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Optimal control for a class of mixed variational problems

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Abstract. The present paper concerns a class of abstract mixed variational problems governed by a strongly monotone Lipschitz continuous operator. With the existence and uniqueness results in the literature for the problem under consideration, we prove a general convergence result, which shows the continuous dependence of the solution with respect to the data by using arguments of monotonicity, compactness, lower semicontinuity and Mosco convergence. Then we consider an associated optimal control problem for which we prove the existence of optimal pairs. The mathematical tools developed in this paper are useful in the analysis and control of a large class of boundary value problems which, in a weak formulation, lead to mixed variational problems. To provide an example, we illustrate our results in the study of a mathematical model which describes the equilibrium of an elastic body in frictional contact with a foundation.

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1. Introduction

Mixed variational problems are widely used in the study of many nonlinear boundary value problems with or without unilateral constraints. Their numerical treatment is efficient and accurate, and for this reason, mixed variational formulations have a large number of applications in engineering sciences and, in particular, in solid and contact mechanics. The literature in the field has been growing rapidly in the last decades. Existence and uniqueness results in the study of stationary mixed variational problems with multipliers, together with various applications in solid mechanics, can be found in [4,5,7,9,12,30,35,46]. References concerning the analysis, numerical treatment and applications in the contact problems of mixed variational problems and their related problems include [2,10,11,13,15,16,18,27,32,37,38,43,49].

The optimal control theory deals with the existence and, when possible, the uniqueness of optimal pairs, as well as their characterization via optimality conditions and their numerical approximation. Optimal control problems for variational and hemivariational inequalities have been discussed in several works, including [3,6,19,23,25,26,29,36,44,45]. Due to their important industrial applications, during the last years, many authors payed attention to the optimal control of mathematical models which arise in contact mechanics in the framework of both variational and hemivariational inequalities. References in the field include [1,22,24] and [33,47], respectively. In contrast, the literature of optimal control of mixed variational problems is quite poor and the need for results in this topic is currently widely recognized.

The first aim of this paper is to contribute to filling this gap. Indeed, here we state and prove the existence of optimal pairs for an optimal control problem associated with a general class of mixed variational problems in Hilbert spaces. The setting is quite general, and its novelty arises in the fact that the control can be involved in all the data of the mixed variational problem. The existence of the optimal pairs is based on an abstract convergence result we state and prove here, which is new and has interest in its own.

Our second aim in this paper is to illustrate the use of our abstract existence result in the study of optimal control problems in contact mechanics. Due to its generality, the existence result we obtain in the present paper can be applied to a wide class of optimal control problems associated with a given mathematical model of contact. Its flexibility consists in the fact that we can choose as control a large number of variables. Among them we mention the density of surface traction, the friction bound and the elasticity coefficients.

The rest of the paper is organized as follows: In Sect. 2 we introduce the state problem, governed by a nonlinear operator A, a bilinear form b, a set of constraints Λ and given data f. Then, we briefly recall its unique solvability in the literature. In Sect. 3 we provide a convergence result which shows the continuous dependence of the solution with respect to the data A, b, f and Λ of the problem. Then, in Sect. 4 we consider an associate optimal control problem for which we prove the existence of the optimal pairs, which is based on Weierstrass-type arguments. Finally, in Sect. 5 we illustrate the application of our abstract results in the study of a mathematical model which describes the equilibrium of an elastic body in frictional contact with a foundation. In particular, we consider three optimal control problems associated with the model and, for each problem, we provide the existence of the optimal pairs and present the corresponding mechanical interpretations.

We end this section with the following version of the Weierstrass theorem.

Theorem 1. *Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, K a nonempty weakly closed subset of X and $J : X \to \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that either K is bounded or J is coercive, i.e., $J(v) \to \infty$ as $\|v\|_X \to \infty$. Then, there exists at least one element u such that*

$$u \in K, \qquad J(u) \leq J(v) \qquad \forall\, v \in K. \tag{1.1}$$

Theorem 1 will be used in Sect. 4 of the paper in order to prove the existence of optimal pairs. Its proof is based on standard arguments which can be found in many books and survey as, for instance, [17,34].

2. The state problem

Everywhere in this paper we assume that X, Y and Z are real Hilbert spaces endowed with the inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ and $(\cdot,\cdot)_Z$. The associated norms will be denoted by $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Moreover, 0_X and 0_Y will represent the zero elements of the spaces X and Y, and $X \times Y$ is their product space endowed with the canonical inner product. A typical element of $X \times Y$ will be denoted by (u, λ). Consider two operators $A : X \to X$ and $\pi : X \to Z$, a form $b : X \times Y \to \mathbb{R}$, a set $\Lambda \subset Y$ and an element $f \in Z$. We associate with these data the following mixed variational problem.

Problem 2. *Find $u \in X$ and $\lambda \in \Lambda$ such that*

$$(Au, v)_X + b(v, \lambda) = (f, \pi v)_Z \qquad \forall\, v \in X, \tag{2.1}$$

$$b(u, \mu - \lambda) \leq 0 \qquad \forall\, \mu \in \Lambda. \tag{2.2}$$

In the study of this problem we consider the following assumptions, some of which could be found in [48].

$$\begin{cases} A : X \to X \text{ is a strongly monotone Lipschitz continuous operator, i.e.,} \\ \text{there exist } m > 0 \text{ and } L > 0 \text{ such that} \\ \text{(a)} \quad (Au - Av, u - v)_X \geq m\|u - v\|_X^2 \quad \forall\, u,\, v \in X; \\ \text{(b)} \quad \|Au - Av\|_X \leq L\,\|u - v\|_X \quad \forall\, u,\, v \in X. \end{cases} \tag{2.3}$$

$$\begin{cases} b: X\times Y\to\mathbb{R} \text{ is a bilinear and continuous form which satisfies} \\ \text{the inf-sup condition, i.e., there exist } M>0 \text{ and } \alpha>0 \text{ such that} \\ \text{(a)}\quad |b(v,\mu)|\le M\|v\|_X\|\mu\|_Y \qquad \forall v\in X,\ \mu\in Y; \\ \text{(b)}\quad \inf\limits_{\mu\in Y,\mu\neq 0_Y}\ \sup\limits_{v\in X, v\neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X\|\mu\|_Y}\ge\alpha. \end{cases} \tag{2.4}$$

$$\begin{cases} \pi \text{ is a linear continuous operator, i.e., there exists } c_0>0 \text{ such that} \\ \|\pi v\|_Z\le c_0\,\|v\|_X \quad \forall\, v\in X. \end{cases} \tag{2.5}$$

$$\Lambda \text{ is a closed convex subset of } Y \text{ such that } 0_Y\in\Lambda. \tag{2.6}$$

Moreover, recall that

$$f\in Z. \tag{2.7}$$

We have the following existence and uniqueness result, which guarantees the unique solvability of Problem 2.

Theorem 3. *Assume that* (2.3)–(2.7) *hold. Then, Problem* 2 *has a unique solution* $(u,\lambda)\in X\times\Lambda$.

Theorem 3 represents a slightly modified version of Theorem 5.2 in [21] (the case when Λ is an unbounded subset) and Theorem 2.1 in [20] (the case when Λ is bounded). Its proof is carried out in several steps, based on arguments of saddle points and the Banach fixed point theorem.

