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On the Weak Solvability Via Lagrange Multipliers for a Bingham Model

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Abstract. We study the stationary flow of an incompressible non-Newtonian fluid of Bingham type, mathematically described by means of a nonlinear boundary value problem governed by PDEs. The variational formulation which we propose is a mixed variational problem with Lagrange multipliers. First, we obtain existence, uniqueness, and stability results into an abstract framework. Then, we discuss the well-posedness of the mechanical model based on the auxiliary abstract results.

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1. Introduction

The Bingham fluid is a medium who enjoys rigidity below a critical value; above the critical value, such a medium behaves like an incompressible viscous fluid. Some clay suspensions, blood in the capillaries, mayonnaise, mustard, or tooth pastes are only a few examples of materials having a Bingham medium behavior. The mathematical form of the Bingham fluid was proposed by Eugene C. Bingham, see [1].

The flow of the Bingham fluid was studied for many decades by applied mathematicians and numerical analysts. The main progress is due to [8]. For a broad discussion about the numerical simulation of Bingham flow, we send the reader to [6]. The literature contains very much references related to the modeling and simulation of Bingham fluid flow. In addition to [6, 8], we mention herein [4, 5, 10–12, 16, 17, 19, 21–23, 26, 29], to give only a few examples.

In the present paper, we focus on a variational approach via Lagrange multipliers for a model describing the stationary flow of an incompressible non-Newtonian fluid of Bingham type, by means of the following boundary value problem.

Problem 1. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$, $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ *and* $p : \bar{\Omega} \to \mathbb{R}$, *such that:*

$$-\operatorname{Div} \boldsymbol{\sigma}'(\boldsymbol{x}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}(\boldsymbol{x}) + \nabla p(\boldsymbol{x}) = \boldsymbol{f}_0(\boldsymbol{x}) \qquad \text{in } \Omega, \tag{1}$$

$$\left.\begin{array}{ll} \boldsymbol{\sigma}'(\boldsymbol{x}) = 2\eta \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) + g\frac{\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})}{\|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3}} & \text{if } \|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3} \neq 0 \\ \|\boldsymbol{\sigma}'(\boldsymbol{x})\|_{\mathbb{S}^3} \leq g & \text{if } \|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3} = 0 \end{array}\right\} \text{ in } \Omega, \tag{2}$$

$$\operatorname{div} \boldsymbol{u}(\boldsymbol{x}) = 0 \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x}) = 0 \qquad \text{on } \Gamma_1, \tag{4}$$

$$u_\nu(\boldsymbol{x}) = 0, \ \ \boldsymbol{\sigma}_\tau(\boldsymbol{x}) = 0 \qquad \text{on } \Gamma_2. \tag{5}$$

Herein, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ partitioned in two measurable parts Γ_1 and Γ_2 with positive measure, $\boldsymbol{u}$ is the velocity, $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{\sigma}'$ is the deviatoric stress tensor ($\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{trace(\boldsymbol{\sigma})}{3}\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity tensor), p is the pressure, $\boldsymbol{f}_0$ is the density of the volume force, $\boldsymbol{\nu}$ is the outward unit normal vector at $\partial\Omega$, $\eta, g > 0$ are parameters of material, $\boldsymbol{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ where $\nabla\boldsymbol{u}^T$ is the transpose of the tensor $\nabla\boldsymbol{u}$, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, where $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$. The operator div is the divergence of a vector, e.g., $\operatorname{div} \boldsymbol{v} = \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}$ for a vector $\boldsymbol{v} \in \mathbb{R}^3$, and Div is the divergence of a tensor, e.g., $\operatorname{Div} \boldsymbol{\theta} = (\sum_{j=1}^{3} \frac{\partial \theta_{ij}}{\partial x_j})$ for a tensor $\boldsymbol{\theta} \in \mathbb{S}^3$, where $\mathbb{S}^3$ is the space of the second-order symmetric tensors. By $\cdot$ and $\| \, \|$, we denote the inner product and the Euclidean norm on $\mathbb{R}^3$, respectively, and by $:$ and $\| \, \|_{\mathbb{S}^3}$, we denote the inner product and the norm on $\mathbb{S}^3$, $\boldsymbol{\theta} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \theta_{ij}\eta_{ij}$ for $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{S}^3$, $\|\boldsymbol{\theta}\|_{\mathbb{S}^3} = \sqrt{\boldsymbol{\theta} : \boldsymbol{\theta}}$.

The weak variational formulation via Lagrange multipliers of Problem 1 drives us to a new mixed variational problem having the form below:

$$\begin{aligned} a(u, v) + b(v, \lambda) + J(u, u, v) &= (f, v)_X \quad \text{for all } v \in X, \\ b(u, \mu - \lambda) &\leq 0 \qquad\quad \text{for all } \mu \in \Lambda \subset Y. \end{aligned}$$

If the flow is very slowly, the convective term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ can be neglected. In this case, we are led to a simplified mixed variational formulation:

$$\begin{aligned} a(u, v) + b(v, \lambda) &= (f, v)_X \quad \text{for all } v \in X, \\ b(u, \mu - \lambda) &\leq 0 \qquad\quad \text{for all } \mu \in \Lambda \subset Y, \end{aligned}$$

which is a saddle point problem; we can associate it the functional $\mathcal{L} : X \times \Lambda \to \mathbb{R}$, $\mathcal{L}(v, \mu) = \frac{1}{2}a(v, v) + b(v, \mu) - (f, v)_X$ (see, e.g., [9,13] for elements of the saddle point theory related to this topic). Such an approach is convenient from the numerical point of view, allowing to apply modern numerical techniques; see, e.g., [15] and the references therein.

In the present work, we are interested on existence, uniqueness, and stability results in the case when the convective term is non-negligible. Hence, we will focus on the solvability of the mixed variational problem governed by the functional J.

The rest of the paper is structured as follows. In Sect. 2, we describe the functional setting. In Sect. 3, we deliver a weak variational formulation with

Lagrange multipliers in a dual space. Auxiliary abstract results are obtained in Sect. 4. In Sect. 5, we focus on the well-posedness of Problem 1 based on the abstract results in Sect. 4.

