; u_h^\gamma), v_h - u_h^\gamma) \leq 0.$$

Thus,

$$a(u_h^\gamma, v_h - u_h^\gamma) \geq (f, v_h - u_h^\gamma) + (p(\psi - u_h^\gamma), v_h - u_h^\gamma), \forall v_h \in K_h. \tag{20}$$

Using the coercivity of a, we get that

$$\begin{aligned}
\beta_2 \left\| u_h^\gamma \right\|_{H^1(\Omega)}^2 &\leq a(u_h^\gamma, u_h^\gamma) \leq a(u_h^\gamma, v_h) - (f, v_h - u_h^\gamma) - (p(\psi - u_h^\gamma), v_h - u_h^\gamma) \\
&\leq \beta_1 \left\| u_h^\gamma \right\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} + \left(\|f\|_{L^2(\Omega)} + \left\| p(\psi - u_h^\gamma) \right\|_{L^2(\Omega)} \right) \left\| v_h - u_h^\gamma \right\|_{L^2(\Omega)} \\
&\leq \beta_1 \left\| u_h^\gamma \right\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} + (\|f\|_{L^2(\Omega)} + \|c_v g\|_{L^2(\Omega)}) \left(\|v_h\|_{L^2(\Omega)} + \left\| u_h^\gamma \right\|_{L^2(\Omega)} \right).
\end{aligned}$$

Therefore, u_h^γ is uniformly bounded in $H_0^1(\Omega)$ with respect to γ.

Since V_h is a finite dimensional space, there is a subsequence of $\left\{u_h^\gamma\right\}$ converging to some $w_h \in V_h$. Without loss of generality, we still denote the subsequence by $\left\{u_h^\gamma\right\}$. Passing to the limit $\gamma \to \infty$ in (20), we have

$$a(w_h, v_h - w_h) - (p(\psi - w_h), v_h - w_h) \geq (f, v_h - w_h), \quad \forall v_h \in K_h.$$

Now, we are in the position to show $w_h \in K_h$. Let $v_h \in K_h$ and put $v_h - u_h^\gamma$ instead of v_h in (19), we obtain

$$a(u_h^\gamma, v_h - u_h^\gamma) - (p(\psi - u_h^\gamma), v_h - u_h^\gamma) + (r(\gamma; u_h^\gamma), v_h - u_h^\gamma) = (f, v_h - u_h^\gamma).$$

Since

$$\begin{aligned}
(r(\gamma; u_h^\gamma), v_h - u_h^\gamma) &= -\gamma(\max\left\{\bar{\psi} - u_h^\gamma, 0\right\}, v_h - \bar{\psi} - (u_h^\gamma - \bar{\psi})) \\
&\leq \gamma(\max\left\{\bar{\psi} - u_h^\gamma, 0\right\}, u_h^\gamma - \bar{\psi}) \\
&= -\gamma\left(\max\left\{\bar{\psi} - u_h^\gamma, 0\right\}, \max\left\{\bar{\psi} - u_h^\gamma, 0\right\}\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left(\max\left\{\bar{\psi} - u_h^\gamma, 0\right\}, \max\left\{\bar{\psi} - u_h^\gamma, 0\right\}\right) \leq -\frac{1}{\gamma}(r(\gamma; u_h^\gamma), v_h - u_h^\gamma) \\
&\leq \frac{1}{\gamma}(a(u_h^\gamma, v_h - u_h^\gamma) - (p(\psi - u_h^\gamma), v_h - u_h^\gamma) - (f, v_h - u_h^\gamma)) \\
&\leq \frac{1}{\gamma}(a(u_h^\gamma, v_h) - (p(\psi - u_h^\gamma), v_h - u_h^\gamma) - (f, v_h - u_h^\gamma)) \\
&\leq \frac{1}{\gamma}\left(\left\| u_h^\gamma \right\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} + (\|f\|_{L^2(\Omega)} + \|c_v g\|_{L^2(\Omega)}) \left(\|v_h\|_{L^2(\Omega)} + \left\| u_h^\gamma \right\|_{L^2(\Omega)} \right) \right).
\end{aligned}$$

By the boundedness of u_h^γ, we can conclude that

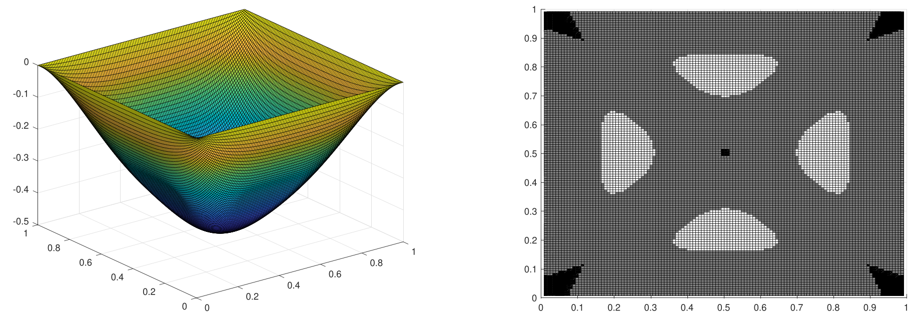
$$(\max\left\{\bar{\psi} - u_h^\gamma, 0\right\}, \max\left\{\bar{\psi} - u_h^\gamma, 0\right\}) \leq \frac{c}{\gamma}.$$