3. A convergence result

The solution (u,λ) obtained in Theorem 3 depends on A, b, f and Λ. In this section we state and prove a convergence result of this solution with respect to these data, which represents a crucial ingredient in the study of the optimal control problem we shall consider in Sect. 4. Unless stated otherwise, all the limits, upper and lower limits, below are considered as $n\to\infty$, even if we do not mention it explicitly. The symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in various spaces which will be specified. Nevertheless, for simplicity, we write $g_n\to g$ for the convergence in $\mathbb{R}$.

The functional framework is as follows. For each $n\in\mathbb{N}$ we consider an operator A_n, a form b_n, an element f_n and a set Λ_n which satisfy assumptions (2.3), (2.4), (2.6) and (2.7), respectively, with constants m_n, L_n, M_n, α_n. To avoid any confusion, when used with n, we refer to these assumptions as assumptions $(2.3)_n$, $(2.4)_n$, $(2.6)_n$ and $(2.7)_n$. Then, if condition (2.5) is satisfied, we deduce from Theorem 3 that for each $n\in\mathbb{N}$ there exists a unique solution u_n for the following mixed variational problem.

Problem 4. *Find* $u_n\in X$ *and* $\lambda_n\in\Lambda_n$ *such that*

$$(A_nu_n,v)_X+b_n(v,\lambda_n)=(f_n,\pi v)_Z \qquad \forall\, v\in X, \tag{3.1}$$

$$b_n(u_n,\mu-\lambda_n)\le 0 \qquad \forall\,\mu\in\Lambda_n. \tag{3.2}$$

We now consider the following additional assumptions.

$$\begin{cases} \text{For any } n\in\mathbb{N} \text{ there exist } F_n\ge 0 \text{ and } \delta_n\ge 0 \text{ such that} \\ \text{(a)}\ \|A_nv-Av\|_X\le F_n(\|v\|_X+\delta_n) \quad \text{for all } v\in X; \\ \text{(b)}\ \lim\limits_{n\to 0} F_n=0; \\ \text{(c) the sequence } \{\delta_n\}\subset\mathbb{R} \text{ is bounded.} \end{cases} \tag{3.3}$$

$$\text{There exists } m_0>0 \text{ such that } m_n\ge m_0 \ \ \forall\, n\in\mathbb{N}. \tag{3.4}$$

$$\begin{cases} \text{For all sequences } \{z_n\}\subset X,\ \{\mu_n\}\subset Y \text{ such that} \\ z_n\rightharpoonup z \text{ in } X,\ \mu_n\rightharpoonup\mu \text{ in } Y, \text{ we have} \\ \limsup b_n(w-z_n,\mu_n)\le b(w-z,\mu) \quad \forall\, w\in X. \end{cases} \tag{3.5}$$

$$\text{There exists } \alpha_0>0 \text{ such that } \alpha_n\ge\alpha_0 \ \ \forall\, n\in\mathbb{N}. \tag{3.6}$$

$$\begin{cases} \text{For all sequence } \{v_n\} \subset X \text{ such that} \\ v_n \rightharpoonup v \text{ in } X, \text{ we have } \quad \pi v_n \to \pi v \quad \text{in} \quad Y. \end{cases} \tag{3.7}$$

$$\begin{cases} \{\Lambda_n\} \text{ converges to } \Lambda \text{ in the sense of Mosco, i.e.,} \\ \text{(a) for each } \mu \in \Lambda \text{ there exists a sequence } \{\mu_n\} \text{ such that} \\ \qquad \mu_n \in \Lambda_n \text{ for each } n \in \mathbb{N} \text{ and } \mu_n \to \mu \text{ in } Y; \\ \text{(b) for each sequence } \{\mu_n\} \text{ such that} \\ \qquad \mu_n \in \Lambda_n \text{ for each } n \in \mathbb{N} \text{ and } \mu_n \rightharpoonup \mu \text{ in } Y, \text{ we have } \mu \in \Lambda. \end{cases} \tag{3.8}$$

$$f_n \rightharpoonup f \quad \text{in} \quad Z. \tag{3.9}$$

Note that assumption (3.7) shows that the linear operator $\pi : X \to Y$ is completely continuous. Details on the convergence of sets in the sense of Mosco can be found in [28]. Such convergence was used in the recent paper [47] in the study of convergence results for elliptic and history variational–hemivariational inequalities, respectively.

The main result of this section is the following.

Theorem 5. *Assume (2.3)–(2.7) and, for each $n \in \mathbb{N}$, assume $(2.3)_n$, $(2.4)_n$, $(2.6)_n$, and $(2.7)_n$. Moreover, assume (3.3)–(3.9) and denote by (u_n, λ_n) and (u, λ) the solutions of Problems 4 and 2, respectively. Then the following convergences hold:*

$$u_n \to u \qquad \text{in} \quad X, \tag{3.10}$$

$$\lambda_n \rightharpoonup \lambda \qquad \text{in} \quad Y. \tag{3.11}$$

The proof of Theorem 5 will be carried out in several steps that we present in what follows. Everywhere below we assume that the hypotheses of Theorem 5 hold. The first step of the proof is the following.

Lemma 6. *There exist a pair $(\widetilde{u}, \widetilde{\lambda}) \in X \times Y$ and a subsequence of the sequence $\{(u_n, \lambda_n)\}$, still denoted by $\{(u_n, \lambda_n)\}$, such that $u_n \rightharpoonup \widetilde{u}$ in X and $\lambda_n \rightharpoonup \widetilde{\lambda}$ in Y.*

Proof. We first establish the boundedness of $\{u_n\}$ in X. Let $n \in \mathbb{N}$. We use assumption $(2.6)_n$ and test in (3.2) with $\mu = 0_Y$ to obtain that

$$b_n(u_n, \lambda_n) \geq 0.$$

We now take $v = u_n$ in (3.1) and use the previous inequality to see that

$$(A_n u_n, u_n)_X \leq (f_n, \pi u_n)_Z.$$

Next, we write $A_n u_n = A_n u_n - A_n 0_X + A_n 0_X$ and use assumptions $(2.3)_n(a)$ and (2.5) to deduce that

$$m_n \|u_n\|_X \leq c_0 \|f_n\|_Z + \|A_n 0_X\|_X. \tag{3.12}$$

On the other hand, writing $A_n 0_X = A_n 0_X - A 0_X + A 0_X$ and using inequality (3.3) (a) yield

$$\|A_n 0_X\|_X \leq F_n \delta_n + \|A 0_X\|_X. \tag{3.13}$$

We now combine inequalities (3.12) and (3.13) and use assumption (3.4) to see that

$$\|u_n\|_X \leq \frac{1}{m_0} \big(c_0 \|f_n\|_Z + F_n \delta_n + \|A 0_X\|_X \big).$$

Finally, we use assumptions (3.3)(b, c) and (3.9) to deduce that the sequence $\{u_n\}$ is bounded, i.e., there exists $K > 0$ which does not depend on n such that

$$\|u_n\|_X \leq K. \tag{3.14}$$

Next, we establish the boundedness of $\{\lambda_n\}$ in Y. To this end, we use (2.1) and assumption (2.5) to see that

$$b_n(v, \lambda_n) = (f_n, \pi v)_Z - (A_n u_n, v)_X \leq \big(c_0 \|f_n\|_Z + \|A_n u_n\|_X \big) \|v\|_X$$

for all $v \in X$, which implies that

$$\sup_{v\in X, v\neq 0_X} \frac{b_n(v,\lambda_n)}{\|v\|_X\|\lambda_n\|_Y} \le \frac{1}{\|\lambda_n\|_Y}\big(c_0\|f_n\|_Z + \|A_nu_n\|_X\big),$$

if $\lambda_n \neq 0_Y$. Therefore,

$$\inf_{\mu\in Y, \mu\neq 0_Y}\ \sup_{v\in X, v\neq 0_X} \frac{b_n(v,\mu)}{\|v\|_X\|\mu\|_Y} \le \frac{1}{\|\lambda_n\|_Y}\big(c_0\|f_n\|_Z + \|A_nu_n\|_X\big),$$

if $\lambda_n \neq 0_Y$. We now use assumption (2.4)$_n$ on the bilinear form b_n together with bound (3.6) to deduce that

$$\alpha_0\|\lambda_n\|_Y \le c_0\|f_n\|_Z + \|A_nu_n\|_X, \tag{3.15}$$

both when $\lambda_n \neq 0_Y$ and when $\lambda_n = 0_Y$.