2. Functional Setting

In this section, we fix the functional setting. Let $2 \leq p < \infty$. The space

$$L^p(\Omega)^3 = \{\boldsymbol{v} = (v_i) \mid v_i \in L^p(\Omega),\ \ 1 \leq i \leq 3\}$$

is a real Banach space endowed with the norm:

$$\|\boldsymbol{u}\|_{L^p(\Omega)^3} = \left(\int_\Omega \sum_{i=1}^3 |u_i|^p \,\mathrm{d}x\right)^{1/p} = \left(\sum_{i=1}^3 \|u_i\|^p_{L^p(\Omega)}\right)^{1/p}.$$

Clearly:

$$\|u_i\|_{L^p(\Omega)} \leq \|\boldsymbol{u}\|_{L^p(\Omega)^3} \quad \text{for all } i \in \{1,2,3\}. \tag{6}$$

The space

$$L^p(\Omega)^{3\times 3} = \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} \in L^p(\Omega),\ \ 1 \leq i,j \leq 3\}$$

is a real Banach space endowed with the norm:

$$\|\boldsymbol{\tau}\|_{L^p(\Omega)^{3\times 3}} = \left(\int_\Omega \sum_{i,j=1}^3 |\tau_{ij}|^p \,\mathrm{d}x\right)^{1/p} = \left(\sum_{i,j=1}^3 \|\tau_{ij}\|^p_{L^p(\Omega)}\right)^{1/p}.$$

Obviously:

$$\|\tau_{ij}\|_{L^p(\Omega)} \leq \|\boldsymbol{\tau}\|_{L^p(\Omega)^{3\times 3}} \quad \text{for all } i,j \in \{1,2,3\}. \tag{7}$$

Let us set $p=2$. The space $(L^2(\Omega)^3, (\ ,\ )_{L^2(\Omega)^3}, \|\ \|_{L^2(\Omega)^3})$ is a Hilbert space, where:

$$(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^3} = \sum_{i=1}^3 \int_\Omega u_i\, v_i \,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x;$$

$$\|\boldsymbol{v}\|_{L^2(\Omega)^3} = \left(\sum_{i=1}^3 \int_\Omega v_i\, v_i \,\mathrm{d}x\right)^{1/2} = \left(\int_\Omega \|\boldsymbol{v}\|^2 \,\mathrm{d}x\right)^{1/2}.$$

Also, the space $(L^2(\Omega)^{3\times 3}, (\ ,\ )_{L^2(\Omega)^{3\times 3}}, \|\ \|_{L^2(\Omega)^{3\times 3}})$ is a Hilbert space, where

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2(\Omega)^{3\times 3}} = \sum_{i,j=1}^3 \int_\Omega \sigma_{ij}\, \tau_{ij} \,\mathrm{d}x = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\,\mathrm{d}x;$$

$$\|\boldsymbol{\tau}\|_{L^2(\Omega)^{3\times 3}} = \left(\sum_{i,j=1}^3 \int_\Omega \tau_{ij}\, \tau_{ij} \,\mathrm{d}x\right)^{1/2} = \left(\int_\Omega \|\boldsymbol{\tau}\|^2_{\mathbb{S}^3} \,\mathrm{d}x\right)^{1/2}.$$

In addition, we introduce the space:

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\tau} \in L^2(\Omega)^{3\times 3} \mid \tau_{ij} = \tau_{ji}\ \ 1 \leq i,j \leq 3\}.$$

The space $(L^2_s(\Omega)^{3\times 3}, (\ ,\ )_{L^2_s(\Omega)^{3\times 3}}, \|\ \|_{L^2_s(\Omega)^{3\times 3}})$ is also a Hilbert space, where:

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2_s(\Omega)^{3\times 3}} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega)^{3\times 3}}; \quad \|\boldsymbol{\sigma}\|_{L^2_s(\Omega)^{3\times 3}} = \|\boldsymbol{\sigma}\|_{L^2(\Omega)^{3\times 3}}.$$

Let us consider now the space

$$H^1(\Omega)^3 = \{\boldsymbol{v} = (v_i) \mid \ v_i \in H^1(\Omega),\ \ 1 \le i \le 3\}$$

endowed with the inner product:

$$(\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega)^3} = \sum_{i=1}^{3} (u_i, v_i)_{L^2(\Omega)} + \sum_{i=1}^{3} (\nabla u_i, \nabla v_i)_{L^2(\Omega)^3},$$

and the associated norm

$$\begin{aligned} \|\boldsymbol{v}\|_{H^1(\Omega)^3} &= \sqrt{\sum_{i=1}^{3} \|v_i\|^2_{L^2(\Omega)} + \sum_{i=1}^{3} \|\nabla v_i\|^2_{L^2(\Omega)^3}} \\ &= \sqrt{\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)^{3\times 3}}}. \end{aligned} \tag{8}$$

The space $(H^1(\Omega)^3, (\ ,\ )_{H^1(\Omega)^3}, \|\ \|_{H^1(\Omega)^3})$ is a Hilbert space. For scalar Lebesgue and Sobolev function spaces, we use standard notation, see, e.g., [2,18].

Let $\boldsymbol{D} : H^1(\Omega)^3 \to L^2_s(\Omega)^{3\times 3}, \quad \boldsymbol{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$. It can be easily observed that $\boldsymbol{D}$ is a linear and continuous operator. By means of the operator $\boldsymbol{D}$, we can consider on $H^1(\Omega)^3$ the following particular inner product and its associated norm:

$$\begin{aligned} ((\boldsymbol{u}, \boldsymbol{v}))_{H^1(\Omega)^3} &= (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^3} + (\boldsymbol{D}\boldsymbol{u}, \boldsymbol{D}\boldsymbol{v})_{L^2(\Omega)^{3\times 3}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)^3, \\ |||\boldsymbol{v}|||_{H^1(\Omega)^3} &= (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{D}\boldsymbol{v}\|^2_{L^2(\Omega)^{3\times 3}})^{1/2} \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^3. \end{aligned}$$

Obviously, $|||\boldsymbol{v}|||_{H^1(\Omega)^3} \le ||\boldsymbol{v}||_{H^1(\Omega)^3}$ for all $\boldsymbol{v} \in H^1(\Omega)^3$. In addition, it can be proved that there exists $c > 0$, such that, for all $\boldsymbol{v} \in H^1(\Omega)^3$, we have:

$$\int_\Omega \|\boldsymbol{v}\|^2 \,\mathrm{d}x + \int_\Omega \|\nabla \boldsymbol{v}\|^2_{\mathbb{S}^3} \,\mathrm{d}x \le c \left(\int_\Omega \|\boldsymbol{v}\|^2 \,\mathrm{d}x + \int_\Omega \|\boldsymbol{D}\boldsymbol{v}\|^2_{\mathbb{S}^3} \,\mathrm{d}x \right). \tag{9}$$

Hence, $\|\ \|_{H^1(\Omega)^3}$ and $|||\ |||_{H^1(\Omega)^3}$ are equivalent norms. Therefore, the space $(H^1(\Omega)^3, ((\ ,\ ))_{H^1(\Omega)^3}, |||\ |||_{H^1(\Omega)^3})$ is a Hilbert space too.