Then, we deduce that

$$\max\left\{\bar{\psi} - w_h, 0\right\} = 0 \text{ a.e. in } \Omega.$$

That is $w_h \in K_h$. Therefore, the limit $w_h = u_h$ is the unique solution of (13), and the whole sequence converges. □

Furthermore, one can prove that u_h^γ converges to the solution u of (11) as $h \to 0$ and $\gamma \to \infty$. The details are similar to [28] and we omit to discuss it here.



(a) The numerical solution. (b) The non-coincidence sets(black).

Fig. 2. The numerical results of example 1.

Table 1
The iterative data of each example.

	Example 1	Example 2	Example 3	Example 4
n	14	16	14	16
e_n	8.7269e-07	6.8182e-07	2.5893e-07	3.6584e-07
γ	3.1623e+07	3.1623e+08	3.1623e+07	3.1623e+08

4. Numerical results

In this section, we report simulation results for four numerical examples to support our theoretical analysis. All the simulations are performed on an Intel Dual-core personal computer with MatLab version R2018b. The solution of the penalized discrete problems is based on a kind of Picard iterative. We take $\Omega = (0,1) \times (0,1)$ and $c_v = 1$ for all the following numerical examples. We adopt the absolute error $e_{n+1} = \left\| u^{n+1} - u^n \right\|_{L^2(\Omega)} \leq \epsilon_0 = 10^{-6}$ as the stop criterion in the following tests, where u^n represents the result of the nth iteration. Unless otherwise specified, we set the penalty parameter as follows: give the initial penalty parameter $\gamma_0 = 50$ and keep updating $\gamma_{n+1} \leftarrow \sqrt{10}\gamma_n$ as the iteration progresses.

Example 1. This example is modified from Brézis and Sibony [4]. The problem is expressed as

$$u = \underset{v \in K}{\mathrm{Arginf}} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega Cv \; dx \right\}, \tag{21}$$

with

$$K = \left\{ v \in H_0^1(\Omega) \middle| \; v \geq \psi - g \text{ a.e. in } \Omega \right\}. \tag{22}$$

Let $f(x, y) = C = -8$, $\psi(x, y) = -\mathrm{dist}((x, y), \partial\Omega)$ and $g(x, y) = 0.1$. We show the numerical results in Fig. 2 and the corresponding iterative data in Table 1.

Example 2. This two-dimensional unilateral obstacle problem has been reported in [5,25]. The difference is that the obstacle is covered by a soft layer of thickness $g(x, y) = 0.1$. Given $\psi(x, y) = -\mathrm{dist}((x, y), \partial\Omega)$ and

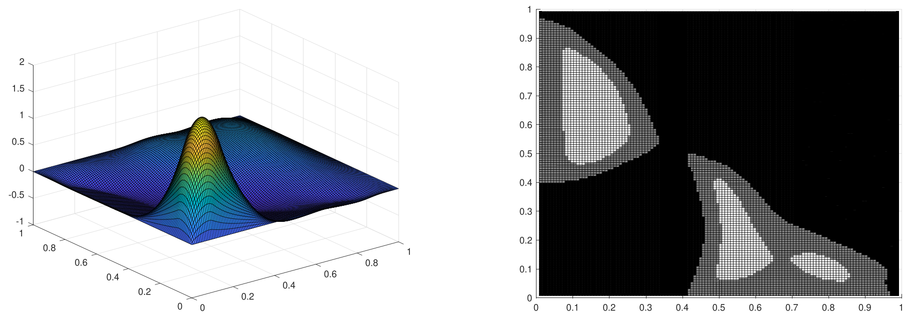
$$f(x, y) = \begin{cases} 300, & \text{if } (x, y) \in S = \{(x, y) \in \Omega : \mid x - y \mid \leq 0.1 \text{ and } x \leq 0.3\}, \\ -70e^y h(x), & \text{if } x \leq 1 - y \text{ and } (x, y) \notin S, \\ 15e^y h(x), & \text{if } x > 1 - y \text{ and } (x, y) \notin S, \end{cases}$$

where

$$h(x) = \begin{cases} 6x, & \text{if } 0 < x \leq 1/6, \\ 2(1 - 3x), & \text{if } 1/6 < x \leq 1/3, \\ 6\left(x - \frac{1}{3}\right), & \text{if } 1/3 < x \leq 1/2, \\ 2\left(1 - 3\left(x - \frac{1}{3}\right)\right), & \text{if } 1/2 < x \leq 2/3, \\ 6\left(x - \frac{2}{3}\right), & \text{if } 2/3 < x \leq 5/6, \\ 2\left(1 - 3\left(x - \frac{2}{3}\right)\right), & \text{if } 5/6 < x \leq 1. \end{cases}$$