Next, we use assumptions (3.3)(a) and (2.3)(b) to see that

$$\begin{aligned}\|A_nu_n\|_X &\le \|A_nu_n - Au_n\|_X + \|Au_n\|_X\\ &\le F_n(\|u_n\|_X + \delta_n) + \|Au_n - A0_X\|_X + \|A0_X\|_X\\ &\le F_n(\|u_n\|_X + \delta_n) + L\|u_n\|_X + \|A0_X\|_X.\end{aligned}$$

We now use assumptions (3.3)(b, c) and bound (3.14) to see that the sequence $\{A_nu_n\}$ is bounded in X. Using this result and assumption (3.9), inequality (3.15) implies that the sequence $\{\lambda_n\}$ is bounded in Y, i.e., there exists $P > 0$ which does not depend on n such that

$$\|\lambda_n\|_Y \le P. \tag{3.16}$$

Lemma 6 is now a direct consequence of inequalities (3.14) and (3.16) combined with a standard reflexivity argument. □

The second step in the proof is given by the following convergence result.

Lemma 7. *Assume that* $u_n \rightharpoonup \widetilde{u}$ *in* X *and* $\lambda_n \rightharpoonup \widetilde{\lambda}$ *in* Y. *Then* $u_n \to \widetilde{u}$ *in* X.

Proof. Let $n \in \mathbb{N}$. We test in (3.1) with $v = u_n - \widetilde{u}$ to obtain that

$$(A_nu_n, u_n - \widetilde{u})_X + b_n(u_n - \widetilde{u}, \lambda_n) = (f_n, \pi u_n - \pi\widetilde{u})_Z$$

and, therefore,

$$(A_nu_n - A_n\widetilde{u}, u_n - \widetilde{u})_X = (f_n, \pi u_n - \pi\widetilde{u})_Z + (A_n\widetilde{u}, \widetilde{u} - u_n)_X + b_n(\widetilde{u} - u_n, \lambda_n).$$

We now use assumptions (2.3)$_n$(a) and (3.4) to see that

$$m_0\|u_n - \widetilde{u}\|_X^2 \le (f_n, \pi u_n - \pi\widetilde{u})_Z + (A_n\widetilde{u}, \widetilde{u} - u_n)_X + b_n(\widetilde{u} - u_n, \lambda_n). \tag{3.17}$$

Note that the convergences $f_n \rightharpoonup f$ in Z, $u_n \rightharpoonup \widetilde{u}$ in X and assumption (3.7) imply that

$$(f_n, \pi u_n - \pi\widetilde{u})_Z \to 0. \tag{3.18}$$

On the other hand, using (3.3)(a) we find that

$$\begin{aligned}(A_n\widetilde{u}, \widetilde{u} - u_n)_X &= (A_n\widetilde{u} - A\widetilde{u}, \widetilde{u} - u_n)_X + (A\widetilde{u}, \widetilde{u} - u_n)_X\\ &\le \|A_n\widetilde{u} - A\widetilde{u}\|_X\|u_n - \widetilde{u}\|_X + (A\widetilde{u}, \widetilde{u} - u_n)_X\\ &\le F_n(\|\widetilde{u}\|_X + \delta_n)\|u_n - \widetilde{u}\|_X + (A\widetilde{u}, \widetilde{u} - u_n)_X.\end{aligned}$$

We now pass to the upper limit in this inequality and use assumptions (3.3)(b, c) and the convergence $u_n \rightharpoonup \widetilde{u}$ in X to see that

$$\limsup\, (A_n\widetilde{u}, \widetilde{u} - u_n)_X \le 0. \tag{3.19}$$

Next, taking $z_n = u_n$, $\mu_n = \lambda_n$ and $w = \widetilde{u}$ in (3.5) yields

$$\limsup\, b_n(\widetilde{u} - u_n, \lambda_n) \leq 0. \tag{3.20}$$

We now pass to the upper limit in inequality (3.17) and use (3.18)–(3.20) to deduce that

$$\limsup\, m_0\|u_n - \widetilde{u}\|_X^2 \leq 0$$

which concludes the proof. □

Lemma 8. *The pair* $(\widetilde{u}, \widetilde{\lambda})$ *is a solution of Problem 2.*

Proof. We first recall that for each $n \in \mathbb{N}$ we have $\lambda_n \in \Lambda_n$. Moreover, using Lemma 6 it follows that passing to a subsequence, still denoted $\{\lambda_n\}$, we have $\lambda_n \rightharpoonup \widetilde{\lambda}$ in Y. Therefore, assumption (3.8)(b) implies that

$$\widetilde{\lambda} \in \Lambda. \tag{3.21}$$

Let $n \in \mathbb{N}$ and $v \in X$. We use assumptions (3.3)(a) and (2.3)(b) to see that

$$\begin{aligned}\|A_n u_n - A\widetilde{u}\|_X &\leq \|A_n u_n - Au_n\|_X + \|Au_n - A\widetilde{u}\|_X\\ &\leq F_n(\|u_n\|_X + \delta_n) + L\,\|u_n - \widetilde{u}\|_X\end{aligned}$$

and, therefore, assumptions (3.3)(b),(c) and Lemma 7 imply that

$$A_n u_n \to A\widetilde{u} \quad \text{in} \quad X. \tag{3.22}$$

On the other hand, we write condition (3.5) with $z_n = 0_X$, $\mu_n = \lambda_n$, $w = v$, and then with $z_n = v$, $\mu_n = \lambda_n$ and $w = 0_X$ to obtain

$$\limsup\, b_n(v, \lambda_n) \leq b(v, \widetilde{\lambda}) \quad \text{and} \quad b(v, \widetilde{\lambda}) \leq \liminf\, b_n(v, \lambda_n),$$

respectively. These inequalities show that

$$b_n(v, \lambda_n) \to b(v, \widetilde{\lambda}). \tag{3.23}$$

Finally, note that convergence (3.9) implies that

$$(f_n, \pi v)_Z \to (f, \pi v)_Z. \tag{3.24}$$

Next, we pass to the limit in equality (3.1) and use convergences (3.22)–(3.24) to see that

$$(A\widetilde{u}, v)_X + b(v, \lambda) = (f, \pi v)_Z. \tag{3.25}$$

Consider now an arbitrary element $\mu \in \Lambda$. Using assumption (3.8) we know that there exists a sequence $\{\mu_n\}$ such that $\mu_n \in \Lambda_n$ for each $n \in \mathbb{N}$ and $\mu_n \to \mu$ in Y. This allows to use inequality (3.2) to see that

$$b_n(u_n, \mu_n - \lambda_n) \leq 0,$$

which implies

$$\liminf\, b_n(u_n, \mu_n - \lambda_n) \leq 0. \tag{3.26}$$

On the other hand, writing condition (3.5) with $w = 0_X, z_n = u_n$ and $\mu_n - \lambda_n$ instead of μ_n we deduce that

$$\limsup\, b_n(-u_n, \mu_n - \lambda_n) \leq b(-\widetilde{u}, \mu - \widetilde{\lambda})$$

or, equivalently,

$$b(\widetilde{u}, \mu - \widetilde{\lambda}) \leq \liminf\, b_n(u_n, \mu_n - \lambda_n). \tag{3.27}$$

We combine inequalities (3.26) and (3.27) to find that

$$b(\widetilde{u}, \mu - \widetilde{\lambda}) \leq 0. \tag{3.28}$$

Finally, we gather (3.21), (3.25) and (3.28) to conclude the proof of the lemma. □

We are now in a position to provide the proof of Theorem 5.