Let us introduce the space:

$$V_0 = \{\boldsymbol{v} \in H^1(\Omega)^3 \mid \boldsymbol{\gamma}\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$

where $\Gamma_1 \subset \partial\Omega,\ meas(\Gamma_1) > 0$ and

$$\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^r(\partial\Omega)^3 \quad (1 \le r < 4) \tag{10}$$

is the Sobolev's trace operator for vectors (a linear, continuous and compact operator), see, e.g., [18] or, more recently, Theorem 2.79 in [3] or Theorem 2.21 in [20]. The space $(V_0, (\ ,\ )_{H^1(\Omega)^3}, \|\ \|_{H^1(\Omega)^3})$ is a Hilbert space being a closed subspace of $H^1(\Omega)^3$. Indeed, let $(\boldsymbol{v}_n)_n \subset V_0$, such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $H^1(\Omega)^3$ as $n \to \infty$. Setting $r = 2$ in (10), as $\boldsymbol{\gamma}$ is a linear and continuous operator:

$$\boldsymbol{\gamma} \boldsymbol{v}_n \to \boldsymbol{\gamma} \boldsymbol{v} \ \text{ in } \ L^2(\partial\Omega)^3 \ \text{ as } \ n \to \infty. \tag{11}$$

According to, e.g., Theorem 4.9 in [2], we deduce that there exists a subsequence of $(\boldsymbol{v}_n)_n \subset V_0$, denoted herein by $(\boldsymbol{v}_{n'})_{n'}$, such that:

$$\boldsymbol{\gamma v}_{n'}(\boldsymbol{x}) \to \boldsymbol{\gamma v}(\boldsymbol{x}) \text{ a.e. on } \partial\Omega \text{ as } n' \to \infty.$$

Because $\boldsymbol{\gamma v}_{n'}(\boldsymbol{x}) = 0$ a.e. on Γ_1 we deduce that $\boldsymbol{\gamma v}(\boldsymbol{x}) = 0$ a.e. on Γ_1.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary and let $\Gamma_1 \subset \partial\Omega$ with positive measure. Then, there exists $c_K = c_K(\Omega, \Gamma_1) > 0$, such that:*

$$\|\boldsymbol{Dv}\|_{L^2_s(\Omega)^{3\times 3}} \geq c_K |||\boldsymbol{v}|||_{H^1(\Omega)^3} \quad \textit{for all } \boldsymbol{v} \in V_0. \tag{12}$$

This is a version of the Korn's inequality; see, e.g., [7,24,25,27,30] for some versions of the Korn's inequality.

We define the following inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{V_0} = \int_\Omega \boldsymbol{Du} : \boldsymbol{Dv} \, \mathrm{d}x = (\boldsymbol{Du}, \boldsymbol{Dv})_{L^2_s(\Omega)^{3\times 3}}$$

and the associated norm

$$\|\boldsymbol{v}\|_{V_0} = \left(\int_\Omega \boldsymbol{Dv} : \boldsymbol{Dv} \, \mathrm{d}x\right)^{1/2} = \|\boldsymbol{Dv}\|_{L^2_s(\Omega)^{3\times 3}}.$$

Keeping in mind Theorem 1, we deduce that $\| \ \|_{V_0}$ and $||| \ |||_{H^1(\Omega)^3}$ are equivalent norms. Consequently, $(V_0, (\ , \ )_{V_0}, \| \ \|_{V_0})$ is a Hilbert space.

Furthermore, we consider the space:

$$V_1 = \{\boldsymbol{v} \in V_0 \mid v_\nu = 0 \quad \text{a.e. on } \Gamma_2\},$$

where $v_\nu = \boldsymbol{\gamma v} \cdot \boldsymbol{\nu}$. This space is a closed subspace of V_0. So, $(V_1, (\ , \ )_{V_0}, \| \ \|_{V_0})$ is a Hilbert space; see, for instance, [30], page 88.

We introduce now the following space:

$$V = \{ \boldsymbol{v} \in V_1 \mid \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}, \tag{13}$$

endowed with the inner product $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{u}, \boldsymbol{v})_{V_0}$ and its associated norm $\|\boldsymbol{u}\|_V = \|\boldsymbol{u}\|_{V_0}$.

Proposition 1. *The space $(V, (\ , \ )_V, \| \ \|_V)$ is a Hilbert space.*

Proof. Let us consider the linear operator $B : H^1(\Omega)^3 \to L^2(\Omega)$ defined by:

$$B(\boldsymbol{v}) = \operatorname{div} \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^3.$$

The operator B is continuous. Indeed:

$$\|B\boldsymbol{v}\|_{L^2(\Omega)} = \|\operatorname{div} \boldsymbol{v}\|_{L^2(\Omega)} \leq \sum_{i=1}^3 \left\|\frac{\partial v_i}{\partial x_i}\right\|_{L^2(\Omega)} \leq \sum_{i=1}^3 \sqrt{\sum_{j=1}^3 \left\|\frac{\partial v_i}{\partial x_j}\right\|^2_{L^2(\Omega)}}$$

$$= \sum_{i=1}^3 \| |\nabla v_i| \|_{L^2(\Omega)} = \sum_{i=1}^3 \|\nabla v_i\|_{L^2(\Omega)^3}.$$

Hence:

$$\|B\boldsymbol{v}\|_{L^2(\Omega)} \leq \sqrt{3} \sqrt{\sum_{i=1}^3 \|\nabla v_i\|^2_{L^2(\Omega)^3}}.$$

Keeping in mind (8), we obtain:

$$\|B\boldsymbol{v}\|_{L^2(\Omega)} \leq \sqrt{3}\|\boldsymbol{v}\|_{H^1(\Omega)^3}.$$