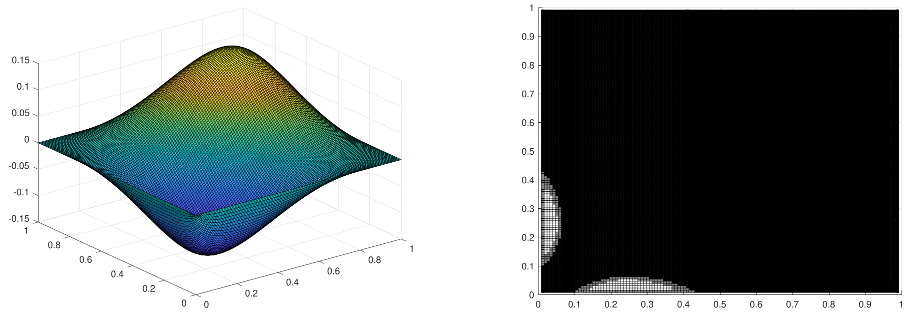
The numerical results and corresponding data are shown in Fig. 3 and Table 1, respectively.

Example 3. Let $\psi(x, y) = -\mathrm{dist}((x, y), \partial\Omega)$, $f(x, y) = 11(x + y - 1)$ [2], and set $g(x, y) = 0.001$. The corresponding results are shown in Fig. 4 and Table 1.



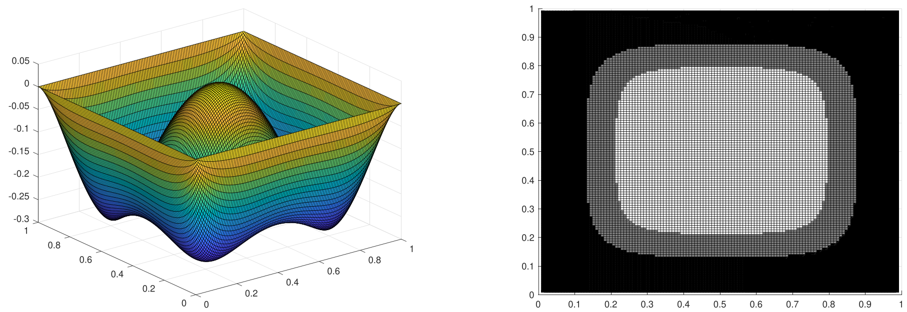
(a) The numerical solution. (b) The non-coincidence sets(black).

Fig. 3. The numerical results of example 2.



(a) The numerical solution. (b) The non-coincidence sets(black).

Fig. 4. The numerical results of example 3.



(a) The numerical solution. (b) The non-coincidence sets(black).

Fig. 5. The numerical results of example 4.

Example 4. The load and obstacle function are $f(x,y) = -20$ and $\psi(x,y) = 10x(1-x)y(1-y) - \frac{1}{2}$, respectively [8]. Let $g(x,y) = 0.1$, we show the corresponding results in Fig. 5 and Table 1.

In Figs. 2–5, (a) shows the numerical solution of each example. In (b), we show the coincidence region of the elastic membrane and the rigid part of the obstacle in white, and the non-coincidence region of the elastic membrane and the obstacle in black. The complement represents the contact region of the membrane with the soft layer of the obstacle, while the penetration does not reach the maximum value g. These four examples visually present the previous theoretical analysis and are consistent with physical reality.

5. Conclusion

The goal of this paper is to study a new obstacle problem for the elastic membrane which is constrained to lie above an elastic-rigid obstacle. Under certain assumptions, we deduced three equivalent descriptions of the problem and proved its existence and uniqueness of the solution. Based on the variational inequality form, we derived an optimal order error estimate for the finite element approximate solution. Several numerical experiments illustrated that the simulation results are in good agreement with the theoretical analysis.

In the future, we can consider some more general elastic-rigid obstacle problem. For example, we take $p(r)$ as in [21]:

$$p(r) = \begin{cases} 0, \ \text{if } r < 0, \\ r, \ \text{if } 0 \le r < 1, \\ 2 - r, \ \text{if } 1 \le r < 2, \\ \sqrt{2r-2} + r - 2, \ \text{if } 2 \le r < 6, \\ r, \ \text{if } r \ge 6, \end{cases}$$

or

$$p(r) = \begin{cases} 0, \ \text{if } r \le 0, \\ c_{\nu_1} r, \ \text{if } 0 < r \le r_{\nu_1}, \\ c_{\nu_1} r + c_{\nu_2}(r - r_{\nu_1}), \ \text{if } r_{\nu_1} < r \le r_{\nu_2}, \\ c_{\nu_1} r + c_{\nu_2}(r_{\nu_2} - r_{\nu_1}) + c_{\nu_3}(r - r_{\nu_2}), \ \text{if } r > r_{\nu_2}, \end{cases}$$

where $c_{\nu_1} > 0, c_{\nu_2} < 0, c_{\nu_3} > 0$ and $0 < r_{\nu_1} < r_{\nu_2}$ are given constants. Then hemi-variational inequalities will be obtained, and we will discuss them in the next work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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