Proof. Recall that Theorem 3 states the existence of a unique solution to Problem 2, denoted (u, λ). Therefore, it follows from Lemma 8 that $\widetilde{u} = u$ and $\widetilde{\lambda} = \lambda$. A careful examination of the proof for Lemmas 6–8 reveals the fact that the sequence $\{(u_n, \lambda_n)\}$ is bounded in $X \times Y$ and every subsequence of $\{(u_n, \lambda_n)\}$ which converges weakly in $X \times Y$ has the same limit (u, λ). Therefore, by a standard argument we deduce that the whole sequence $\{(u_n, \lambda_n\}\}$ converges weakly in $X \times Y$ to (u, λ) or, equivalently, $u_n \rightharpoonup \widetilde{u}$ in X and $\lambda_n \rightharpoonup \widetilde{\lambda}$ in Y. This implies that (3.11) holds. Moreover, Lemma 7 shows that the strong convergence (3.10) holds, which concludes the proof of the theorem. □

4. The optimal control problem

In this section we apply Theorem 5 in the study of a general optimal control problem associated with Problem 2. To this end, we consider a reflexive Banach space W endowed with the norm $\|\cdot\|_W$ and a nonempty subset $U \subset W$. For a pair (θ, p) with $\theta = (u, \lambda) \in X \times Y$ and $p \in W$ we use the short-hand notation (u, v, λ) and we still refer to this triple as a pair. For each $p \in U$ we consider an operator A_p, a form b_p, a set Λ_p and an element f_p which satisfy assumptions (2.3), (2.4), (2.6) and (2.7), respectively, with constants m_p, L_p, M_p, α_p. To avoid any confusion, when used with p, we refer to these assumptions as assumptions $(2.3)_p$, $(2.4)_p$, $(2.6)_p$ and $(2.7)_p$. Then, if condition (2.5) is satisfied, we deduce from Theorem 3 that for each $p \in U$ there exists a unique solution (u_p, λ_p) for the following problem.

Problem 9. *Find $u_p \in X$ and $\lambda_p \in \Lambda_p$ such that*

$$(A_p u_p, v)_X + b_p(v, \lambda_p) = (f_p, \pi v)_Z \qquad \forall\, v \in X, \tag{4.1}$$

$$b_p(u_p, \mu - \lambda_p) \le 0 \qquad \forall\, \mu \in \Lambda_p. \tag{4.2}$$

We now define the set of admissible pairs for Problem 9 as follows:

$$\mathcal{V}_{ad} = \{\, (u, \lambda, p) \;:\; p \in U,\ u = u_p,\ \lambda = \lambda_p \,\}. \tag{4.3}$$

In other words, an element (u, λ, p) belongs to $\mathcal{V}_{ad}$ if and only if $p \in U$ and, moreover, (u, λ) is the solution to Problem 9, i.e., $u = u_p$ and $\lambda = \lambda_p$. Consider also a cost function $\mathcal{L} : X \times Y \times U \to \mathbb{R}$. Then, the optimal control problem we are interested in is the following.

Problem 10. *Find $(u^*, \lambda^*, p^*) \in \mathcal{V}_{ad}$ such that*

$$\mathcal{L}(u^*, \lambda^*, p^*) = \min_{(u,\lambda,p) \in \mathcal{V}_{ad}} \mathcal{L}(u, \lambda, p). \tag{4.4}$$

Following the terminology used in the optimal control theory, a pair $((u^*, \lambda^*), p^*) \in (X \times \Lambda) \times U$ such that (u^*, λ^*, p^*) is a solution to Problem 10 is called an optimal pair. In this case p^* represents an optimal control and (u^*, λ^*) is an optimal state.

To solve Problem 10 we consider the following assumptions.

$$U \;\text{ is a nonempty weakly closed subset of }\; W. \tag{4.5}$$

$$\begin{cases} \text{For all sequences } \{u_n\} \subset X, \{\lambda_n\} \subset Y \text{ and } \{p_n\} \subset U \text{ such that} \\ u_n \to u \quad \text{in} \quad X, \quad \lambda_n \rightharpoonup \lambda \quad \text{in} \quad Y, \quad p_n \rightharpoonup p \quad \text{in} \quad W, \text{ we have} \\ \liminf\limits_{n\to\infty} \mathcal{L}(u_n, \lambda_n, p_n) \ge \mathcal{L}(u, \lambda, p), \end{cases} \tag{4.6}$$

$$\begin{cases} \text{There exists } h : U \to \mathbb{R} \text{ such that} \\ \text{(a)} \quad \mathcal{L}(u, \lambda, p) \ge h(p) \quad \forall\, u \in X,\ \lambda \in Y,\ p \in U, \\ \text{(b)} \quad \|p_n\|_W \to +\infty \implies h(p_n) \to \infty. \end{cases} \tag{4.7}$$

$$U \;\text{ is a bounded subset of }\; W. \tag{4.8}$$

Example 11. A typical example of function $\mathcal{L}$ which satisfies conditions (4.6) and (4.7) is obtained by taking

$$\mathcal{L}(u,\lambda,p) = g(u) + k(\lambda) + h(p) \qquad \forall\, u \in X,\ \lambda \in Y,\ p \in U,$$

where $g : X \to \mathbb{R}_+$ is a lower semicontinuous function, $k : Y \to \mathbb{R}_+$ is a weakly lower semicontinuous function, and $h : U \to \mathbb{R}$ is a weakly lower semicontinuous and coercive function, i.e., it satisfies condition (4.7)(b).

Our main result in this section is the following.