Due to the linearity of the operator B, we immediately observe that V is a subspace of V_1. In addition, due to the continuity of the operator B, we can prove that V is a closed subspace of V_1. To this end in view, let $(\boldsymbol{v}_n)_n \subset V$ be such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in V_1 as $n \to \infty$. As B is a continuous operator, we obtain:

$$B\boldsymbol{v}_n \to B\boldsymbol{v} \ \text{ in } \ L^2(\Omega) \ \text{ as } \ n \to \infty. \tag{14}$$

By Theorem 4.9 in [2], we deduce that there exists a subsequence of $(\boldsymbol{v}_n)_n \subset V$, $(\boldsymbol{v}_{n'})_{n'}$, such that:

$$B\boldsymbol{v}_{n'}(\boldsymbol{x}) \to B\boldsymbol{v}(\boldsymbol{x}) \ \text{ a.e. in } \ \Omega \ \text{ as } \ n' \to \infty.$$

Because $B\boldsymbol{v}_{n'}(\boldsymbol{x}) = 0$ a.e. in Ω, we deduce that $B\boldsymbol{v}(\boldsymbol{x}) = 0$ a.e. in Ω. Therefore, $\boldsymbol{v} \in V$. Therefore, V is a closed subspace of the space V_1. As V_1 is a Hilbert space, we conclude that V is a Hilbert space too. $\square$

Using the operator $\boldsymbol{D}$ and the space V, we introduce the following space:

$$S = \boldsymbol{D}(V) = \{\boldsymbol{\tau} \in L^2_s(\Omega)^{3\times 3} \,|\, \exists\, \boldsymbol{v} \in V \text{ such that } \boldsymbol{\tau} = \boldsymbol{D}\boldsymbol{v}\}. \tag{15}$$

Proposition 2. *The space $(S, (\,,\,)_S, \|\,\|_S)$ is a Hilbert space, where $(\,,\,)_S = (\,,\,)_{L^2_s(\Omega)^{3\times 3}}$ and $\|\,\|_S = \|\,\|_{L^2_s(\Omega)^{3\times 3}}$.*

Proof. Due to the linearity of the operator $\boldsymbol{D}$, we observe that S is a subspace of $L^2_s(\Omega)^{3\times 3}$. In addition, due to the continuity of the operator $\boldsymbol{D}$, we can prove that S is a closed subspace of $L^2_s(\Omega)^{3\times 3}$. Indeed, let $(\boldsymbol{\tau}_n)_n \subset S$, such that:

$$\boldsymbol{\tau}_n \to \boldsymbol{\tau} \text{ in } L^2_s(\Omega)^{3\times 3} \text{ as } n \to \infty.$$

Let $(\boldsymbol{v}_n)_n \subset V$, such that $\boldsymbol{\tau}_n = \boldsymbol{D}\boldsymbol{v}_n$ for all $n \in \mathbb{N}$. Since $(\boldsymbol{D}\boldsymbol{v}_n)_n$ is a convergent sequence in $L^2_s(\Omega)^{3\times 3}$, then $(\boldsymbol{D}\boldsymbol{v}_n)_n$ is a Cauchy sequence too. By (12), we deduce that $(\boldsymbol{v}_n)_n$ is a Cauchy sequence in V. As V is a Hilbert space, we conclude that $(\boldsymbol{v}_n)_n$ is a convergent sequence in V. Let us denote by $\boldsymbol{v} \in V$ its limit. Therefore:

$$\boldsymbol{v}_n \to \boldsymbol{v} \text{ in } V \text{ as } n \to \infty.$$

Since $\boldsymbol{D}$ is a continuous operator, then:

$$\boldsymbol{D}\boldsymbol{v}_n \to \boldsymbol{D}\boldsymbol{v} \text{ in } L^2_s(\Omega)^{3\times 3} \text{ as } n \to \infty.$$

However:

$$\boldsymbol{D}\boldsymbol{v}_n \to \boldsymbol{\tau} \text{ in } L^2_s(\Omega)^{3\times 3} \text{ as } n \to \infty.$$

Due to the uniqueness of the limit, we obtain:

$$\boldsymbol{\tau} = \boldsymbol{D}\boldsymbol{v}.$$

Therefore, $\boldsymbol{\tau} \in S$. So, S is a closed subspace of a Hilbert space. $\square$

Everywhere below, we will denote by S' the Hilbert space which is the dual of the space S.

3. A Weak Formulation with Lagrange Multipliers

We assume that the data $\boldsymbol{f}_0$, g, η fulfill the following hypothesis:

$$(H)\ \boldsymbol{f}_0 \in L^2(\Omega)^3 \text{ and } g, \eta > 0.$$

Let $\boldsymbol{u}$, $\boldsymbol{\sigma}$, and p be regular enough functions satisfying Problem 1. For all $\boldsymbol{v} \in V$:

$$-\int_\Omega \mathrm{Div}\ \boldsymbol{\sigma}' \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega \nabla p \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x. \tag{16}$$

Keeping in mind the definition of the deviatoric tensor $\boldsymbol{\sigma}'$ and using the integration by parts formula, we deduce that, for all $v \in V$:

$$\begin{aligned} -\int_\Omega \mathrm{Div}\ \boldsymbol{\sigma}' \cdot \boldsymbol{v}\,\mathrm{d}x &= -\int_\Omega \mathrm{Div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathrm{Div}\left(\frac{trace(\boldsymbol{\sigma})}{3}\boldsymbol{I}\right) \cdot \boldsymbol{v}\,\mathrm{d}x \\ &= -\int_\Omega \mathrm{Div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\,\mathrm{d}x. \end{aligned}$$

Using again the integration by parts formula:

$$\int_\Omega \nabla p \cdot \boldsymbol{v}\,\mathrm{d}x = \int_{\partial\Omega} p\, v_\nu\, d\Gamma - \int_\Omega p\ \mathrm{div}\ \boldsymbol{v}\,\mathrm{d}x = 0. \tag{17}$$

Hence:

$$-\int_\Omega \mathrm{Div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x \quad \text{for all } \boldsymbol{v} \in V.$$

And from this, using a Green formula:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{D}\boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{\sigma\nu} \cdot \gamma\,\boldsymbol{v}\,d\Gamma \quad \text{for all } v \in V.$$

Since

$$\boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma}\,\boldsymbol{v} = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau,$$

where $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, then

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{D}\boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x &= \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x \\ + \int_{\Gamma_1} \boldsymbol{\sigma\nu} \cdot \boldsymbol{\gamma v}\,d\Gamma + \int_{\Gamma_2} (\sigma_\nu\, v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau)\,d\Gamma &\quad \text{for all } v \in V. \end{aligned}$$