Theorem 12. *Assume* $(2.3)_p$, $(2.4)_p$, $(2.6)_p$ *and* $(2.7)_p$ *for any* $p \in U$. *Moreover, assume* (2.5), (4.5), (4.6) *and either* (4.7) *or* (4.8). *In addition, assume that for any sequence* $\{p_n\} \subset U$ *such that* $p_n \rightharpoonup p$ *in* W, *conditions* (3.3)–(3.9) *are satisfied with* $A_n = A_{p_n}$, $A = A_p$, $m_n = m_{p_n}$, $m = m_p$, $b_n = b_{p_n}$, $\alpha_n = \alpha_{p_n}$, $\alpha = \alpha_p$, $\Lambda_n = \Lambda_{p_n}$, $\Lambda = \Lambda_p$, $f_n = f_{p_n}$, $f = f_p$. *Then Problem* 10 *has at least one solution* (u^*, λ^*, p^*). *Moreover, the set of solutions is weakly sequentially compact.*

Proof. We consider the function $J : U \to \mathbb{R}$ defined by

$$J(p) = \mathcal{L}(u_p, \lambda_p, p) \qquad \forall\, p \in U \tag{4.9}$$

together with the problem of finding $p^* \in U$ such that

$$J(p^*) = \min_{p\in U} J(p). \tag{4.10}$$

We shall use Theorem 1 in order to see that the optimization problem (4.10) has at least one solution. Assume that $\{p_n\} \subset U$ is such that $p_n \rightharpoonup p$ in W. Then, since conditions (3.3)–(3.9) are satisfied in the sense prescribed in the statement of Theorem 12, we are in a position to apply Theorem 5 in order to obtain that $u_{p_n} \to u_p$ in X and $\lambda_{p_n} \rightharpoonup \lambda_p$ in Y. Therefore, using definition (4.9) and assumption (4.6) we deduce that

$$\liminf J(p_n) = \liminf \mathcal{L}(u_{p_n}, \lambda_{p_n}, p_n) \geq \mathcal{L}(u_p, \lambda_p, p) = J(p).$$

It follows from here that the function $J : U \to \mathbb{R}$ is weakly lower semicontinuous.

Assume now that (4.7) holds. Then, for any sequence $\{p_n\} \subset U$, we have

$$J(p_n) = \mathcal{L}(u_{p_n}, \lambda_{p_n}, p_n) \geq h(p_n).$$

Therefore, if $\|p_n\|_W \to \infty$ we deduce that $J(p_n) \to \infty$ which shows that $J : U \to \mathbb{R}$ is coercive. Recall also assumption (4.5) and the reflexivity of the space W. The existence of at least one solution to problem (4.10) is now a direct consequence of Theorem 1. On the other hand, if we assume that condition (4.8) is satisfied we are still in a position to apply Theorem 1. We deduce from here that, if either (4.7) or (4.8) holds, then there exists at least one solution $p^* \in U$ to the optimization problem (4.10).

Denote by u^* and λ^* the elements of X and Λ given by $u^* = u_{p^*}$ and $\lambda^* = \lambda_{p^*}$, respectively. Then, (4.3) implies that

$$(u^*, \lambda^*, p^*) \in \mathcal{V}_{ad}. \tag{4.11}$$

On the other hand, (4.10) and (4.9) yield

$$\mathcal{L}(u^*, \lambda^*, p^*) \leq \mathcal{L}(u, \lambda, p) \qquad \forall\, (u, \lambda, p) \in \mathcal{V}_{ad}. \tag{4.12}$$

Inclusion (4.11) and inequality (4.12) show that the triple (u^*, λ^*, p^*) is a solution of Problem 10.

Assume now that $\{(u_n^*, \lambda_n^*, p_n^*)\}$ is a sequence of solutions to Problem 10. We claim that the sequence $\{p_n^*\}$ is bounded in W. Indeed, the claim is obviously satisfied if (4.8) holds. Assume now that (4.7) holds. Then, arguing by contradiction, if $\{p_n^*\}$ is not bounded in W we can find a subsequence, still denoted by $\{p_n^*\}$, such that $\|p_n^*\|_W \to \infty$ and, using assumption (4.7) we deduce that

$$\mathcal{L}(u_n^*, \lambda_n^*, p_n^*) \to \infty. \tag{4.13}$$

On the other hand, by the optimality of $(u_n^*, \lambda_n^*, p_n^*)$ we have

$$\mathcal{L}(u_n^*, \lambda_n^*, p_n^*) \le \mathcal{L}(u_0, \lambda_0, p_0) \qquad \forall\, n \in \mathbb{N}, \tag{4.14}$$

where (u_0, λ_0, p_0) is a given arbitrary element of $\mathcal{V}_{ad}$. Relations (4.13) and (4.14) lead to a contradiction, which proves the claim.

The claim allows us to find an element $p^* \in W$ such that, passing to a subsequence still denoted by $\{p_n^*\}$ we have

$$p_n^* \rightharpoonup p^* \quad \text{in} \quad W. \tag{4.15}$$

Therefore, Theorem 5 implies that

$$u_n^* \to u^* \quad \text{in} \quad X, \tag{4.16}$$

$$\lambda_n^* \rightharpoonup \lambda^* \quad \text{in} \quad Y, \tag{4.17}$$

where, here and below, $u^* = u_{p^*}$, $\lambda^* = \lambda_{p^*}$. Note that assumption (4.5) guarantees that $p^* \in U$. Therefore, definition (4.3) implies that

$$(u^*, \lambda^*, p^*) \in \mathcal{V}_{ad}. \tag{4.18}$$

We now use convergences (4.15)–(4.18), assumption (4.6) and inequality (4.14) to see that

$$\mathcal{L}(u^*, \lambda^*, p^*) \le \mathcal{L}(u_0, \lambda_0, p_0) \qquad \forall\, (u_0, \lambda_0, p_0) \in \mathcal{V}_{ad}. \tag{4.19}$$

We now combine (4.18) and (4.19) to see that (u^*, λ^*, p^*) is a solution of Problem 10, which concludes the proof. □

5. An elastic frictional contact problem

The abstract results in Sects. 3–4 are useful in the study of various mathematical models which describe the equilibrium of elastic bodies in contact with a rigid or a deformable foundation. In this section we illustrate their use in the study of a very simple contact model. For the description of more complicate models as well as for details on the notations and preliminaries introduced below we refer the reader to [8,14,31,34,39–42].

We consider an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The body is fixed on Γ_1, is acted by given body forces and given surface tractions on Γ_2, and is in frictional contact with an obstacle on Γ_3. Then, under additional mechanical assumptions which will be described below, the equilibrium of the elastic body in the physical setting above is described by the following boundary problem.

Problem 13. *Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big) \quad \text{in } \Omega, \tag{5.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{5.4}$$

$$u_\nu = 0, \quad \|\boldsymbol{\sigma}_\tau\| \le g, \quad \boldsymbol{\sigma}_\tau = -g\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \ne \boldsymbol{0} \text{ on}\Gamma_3 \quad \text{on } \Gamma_3. \tag{5.5}$$

Here and below in this section we do not mention the dependence of various functions and data with respect to the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Notation $\mathbb{S}^d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i, \qquad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,\\ &\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d,\end{aligned}$$

and the zero element of these spaces will be denoted by $\mathbf{0}$. Also, $\boldsymbol{\nu}$ is the outward unit normal at Γ and u_ν, $\boldsymbol{u}_\tau$ will represent the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu=\boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau=\boldsymbol{u}-u_\nu\boldsymbol{\nu}$, respectively. Finally, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the normal and tangential stress on Γ, that is $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$.

We now provide a short description of the equations and boundary conditions in Problem 13. First, Eq. (5.1) represents the elastic constitutive law of the material in which $\mathcal{F}$ is assumed to be a nonlinear constitutive operator, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain field, ω is a given elasticity coefficient, and P_B denotes the projector on the convex set $B\subset\mathbb{S}^d$, which is assumed to be closed and nonempty. Equation (5.2) is the equation of equilibrium in which $\boldsymbol{f}_0$ represents the density of body forces and Div denotes the divergence operator. We use it here since the contact process is assumed to be static and, therefore, the inertial term in the equation of motion is neglected. Conditions (5.3), (5.4) represent the displacement and traction boundary conditions, respectively. Here, $\boldsymbol{f}_2$ denotes the density of given surface tractions which act on the part Γ_2 of the boundary. Finally, condition (5.5) represents the interface law on the contact surface. Equality $u_\nu=0$ shows that there is no separation between the body and the obstacle, i.e., the contact is bilateral. The rest of the condition in (5.5) represent the static version of the Tresca's friction law, in which $g\geq 0$ denotes the friction bound, assumed to be given.

In the study of the contact problems (5.1)–(5.5) we assume that the elasticity operator $\mathcal{F}$ and the set B satisfy the following conditions.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{F}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{F}}>0 \text{ such that}\\ \qquad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\leq L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) There exists } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geq m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad\quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \qquad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(e) } \mathcal{F}(\boldsymbol{x},\mathbf{0})=\mathbf{0},\ \text{a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{5.6}$$

$$B(\boldsymbol{x})\ \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0}\in B(x),\ \text{a.e. } \boldsymbol{x}\in\Omega. \tag{5.7}$$

Moreover, we assume that the elasticity coefficient, the densities of body forces and tractions, and the friction bound are such that

$$\omega\geq 0, \tag{5.8}$$
$$\boldsymbol{f}_0\in L^2(\Omega)^d, \tag{5.9}$$
$$\boldsymbol{f}_2\in L^2(\Gamma_2)^d, \tag{5.10}$$
$$g\geq 0. \tag{5.11}$$

Everywhere in this section we use the standard notation for Sobolev and Lebesgue spaces associated with Ω and Γ, endowed with their canonical inner products and associated norms. We denote by $\gamma: H^1(\Omega)^d\to L^2(\Gamma)^d$ the trace operator, and for an element $\boldsymbol{v}\in H^1(\Omega)^d$, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on Γ, that is, $v_\nu=\gamma\boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau=\gamma\boldsymbol{v}-v_\nu\boldsymbol{\nu}$, respectively. Moreover, we consider the space

$$X=\{\,\boldsymbol{v}\in H^1(\Omega)^d:\ \gamma\boldsymbol{v}=\mathbf{0}\ \text{ on }\ \Gamma_1,\quad v_\nu=0\ \text{ on }\ \Gamma_3\,\},$$

which is a real Hilbert space endowed with the canonical inner product

$$(\boldsymbol{u},\boldsymbol{v})_X = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \tag{5.12}$$

and the associated norm $\|\cdot\|_X$. Recall that $\boldsymbol{\varepsilon}$ represents the linearized strain operator, that is

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j}+u_{j,i}),$$

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. The completeness of the space X follows from the assumption $meas\,(\Gamma_1) > 0$ which allows the use of Korn's inequality.

We now follow [21] and recall that the space $\gamma(X)$ is a closed subspace of the Hilbert space $\gamma(H^1(\Omega)^d)$ and, therefore, is a Hilbert space. Let Y be its dual (which, in turn, can be organized as a real Hilbert space) and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between Y and $\gamma(X)$. Recall also that $\gamma(X)$ is continuously embedded in $L^2(\Gamma)^d$. Finally, we need the space $Z = L^2(\Omega)^d\times L^2(\Gamma_2)^d$ equipped with the canonical inner product.

Next, we introduce the operators $A : X\to X$ and $\pi : X\to Z$, the form $b : X\times Y\to\mathbb{R}$, the set $\Lambda\subset Y$ and the element $\boldsymbol{f}\in Z$ as follows:

$$(A\boldsymbol{u},\boldsymbol{v})_X = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \omega\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx$$

$$\forall\,\boldsymbol{u},\,\boldsymbol{v}\in X, \tag{5.13}$$

$$b(\boldsymbol{v},\,\boldsymbol{\mu}) = \langle\boldsymbol{\mu},\gamma\boldsymbol{v}\rangle,\qquad \forall\,\boldsymbol{v}\in X,\ \mu\in Y, \tag{5.14}$$

$$\pi\boldsymbol{v} = (\boldsymbol{v},\gamma_2\boldsymbol{v})\qquad \forall\,\boldsymbol{v}\in X, \tag{5.15}$$

$$\Lambda = \Big\{\boldsymbol{\mu}\in Y\,:\,\langle\boldsymbol{\mu},\,\boldsymbol{\xi}\rangle\le g\int_{\Gamma_3}\|\boldsymbol{\xi}\|da\quad\forall\,\boldsymbol{\xi}\in\gamma(X)\Big\}, \tag{5.16}$$

$$\boldsymbol{f} = (\boldsymbol{f}_0,\boldsymbol{f}_2). \tag{5.17}$$

Here and below, $\gamma_2\boldsymbol{v}\in L^2(\Gamma_2)^d$ denotes the restriction to Γ_2 of the trace $\gamma\boldsymbol{v}\in L^2(\Gamma)^d$, for any $\boldsymbol{v}\in X$. Moreover, note that definition (5.15) implies that

$$(\boldsymbol{f},\pi\boldsymbol{v})_Z = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2}\boldsymbol{f}_2\cdot\gamma_2\boldsymbol{v}\,da\qquad\forall\,\boldsymbol{v}\in X. \tag{5.18}$$

We now introduce a new variable, the Lagrange multiplier, denoted by $\boldsymbol{\lambda}$. It is related to the friction force $\boldsymbol{\sigma}_\tau$ on the contact zone Γ_3 by equality

$$\langle\boldsymbol{\lambda},\widetilde{\boldsymbol{v}}\rangle = -\int_{\Gamma_3}\boldsymbol{\sigma}_\tau\cdot\widetilde{\boldsymbol{v}}\,da\qquad\forall\,\widetilde{\boldsymbol{v}}\in\gamma(X). \tag{5.19}$$

The variational formulation of Problem 13 in terms of Lagrange multipliers was derived in [21] in the case $\omega = 0$, based on integration by parts and equalities (5.18) and (5.19). Using the same arguments, it is easy to derive the following mixed formulation of Problem 13.

Problem 14. *Find $\boldsymbol{u}\in X$ and $\boldsymbol{\lambda}\in\Lambda$ such that*

$$(A\boldsymbol{u},\boldsymbol{v})_X + b(\boldsymbol{v},\boldsymbol{\lambda}) = (\boldsymbol{f},\,\pi\boldsymbol{v})_Z\qquad\forall\,\boldsymbol{v}\in V, \tag{5.20}$$

$$b(\boldsymbol{u},\boldsymbol{\mu}-\boldsymbol{\lambda})\le 0\qquad\forall\,\boldsymbol{\mu}\in\Lambda. \tag{5.21}$$

The unique solvability of Problem 14 is given by the following existence and uniqueness result.

Theorem 15. *Assume (5.6)–(5.11). Then, Problem 14 has a unique solution $(\boldsymbol{u},\boldsymbol{\lambda})\in X\times\Lambda$.*

Proof. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in X$. We use definition (5.13), assumptions (5.6)(b), (5.7), (5.8) and the nonexpansivity of the projection operator P_B to see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w})_X \leq (L_{\mathcal{F}} + 2\omega)\|\boldsymbol{u} - \boldsymbol{v}\|_X \|\boldsymbol{w}\|_X.$$

This proves that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_X \leq (L_{\mathcal{F}} + 2\omega)\|\boldsymbol{u} - \boldsymbol{v}\|_X,$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$, which implies that A is a Lipschitz continuous operator, i.e., it satisfies condition (2.3)(b). On the other hand, using assumption (5.6)(c) and the nonexpansivity of P_B, again, we find that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_X \geq m_{\mathcal{F}}\|\boldsymbol{u} - \boldsymbol{v}\|_X^2, \tag{5.22}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$. This shows that condition (2.3)(a) holds with $m = m_{\mathcal{F}}$. We conclude from above that the operator A defined by (5.13) satisfies condition (2.3).