Because $\boldsymbol{\gamma v} = \boldsymbol{0}$ a.e. on Γ_1, $v_\nu = 0$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$ a.e. on Γ_2, we obtain:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{D}\boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\,\mathrm{d}x.$$

On the other hand, keeping in mind the definition of the deviatoric stress tensor $\boldsymbol{\sigma}'$ and taking into account that $div\,\boldsymbol{v} = 0$ a.e. in Ω, we deduce that:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{D}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}' : \boldsymbol{D}\boldsymbol{v}\,\mathrm{d}x \text{ for all } \boldsymbol{v} \in V.$$

Therefore:

$$2\eta \int_\Omega \boldsymbol{Du} : \boldsymbol{Dv}\, \mathrm{d}x + \int_\Omega (\boldsymbol{\sigma}' - 2\eta\, \boldsymbol{Du}) : \boldsymbol{Dv}\, \mathrm{d}x + \int_\Omega (\boldsymbol{u}\cdot\nabla)\, \boldsymbol{u}\cdot\boldsymbol{v}\, \mathrm{d}x = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, \mathrm{d}x \quad \text{for all } \boldsymbol{v} \in V. \tag{18}$$

Let us define:

$$a : V\times V \to \mathbb{R}, \quad a(\boldsymbol{u},\boldsymbol{v}) = 2\eta \int_\Omega \boldsymbol{Du} : \boldsymbol{Dv}\, \mathrm{d}x, \tag{19}$$

$$J : V\times V\times V \to \mathbb{R}, \quad J(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\, \boldsymbol{v}\cdot\boldsymbol{w}\, \mathrm{d}x, \tag{20}$$

$$\boldsymbol{f} \in V, \quad (\boldsymbol{f},\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\, \mathrm{d}x \quad \text{for all } \boldsymbol{v}\in V, \tag{21}$$

$$\boldsymbol{\lambda} \in S', \quad \langle \boldsymbol{\lambda}, \boldsymbol{\tau}\rangle_{S',S} = \int_\Omega (\boldsymbol{\sigma}' - 2\eta\, \boldsymbol{Du}) : \boldsymbol{\tau}\, \mathrm{d}x \quad \text{for all } \boldsymbol{\tau}\in S, \tag{22}$$

where, by $\langle , \rangle_{S',S}$, we denote the duality product between S' and S. Also, we introduce the set of the Lagrange multipliers:

$$\Lambda = \{\boldsymbol{\mu} \in S' |\ \langle \boldsymbol{\mu}, \boldsymbol{\tau}\rangle_{S',S} \le \int_\Omega g\, \|\boldsymbol{\tau}\|_{\mathbb{S}^3}\, \mathrm{d}x \quad \text{for all } \boldsymbol{\tau}\in S\}. \tag{23}$$

The Lagrange multiplier $\boldsymbol{\lambda}$ introduced by (22) is an element of Λ. To justify this, we have to prove that:

$$\|\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{Du}(\boldsymbol{x})\|_{\mathbb{S}^3} \le g \qquad \text{a.e. } \boldsymbol{x}\in\Omega.$$

Indeed, let $\boldsymbol{x}\in\Omega$. If $\boldsymbol{Du}(\boldsymbol{x}) \neq 0$, then:

$$\|\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{Du}(\boldsymbol{x})\|_{\mathbb{S}^3} = \left\| g \frac{\boldsymbol{Du}(\boldsymbol{x})}{\|\boldsymbol{Du}(\boldsymbol{x})\|_{\mathbb{S}^3}} \right\|_{\mathbb{S}^3} = g.$$

Otherwise, if $\boldsymbol{Du}(\boldsymbol{x}) = 0$, then:

$$\|\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{Du}(\boldsymbol{x})\|_{\mathbb{S}^3} = \|\boldsymbol{\sigma}'(\boldsymbol{x})\|_{\mathbb{S}^3} \le g.$$

Thus, $\|\boldsymbol{\sigma}' - 2\eta\, \boldsymbol{Du}\|_{\mathbb{S}^3} \le g$ a.e. on Ω. Therefore, $\boldsymbol{\lambda}\in\Lambda$.

Let us define a bilinear form:

$$b : V\times S' \to \mathbb{R}, \quad b(\boldsymbol{v},\boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{Dv}\rangle_{S',S}. \tag{24}$$

Using (18)–(24), we obtain the following variational equation:

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},\boldsymbol{\lambda}) + J(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_V \quad \text{for all } \boldsymbol{v}\in V. \tag{25}$$

Furthermore, we have:

$$b(\boldsymbol{u}, \boldsymbol{\mu}-\boldsymbol{\lambda}) = b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\lambda}) = \langle \boldsymbol{\mu}, \boldsymbol{Du}\rangle_{S',S} - \langle \boldsymbol{\lambda}, \boldsymbol{Du}\rangle_{S',S}. \tag{26}$$

Obviously, for all $\boldsymbol{\mu}\in\Lambda$:

$$\langle \boldsymbol{\mu}, \boldsymbol{Du}\rangle_{S',S} \le \int_\Omega g\, \|\boldsymbol{Du}\|_{\mathbb{S}^3}\, \mathrm{d}x. \tag{27}$$

On the other hand:

$$\langle \boldsymbol{\lambda}, \boldsymbol{Du}\rangle_{S',S} = \int_\Omega g\, \|\boldsymbol{Du}\|_{\mathbb{S}^3}\, \mathrm{d}x. \tag{28}$$

To justify (28), we have to prove that, a.e. $\boldsymbol{x} \in \Omega$:

$$(\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) = g\|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3}.$$

Indeed, let $\boldsymbol{x} \in \Omega$. If $\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) = 0$, then:

$$(\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) = 0 = g\,\|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3}.$$

Otherwise, if $\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) \neq 0$, then:

$$\begin{aligned}(\boldsymbol{\sigma}'(\boldsymbol{x}) - 2\eta\, \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})) : \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) &= g\frac{\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})}{\|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3}} : \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) \\ &= g\,\|\boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})\|_{\mathbb{S}^3}.\end{aligned}$$

Consequently, by (26)–(28), we get the following variational inequality:

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda. \tag{29}$$

Thus, we obtain the following weak formulation of Problem 1.