The form b given by (5.14) satisfies condition (2.4). For the proof of this statement we refer the reader to [21], for instance. Moreover, it is obvious to see that the operator π defined by (5.15) satisfies condition (2.5). On the other hand, assumption (5.11) shows that the set Λ defined by (5.16) satisfies condition (2.6) and, finally, assumptions (5.9), (5.10) imply (2.7) for the element $\boldsymbol{f}$ given by (5.17). Therefore, Theorem 15 is now a direct consequence of Theorem 3. □

A pair $(\boldsymbol{u}, \boldsymbol{\lambda}) \in X \times \Lambda$ which satisfies (5.20) and (5.21) is called a weak solution to Problem 13. We conclude from here that Theorem 15 provides sufficient conditions which guarantee the weak solvability of the contact problems (5.1)–(5.5).

We now study the continuous dependence of the solution to Problem 14 with respect to the data ω, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and g. To this end, we consider the product space $W = \mathbb{R} \times L^2(\Omega)^d \times L^2(\Gamma_2)^d \times \mathbb{R}$ endowed with the canonical Hilbertian norm and let $\widetilde{U}$ be the subset of W defined by

$$\widetilde{U} = \{\, p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \in W \;:\; \omega \geq 0, \quad g \geq 0 \,\}. \tag{5.23}$$

Theorem 16. *Assume* (5.6) *and* (5.7), *and for each* $p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \in \widetilde{U}$, *denote by* $(\boldsymbol{u}_p, \boldsymbol{\lambda}_p)$ *the solution of Problem* 14 *obtained in Theorem* 15. *Then, for each sequence* $\{p_n\} \subset \widetilde{U}$ *such that* $p_n \rightharpoonup p$ in W, *the following convergences hold:*

$$\boldsymbol{u}_{p_n} \to \boldsymbol{u}_p \qquad \text{in} \quad X, \tag{5.24}$$

$$\boldsymbol{\lambda}_{p_n} \rightharpoonup \boldsymbol{\lambda}_p \qquad \text{in} \quad Y. \tag{5.25}$$

Proof. Let $\{p_n\} \subset \widetilde{U}$ be a sequence of elements in $\widetilde{U}$ such that $p_n = (\omega_n, \boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, g_n)$. For each $n \in \mathbb{N}$, let $A_n : X \to X$, $\Lambda_n \subset Y$ and $\boldsymbol{f}_n \in Z$ be defined by equalities

$$(A_n\boldsymbol{u}, \boldsymbol{v})_X = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \omega_n \int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B\boldsymbol{\varepsilon}(\boldsymbol{u})\big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx$$

$$\forall\, \boldsymbol{u}, \boldsymbol{v} \in X, \tag{5.26}$$

$$\Lambda_n = \Big\{ \boldsymbol{\mu} \in Y \;:\; \langle \boldsymbol{\mu}, \boldsymbol{\xi} \rangle \leq g_n \int_{\Gamma_3} \|\boldsymbol{\xi}\|\, da \quad \forall\, \boldsymbol{\xi} \in \gamma(X) \Big\}, \tag{5.27}$$

$$\boldsymbol{f}_n = (\boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}). \tag{5.28}$$

Moreover, for simplicity, denote $\boldsymbol{u}_{p_n} = \boldsymbol{u}_n$ and $\boldsymbol{\lambda}_{p_n} = \boldsymbol{\lambda}_n$. Then it follows that $\boldsymbol{u}_n \in X$, $\boldsymbol{\lambda}_n \in \Lambda_n$ and, in addition,

$$(A_n\boldsymbol{u}_n, \boldsymbol{v})_X + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}_n, \pi\boldsymbol{v})_Z \qquad \forall\, \boldsymbol{v} \in X, \tag{5.29}$$

$$b(\boldsymbol{u}_n, \boldsymbol{\mu} - \boldsymbol{\lambda}_n) \leq 0 \qquad \forall\, \boldsymbol{\mu} \in \Lambda_n. \tag{5.30}$$

Assume now that

$$p_n = (\omega_n, \boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}, g_n) \rightharpoonup p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \quad \text{in} \quad W,$$

which implies that

$$\omega_n \to \omega, \tag{5.31}$$

$$\boldsymbol{f}_{0n} \rightharpoonup \boldsymbol{f}_0 \quad \text{in} \quad L^2(\Omega)^d, \tag{5.32}$$

$$\boldsymbol{f}_{02} \rightharpoonup \boldsymbol{f}_2 \quad \text{in} \quad L^2(\Gamma_2)^d, \tag{5.33}$$

$$g_n \to g. \tag{5.34}$$

Our aim in what follows is to apply Theorem 5 in the study of the mixed variational problems (5.29)–(5.30) and (5.20)–(5.21), and to this end, we check the validity of conditions (3.3)–(3.9).

First, we use convergence (5.31) to see that condition (3.3) is satisfied with $F_n = 2\,|\omega_n - \omega|$ and $\delta_n = 0$. Moreover, inequality (5.22) shows that condition (3.4) holds, too. Next, we note that conditions (3.5) and (3.6) are obviously satisfied, since $b_n = b$ for each $n \in \mathbb{N}$ and b satisfies condition (2.4). On the other hand, the compactness of the embedding $X \subset L^2(\Omega)^d$ combined with the compactness of the trace operator $\gamma_2 : X \to L^2(\Gamma_2)^d$ shows that operator (5.15) satisfies condition (3.7). Assume that $g > 0$. Then, for n large enough we have $g_n > 0$ and, therefore, $\Lambda_n = \frac{g_n}{g}\Lambda$. Using now the convergence (5.34) it is easy to see that condition (3.8) holds. On the other hand, if $g = 0$ we have $\Lambda \subset \Lambda_n$, for each $n \in \mathbb{N}$. Using again convergence (5.34) we deduce that (3.8) still holds. We conclude from above that, in any case, condition (3.8) is satisfied. Finally, we note that convergences (5.32) and (5.33) imply (3.9) for $\boldsymbol{f}_n$ and $\boldsymbol{f}$ given by (5.28) and (5.17), respectively.

Now we are in position to use Theorem 5 in order to deduce that $\boldsymbol{u}_n \to \boldsymbol{u}$ in X and $\boldsymbol{\lambda}_n \rightharpoonup \boldsymbol{\lambda}$ in Y. This shows that convergences (5.24) and (5.25) hold, which concludes the proof. □

Besides the mathematical interest, the convergence results (5.24) and (5.25) are important from mechanical point of view since they provide the continuous dependence of the weak solution of Problem 13 with respect to the elasticity coefficient, the densities of the body forces and surface tractions, and the friction bound.

We now provide three examples of optimal control problems associated with Problem 14 for which the abstract result in Theorem 12 holds. Everywhere below we assume that (5.6)–(5.7) hold and $\widetilde{U}$ represents the set given by (5.23). The three problems we consider below have a common feature and can be casted in the following general form.

Problem 17. *Find* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*, p^*) \in \mathcal{V}_{ad}$ *such that*

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{\lambda}^*, p^*) = \min_{(\boldsymbol{u},\boldsymbol{\lambda},p)\in\mathcal{V}_{ad}} \mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}, p). \tag{5.35}$$

Here $\mathcal{V}_{ad}$ is the set of admissible pairs defined by

$$\mathcal{V}_{ad} = \{\, (\boldsymbol{u}, \boldsymbol{\lambda}, p) \;:\; p \in U,\; \boldsymbol{u} = \boldsymbol{u}_p,\; \boldsymbol{\lambda} = \boldsymbol{\lambda}_p \,\}, \tag{5.36}$$

where $U \subset \widetilde{U}$ represents the set of controls, assumed to be nonempty. Moreover, $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ is the cost functional. Both U and $\mathcal{L}$ will change from example to example and, therefore, will be described below. Here we restrict ourselves to stress that a triple $(\boldsymbol{u}, \boldsymbol{\lambda}, p)$ belongs to $\mathcal{V}_{ad}$ if and only if $p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \in U$ and, moreover, the pair $(\boldsymbol{u}, \boldsymbol{\lambda})$ is the solution of Problem 14 with the data ω, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and g. Assumptions (5.6)–(5.7), inclusion $U \subset \widetilde{U}$, condition $U \neq \emptyset$ and Theorem 15 guarantee that the set $\mathcal{V}_{ad}$ is not empty and, therefore, Problem 17 makes sense. Moreover, it follows from the proof of Theorem 16 that conditions (3.3)–(3.9) are satisfied. Therefore, the solvability of Problem 17 follows from Theorem 12, provided that conditions (4.5), (4.6) and either (4.7) or (4.8) are satisfied.

Example 18. Let ω_1, ω_2, M_0, M_2, g_1, g_2, be positive constants such that $\omega_1 \leq \omega_2$, $g_1 \leq g_2$ and consider a function $\sigma_0 \in L^2(\Omega, \mathbb{S}^d)$. Let $U \subset \widetilde{U}$ and $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ be defined by

$$\begin{aligned} U = \{\, p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \in \widetilde{U} \ &: \ \omega \in [\omega_1, \omega_2],\ \|\boldsymbol{f}_0\|_{L^2(\Omega)^d} \leq M_0, \\ &\|\boldsymbol{f}_2\|_{L^2(\Gamma_2)^d} \leq M_2,\ g \in [g_1, g_2]\,\}, \\ \mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}, p) = \int_\Omega &\|\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{\lambda}, p) - \boldsymbol{\sigma}_0\|^2 \, dx \qquad \forall\, \boldsymbol{u} \in X,\ \boldsymbol{\lambda} \in \Lambda,\ p \in U. \end{aligned}$$

Here $\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{\lambda}, p)$ represents the stress field given by the constitutive law (5.1), i.e., $\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{\lambda}, p) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \omega\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - P_B \boldsymbol{\varepsilon}(\boldsymbol{u})\big)$.

With this choice, the mechanical interpretation of Problem 17 is the following: Given a contact process of form (5.1) to (5.5) we are looking for a set of data $p = (\omega, \boldsymbol{f}_0, \boldsymbol{f}_2, g) \in U$ such that the corresponding stress in the body is as close as possible to the "desired stress" $\boldsymbol{\sigma}_0$.

Note that, in this case, assumptions (4.5), (4.6) and (4.8) are satisfied. Therefore, Theorem 12 guarantees the existence of solutions to the corresponding optimal control problem 17.

Example 18 shows that each of the data ω, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and g can be used (simultaneously, separately or in whatever combination) to control the weak solution of Problem 13. For this reason, this example is important from theoretical point of view. Nevertheless, in real-world applications the role of control is usually played by a single parameter. The following two examples illustrate this situation. There, the control is the density of surface tractions and the friction bound, respectively.

Example 19. Let $\widetilde{\omega} \geq 0$, $\widetilde{\boldsymbol{f}}_0 \in L^2(\Omega)^d$, $\widetilde{g} \geq 0$, $\boldsymbol{u}_0 \in X$ and $\lambda_0 \in Y$ be given, and let c_1, c_2, c_3 be strictly positive constants. Moreover, consider the set $U \subset \widetilde{U}$ and the cost functional $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ defined by

$$\begin{aligned} &U = \{\, p = (\widetilde{\omega}, \widetilde{\boldsymbol{f}}_0, \boldsymbol{f}_2, \widetilde{g}) \in \widetilde{U}\,\}, \\ &\mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}, p) = c_1 \|\boldsymbol{u} - \boldsymbol{u}_0\|_X^2 + c_2 \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\|_Y^2 + c_3 \|\boldsymbol{f}_2\|^2_{L^2(\Gamma_2)^d} \\ &\qquad\qquad\qquad \forall\, \boldsymbol{u} \in X,\ \boldsymbol{\lambda} \in \Lambda,\ p \in U. \end{aligned}$$

With this choice, the mechanical interpretation of Problem 17 is the following: Given a contact process of form (5.1) to (5.5) with the data $\omega = \widetilde{\omega}$, $\boldsymbol{f}_0 = \widetilde{\boldsymbol{f}}_0$, $g = \widetilde{g}$, we are looking for a density of surface traction $\boldsymbol{f}_2$ such that the corresponding state of the body is as close as possible to the "desired state" $(\boldsymbol{u}_0, \boldsymbol{\lambda}_0)$. Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in the functional $\mathcal{L}$. In fact, a compromise policy between the two aims ("$\boldsymbol{u}$ close to $\boldsymbol{u}_0$," "$\boldsymbol{\lambda}$ close to $\boldsymbol{\lambda}_0$" and "minimal data $\boldsymbol{f}_2$") has to be found and the relative importance of each criterion with respect to the other is expressed by the choice of the weight coefficients c_1, c_2 and c_3.

Note that, in this case, assumptions (4.5), (4.6) and (4.7) are satisfied. Therefore, Theorem 12 guarantees the existence of the solutions to the corresponding optimal control problem 17.

Example 20. Let g_1, g_2 be positive constants such that $g_1 \leq g_2$, and let $\boldsymbol{\lambda}_0 \in Y$. In addition, let $\widetilde{\omega} \geq 0$ and consider two elements $\widetilde{\boldsymbol{f}}_0$ and $\widetilde{\boldsymbol{f}}_2$ such that $\widetilde{\boldsymbol{f}}_0 \in L^2(\Omega)^d$ and $\widetilde{\boldsymbol{f}}_2 \in L^2(\Gamma_2)^d$. Define $U \subset \widetilde{U}$ and $\mathcal{L} : X \times Y \times U \to \mathbb{R}$ as follows:

$$\begin{aligned} &U = \{\, p = (\widetilde{\omega}, \widetilde{\boldsymbol{f}}_0, \widetilde{\boldsymbol{f}}_2, g) \in \widetilde{U} \ :\ \ g \in [g_1, g_2]\,\}, \\ &\mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}, p) = \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0\|_Y^2 \qquad \forall\, \boldsymbol{u} \in X,\ \boldsymbol{\lambda} \in \Lambda,\ p \in U. \end{aligned}$$

With this choice, the mechanical interpretation of Problem 17 is the following: Given a contact process of form (5.1)–(5.5) with the data $\omega = \widetilde{\omega}$, $\boldsymbol{f}_0 = \widetilde{\boldsymbol{f}}_0$, $\boldsymbol{f}_2 = \widetilde{\boldsymbol{f}}_2$, we are looking for a friction bound $g \in [g_1, g_2]$ such that the corresponding tangential shear on the contact surface of the body is as close as possible to the "desired shear" $\boldsymbol{\lambda}_0$.

Note that, in this case, assumptions (4.5), (4.6) and (4.8) are satisfied. Therefore, Theorem 12 guarantees the existence of the solutions to the corresponding optimal control problem 17.

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