Problem 2. *Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda \subseteq S'$, such that (25) and (29) hold true.*

4. Abstract Results

Let $(X, (\,,\,)_X, \|\,\|_X)$, $(Y, (\,,\,)_Y, \|\,\|_Y)$ be two Hilbert spaces, $a : X \times X \to \mathbb{R}$, $b : X \times Y \to \mathbb{R}$ two bilinear forms, $J : X \times X \times X \to \mathbb{R}$ a trilinear form, f a given element in X, and Λ a subset of Y.

We draw the attention to the following mixed variational problem.

Problem 3. *Given $f \in X$, find $(u, \lambda) \in X \times \Lambda$, such that:*

$$\begin{aligned} a(u, v) + b(v, \lambda) + J(u, u, v) &= (f, v)_X \quad && \textit{for all } v \in X, \\ b(u, \mu - \lambda) &\leq 0 && \textit{for all } \mu \in \Lambda \subseteq Y. \end{aligned}$$

We focus on existence, uniqueness, and stability results under the following hypothesis.

Assumption 1. (Sp) $(X, (\,,\,)_X, \|\,\|_X)$, $(Y, (\,,\,)_Y, \|\,\|_Y)$ are two real Hilbert spaces.

- (Λ) $\Lambda \subset Y$ is a closed, convex subset that contains 0_Y.
- (a) The bilinear form $a : X \times X \to \mathbb{R}$ is continuous of rank $M_a > 0$ and X-elliptic of rank $m_a > 0$.
- (J1) The trilinear form $J : X \times X \times X \to \mathbb{R}$ is continuous of rank $M_J > 0$.
- (J2) $J(u, v, w) + J(u, w, v) = 0$ for all $u, v, w \in X$.
- (b1) The bilinear form $b : X \times Y \to \mathbb{R}$ is continuous of rank $M_b > 0$.
- (b2) There exists $\alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X\, \|\mu\|_Y} \geq \alpha$.
- (i) $M_J \|f\|_X < m_a^2$.

Theorem 2. *Under Hypothesis 1, Problem 3 has an unique solution. Moreover:*

$$\|u\|_X \leq \frac{1}{m_a}\, \|f\|_X, \tag{30}$$

$$\|\lambda\|_Y \leq \frac{\|f\|_X}{\alpha} \left(1 + \frac{M_a}{m_a} + \frac{M_J}{m_a^2}\, \|f\|_X\right). \tag{31}$$

Let $0 < \varepsilon < m_a^2$ *and* $\mathcal{X} = \left\{ f \in X \,:\, \|f\|_X \leq \dfrac{m_a^2 - \varepsilon}{M_J} \right\}$. *Let* (u_1, λ_1), $(u_2, \lambda_2) \in X \times \Lambda$ *be the solutions of Problem 3 corresponding to the data* $f_1, f_2 \in \mathcal{X}$, *respectively. Then, there exists* $Q = Q(\varepsilon, m_a, M_a) > 0$, *such that:*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq Q\,\|f_1 - f_2\|_X. \tag{32}$$

Proof. Let $\rho \in X$. We consider the following auxiliary problem: given $f, \rho \in X$, find $(u_\rho, \lambda_\rho) \in X \times \Lambda$, such that:

$$a(u_\rho, v) + b(v, \lambda_\rho) + J(\rho, u_\rho, v) = (f, v)_X \quad \text{for all} \quad v \in X, \tag{33}$$

$$b(u_\rho, \mu - \lambda_\rho) \leq 0 \qquad \text{for all} \quad \mu \in \Lambda \subset Y. \tag{34}$$

Let us define $a_\rho : X \times X \to \mathbb{R}$ as follows:

$$a_\rho(u, v) = a(u, v) + J(\rho, u, v).$$

We observe that a_ρ is a continuous bilinear form of rank $M_a + M_J\|\rho\|_X$. Moreover, keeping in mind Hypothesis 1, we observe that $a_\rho(w, w) = a(w, w)$ for all $w \in X$. Hence, a_ρ is X-elliptic of rank m_a.

Because a_ρ is not a symmetric form, then the auxiliary problem is not a saddle point problem. However, according to Theorem 3.1 in [14], the problem (33)–(34) has an unique solution $(u_\rho, \lambda_\rho) \in X \times \Lambda$.

Notice that $u_\rho \leq \dfrac{1}{m_a}\|f\|_X$. This follows from (33) and (34) by setting $v = u_\rho$ and $\mu = 0_Y$. Let us introduce the set:

$$K = \left\{ v \in X \,:\, \|v\|_X \leq \frac{1}{m_a}\|f\|_X \right\},$$

which is a nonempty, closed, convex subset of X.

We define an operator $T : K \to K$, $T\rho = u_\rho$ (u_ρ is the first component of the unique pair solution of the problem (33)–(34), $(u_\rho, \lambda_\rho) \in K \times \Lambda$).

The operator T is a contraction. Indeed, let $\rho_1, \rho_2 \in K$. To simplify the writing, we make the following notation $u_{\rho_i} = u_i$, $\lambda_{\rho_i} = \lambda_i$, for $i \in \{1, 2\}$. Since:

$$a(u_i, v) + b(v, \lambda_i) + J(\rho_i, u_i, v) = (f, v)_X \qquad \text{for all } v \in X,$$

$$b(u_i, \mu - \lambda_i) \leq 0 \qquad \text{for all } \mu \in \Lambda,$$

observing that $J(\rho_1, u_1, u_1 - u_2) - J(\rho_2, u_2, u_1 - u_2) = J(\rho_1 - \rho_2, u_1, u_1 - u_2)$, we deduce $m_a\,\|u_1 - u_2\|_X^2 \leq \dfrac{M_J}{m_a}\,\|f\|_X\,\|\rho_1 - \rho_2\|_X\,\|u_1 - u_2\|_X$. Hence:

$$\|T\rho_1 - T\rho_2\|_X \leq \frac{M_J}{m_a^2}\,\|f\|_X\,\|\rho_1 - \rho_2\|_X.$$

Thus, keeping in mind Hypothesis 1 (i), we conclude that T is a contraction.

Let ρ^* be the unique fixed point of T. The pair $(u_{\rho^*}, \lambda_{\rho^*})$ is a solution of Problem 3. In fact, this is the unique solution. Indeed, let (u_1, λ_1), $(u_2, \lambda_2) \in X \times \Lambda$ be two solutions of Problem 3. Therefore:

$$a(u_1 - u_2, v) + b(v, \lambda_1 - \lambda_2) + J(u_1, u_1, v) - J(u_2, u_2, v) = 0. \tag{35}$$

However:

$$J(u_1,u_1,v)-J(u_2,u_2,v)=J(u_1,u_1-u_2,v)$$
$$-J(u_1-u_2,u_1-u_2,v)+J(u_1-u_2,u_1,v). \tag{36}$$

Setting $v=u_1-u_2$ in (35), since $b(u_1-u_2,\lambda_2-\lambda_1)\leq 0$, we obtain:

$$m_a\|u_1-u_2\|_X^2\leq \frac{M_J}{m_a}\|f\|_X\|u_1-u_2\|_X^2.$$

Keeping in mind that $m_a^2>M_J\|f\|_X$, we obtain $u_1=u_2$. Then, for all $v\in X$:

$$b(v,\lambda_1-\lambda_2)=-a(u_1-u_2,v)-[J(u_1,u_1,v)-J(u_2,u_2,v)]=0.$$

Using the inf–sup property of the form b, see Hypothesis 1 (b2), we deduce that $\|\lambda_1-\lambda_2\|_Y\leq 0$. Therefore, $\lambda_1=\lambda_2$. Therefore, Problem 3 has a unique solution $(u,\lambda)\in X\times\Lambda$, $u=u_{\rho^*}$, $\lambda=\lambda_{\rho^*}$. Since $u_{\rho^*}\in K$, we get (30). Moreover, as:

$$b(v,\lambda)=(f,v)_X-a(u,v)-J(u,u,v)$$
$$\leq\left(\|f\|_X+\frac{M_a}{m_a}\|f\|_X+\frac{M_J\,\|f\|_X^2}{m_a^2}\right)\|v\|_X,$$

we get (31) based on the inf–sup property of the form b.

Let $f_1,f_2\in\mathcal{X}$ and let (u_i,λ_i) be the corresponding solution to f_i, $i\in\{1,2\}$. For all $v\in X$:

$$a(u_1-u_2,v)+b(v,\lambda_1-\lambda_2)+J(u_1,u_1,v)-J(u_2,u_2,v)=(f_1-f_2,v)_X.$$

Setting $v=u_1-u_2$, since

$$b(u_1-u_2,\lambda_2-\lambda_1)\leq 0$$

and

$$J(u_1,u_1,u_1-u_2)-J(u_2,u_2,u_1-u_2)=J(u_1-u_2,u_1,u_1-u_2),$$

we obtain:

$$m_a\,\|u_1-u_2\|_X^2\leq\|f_1-f_2\|_X\,\|u_1-u_2\|_X+\frac{M_J}{m_a}\,\|f_1\|_X\,\|u_1-u_2\|_X^2.$$

Hence,

$$\left(m_a-\frac{M_J\,\|f_1\|_X}{m_a}\right)\|u_1-u_2\|_X^2\leq\frac{\|f_1-f_2\|_X^2}{2\,k}+\frac{k\,\|u_1-u_2\|_X^2}{2}$$

with $k>0$. Choosing $k=\dfrac{\varepsilon}{m_a}$, we are led to:

$$\|u_1-u_2\|_X\leq\frac{m_a}{\varepsilon}\,\|f_1-f_2\|_X. \tag{37}$$

On the other hand:

$$\begin{aligned} b(v,\lambda_1-\lambda_2) &\le \|f_1-f_2\|_X\,\|v\|_X + M_a\,\|u_1-u_2\|_X\,\|v\|_X \\ &\quad +|J(u_1,u_1-u_2,v)|+|J(u_1-u_2,u_2,v)| \\ &\le \|f_1-f_2\|_X\,\|v\|_X + M_a\,\|u_1-u_2\|_X\,\|v\|_X \\ &\quad +M_J\frac{\|f_1\|_X+\|f_2\|_X}{m_a}\,\|u_1-u_2\|_X\,\|v\|_X. \end{aligned}$$

Thus, by the inf–sup property of the form b, we obtain:

$$\alpha\,\|\lambda_1-\lambda_2\|_Y \le \|f_1-f_2\|_X + M_a\,\|u_1-u_2\|_X + M_J\frac{\|f_1\|_X+\|f_2\|_X}{m_a}\,\|u_1-u_2\|_X.$$

Consequently, using (37) and taking into account that $f_1,\ f_2\in\mathcal{X}$, we get:

$$\|\lambda_1-\lambda_2\|_Y \le \widetilde{c}\,\|f_1-f_2\|_X, \tag{38}$$

where $\widetilde{c}=\alpha^{-1}\left[1+\dfrac{m_a\,M_a+2(m_a^2-\varepsilon)}{\varepsilon}\right]$. Taking into account (37) and (38), we obtain (32) with: $Q=\dfrac{m_a}{\varepsilon}+\widetilde{c}$. □

5. The Well-Posedness of Problem 1

In this section, we study the well-posedness of Problem 1 based on the abstract results obtained in Sect. 4, Theorem 2.

Theorem 3. *Under the hypothesis* (H)*, if* $\boldsymbol{f}_0$ *is small enough, then Problem 1 has an unique weak solution* $(\boldsymbol{u},\boldsymbol{\lambda})\in V\times S'$*. Moreover, if* $\boldsymbol{f}_0^1,\boldsymbol{f}_0^2$ *are small enough, then there exists* $M>0$*, such that:*

$$\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_V+\|\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2\|_{S'}\le M\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3}, \tag{39}$$

where $(\boldsymbol{u}_1,\boldsymbol{\lambda}_1)$*,* $(\boldsymbol{u}_2,\boldsymbol{\lambda}_2)\in V\times S'$ *are the solutions of Problem 1 corresponding to the data* $\boldsymbol{f}_0^1$*,* $\boldsymbol{f}_0^2\in L^2(\Omega)^3$*, respectively.*

Proof. Notice that $V(=X)$ and $S'(=Y)$ are Hilbert spaces, and so, Hypothesis 1 (Sp) takes place. Obviously, a in (19) is a bilinear, continuous and X-elliptic form, with $M_a=m_a=2\,\eta$. Therefore, (a) in Hypothesis 1 holds true. Also, keeping in mind (23), it is easy to check Hypothesis 1 (Λ).

Let us show that (J1) in Hypothesis 1 holds true. Clearly, J in (20) is a trilinear form. Let us prove the continuity of J. For all $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\in V$:

$$|J(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|=\left|\int_\Omega(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x\right|\le\sum_{i,j=1}^{3}\int_\Omega|u_j|\left|\frac{\partial v_i}{\partial x_j}\right||w_i|\,\mathrm{d}x.$$

Setting $p=4$ in (18) and $\boldsymbol{\tau}=\nabla\boldsymbol{v}$ in (18), we observe that:

$$|J(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|\le 9\|\boldsymbol{u}\|_{L^4(\Omega)^3}\|\boldsymbol{w}\|_{L^4(\Omega)^3}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)^{3\times3}}.$$

Therefore:

$$\begin{aligned} |J(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le 9c_1^2\|\boldsymbol{u}\|_{H^1(\Omega)^3}\|\boldsymbol{w}\|_{H^1(\Omega)^3}\|\boldsymbol{v}\|_{H^1(\Omega)^3} \\ &\le 9c_1^2\,c\sqrt{c}|||\boldsymbol{u}|||_{H^1(\Omega)^3}|||\boldsymbol{v}|||_{H^1(\Omega)^3}|||\boldsymbol{w}|||_{H^1(\Omega)^3} \\ &\le 9c_1^2\,c\sqrt{c}\,c_K^{-3}\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V, \end{aligned}$$

where $c_1 > 0$ is the constant corresponding to the continuous embedding $H^1(\Omega)^3 \subset L^4(\Omega)^3$, $c > 0$ is the constant in (9), and $c_K > 0$ is the constant in the Korn's inequality in Theorem 1. We can choose $M_J = 9c_1^2\, c\sqrt{c}\, c_K^{-3}$.

To proceed, we check the validity of (J2) in Hypothesis 1:

$$
\begin{aligned}
J(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) + J(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) &= \sum_{i,j=1}^{3} \int_\Omega \left(u_j \frac{\partial v_i}{\partial x_j} w_i + u_j \frac{\partial w_i}{\partial x_j} v_i \right) \mathrm{d}x \\
&= \sum_{i,j=1}^{3} \int_\Omega u_j \frac{\partial}{\partial x_j}(v_i\, w_i)\, \mathrm{d}x \\
&= \sum_{i,j=1}^{3} \int_{\partial\Omega} u_j\, v_i\, w_i\, \nu_j\, d\Gamma - \sum_{i,j} \int_\Omega \frac{\partial u_j}{\partial x_j}\, v_i\, w_i\, \mathrm{d}x \\
&= \int_{\partial\Omega} u_\nu \boldsymbol{\gamma v} \cdot \boldsymbol{\gamma w}\, d\Gamma - \int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}\ \mathrm{div}\ \boldsymbol{u}\, \mathrm{d}x = 0.
\end{aligned}
$$

Remark that the term $\int_{\partial\Omega} u_\nu \boldsymbol{\gamma v}\cdot\boldsymbol{\gamma w}\, d\Gamma$ is well defined. Indeed, setting $r = 3$ in (10), we have $\boldsymbol{\gamma u}$, $\boldsymbol{\gamma v}$, $\boldsymbol{\gamma w} \in L^3(\partial\Omega)^3$. Since $\boldsymbol{\nu} \in L^\infty(\partial\Omega)^3$, then $(\boldsymbol{\gamma u}\cdot\boldsymbol{\nu})(\boldsymbol{\gamma v}\cdot\boldsymbol{\gamma w}) \in L^1(\partial\Omega)$.

It is easy to see that the form b in (24) is bilinear and continuous with $M_b = 1$. Thus, Hypothesis 1 (b1) is fulfilled. Moreover, as:

$$
\begin{aligned}
\|\boldsymbol{\mu}\|_{S'} &= \sup_{\boldsymbol{\tau}\in S,\, \boldsymbol{\tau}\neq 0_S} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\tau}\rangle_{S',S}}{\|\boldsymbol{\tau}\|_S} \\
&\leq \sup_{\boldsymbol{v}\in V,\, \boldsymbol{v}\neq 0_V} \frac{\langle \boldsymbol{\mu}, \boldsymbol{Dv}\rangle_{S',S}}{\|\boldsymbol{Dv}\|_S} \\
&= \sup_{\boldsymbol{v}\in V,\, \boldsymbol{v}\neq 0_V} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V}.
\end{aligned}
$$

Hence:

$$
\inf_{\boldsymbol{\mu}\in S',\, \boldsymbol{\mu}\neq 0_{S'}} \sup_{\boldsymbol{v}\in V,\, \boldsymbol{v}\neq 0_V} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_{S'}} \geq 1.
$$

Let us take $\alpha = 1$. Therefore, (b2) in Hypothesis 1 holds true. Therefore, due to Theorem 2, Problem 1 has an unique weak solution.

Due to (21), if $\boldsymbol{f}_0$ is small enough, then $\|\boldsymbol{f}\|_V$ is small enough, such that (i) in Hypothesis 1 is fulfilled.

Let us define now $\boldsymbol{f}_i \in V$, such that:

$$
(\boldsymbol{f}_i, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0^i \cdot \boldsymbol{v}\, \mathrm{d}x \quad \text{for all } \boldsymbol{v}\in V,\ i\in\{1,2\}.
$$

Since

$$\begin{aligned}\frac{(\boldsymbol{f}_1-\boldsymbol{f}_2,\boldsymbol{v})_V}{\|\boldsymbol{v}\|_V} &= \frac{\int_\Omega(\boldsymbol{f}_0^1-\boldsymbol{f}_0^2)\cdot\boldsymbol{v}\,\mathrm{d}x}{\|\boldsymbol{v}\|_V} \leq \frac{\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3}\|\boldsymbol{v}\|_{L^2(\Omega)^3}}{\|\boldsymbol{v}\|_V}\\ &\leq \frac{\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3}|||\boldsymbol{v}|||_{H^1(\Omega)^3}}{\|\boldsymbol{v}\|_V}\\ &\leq \frac{c_K^{-1}\,\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3}\|\boldsymbol{D}\boldsymbol{v}\|_{L_s^2(\Omega)^{3\times 3}}}{\|\boldsymbol{v}\|_V}\\ &= c_K^{-1}\,\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3},\end{aligned}$$

we get

$$\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_V \leq c_K^{-1}\,\|\boldsymbol{f}_0^1-\boldsymbol{f}_0^2\|_{L^2(\Omega)^3}.$$

Using now Theorem 2, we obtain immediately (39), with $M = c_K^{-1}\,Q$. □

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