Dependence

Roger cooke
RFF
Jan 25, 2018

WIRED MAGAZINE: 17.03

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09

$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of* Wired.

Probability
Specifically, this is a joint default probability—the likelihood that any two members of the pool (A and B) will both default. It's what investors are looking for, and the rest of the formula provides the answer.

Survival times
The amount of time between now and when A and B can be expected to default. Li took the idea from a concept in actuarial science that charts what happens to someone's life expectancy when their spouse dies.

Equality
A dangerously precise concept, since it leaves no room for error. Clean equations help both quants and their managers forget that the real world contains a surprising amount of uncertainty, fuzziness, and precariousness.

Copula
This couples (hence the Latinate term copula) the individual probabilities associated with A and B to come up with a single number. Errors here massively increase the risk of the whole equation blowing up.

Distribution functions
The probabilities of how long A and B are likely to survive. Since these are not certainties, they can be dangerous: Small miscalculations may leave you facing much more risk than the formula indicates.

Gamma
The all-powerful correlation parameter, which reduces correlation to a single constant—something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.

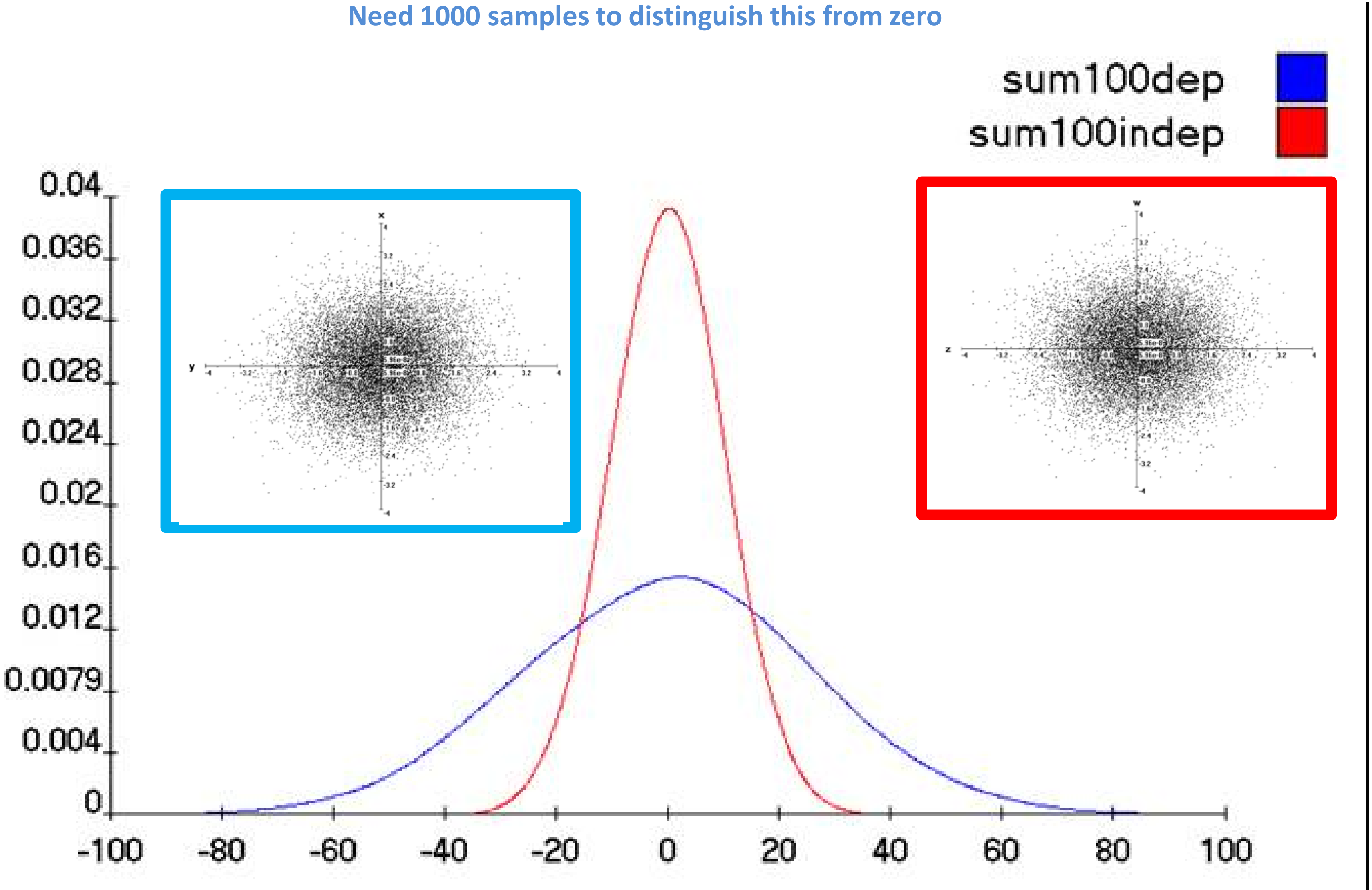
Gaussian Copula

Effect of dependence amplified by summing

100 independent standard normals

100 standard normals pairwise correlated 0.05

Need 1000 samples to distinguish this from zero

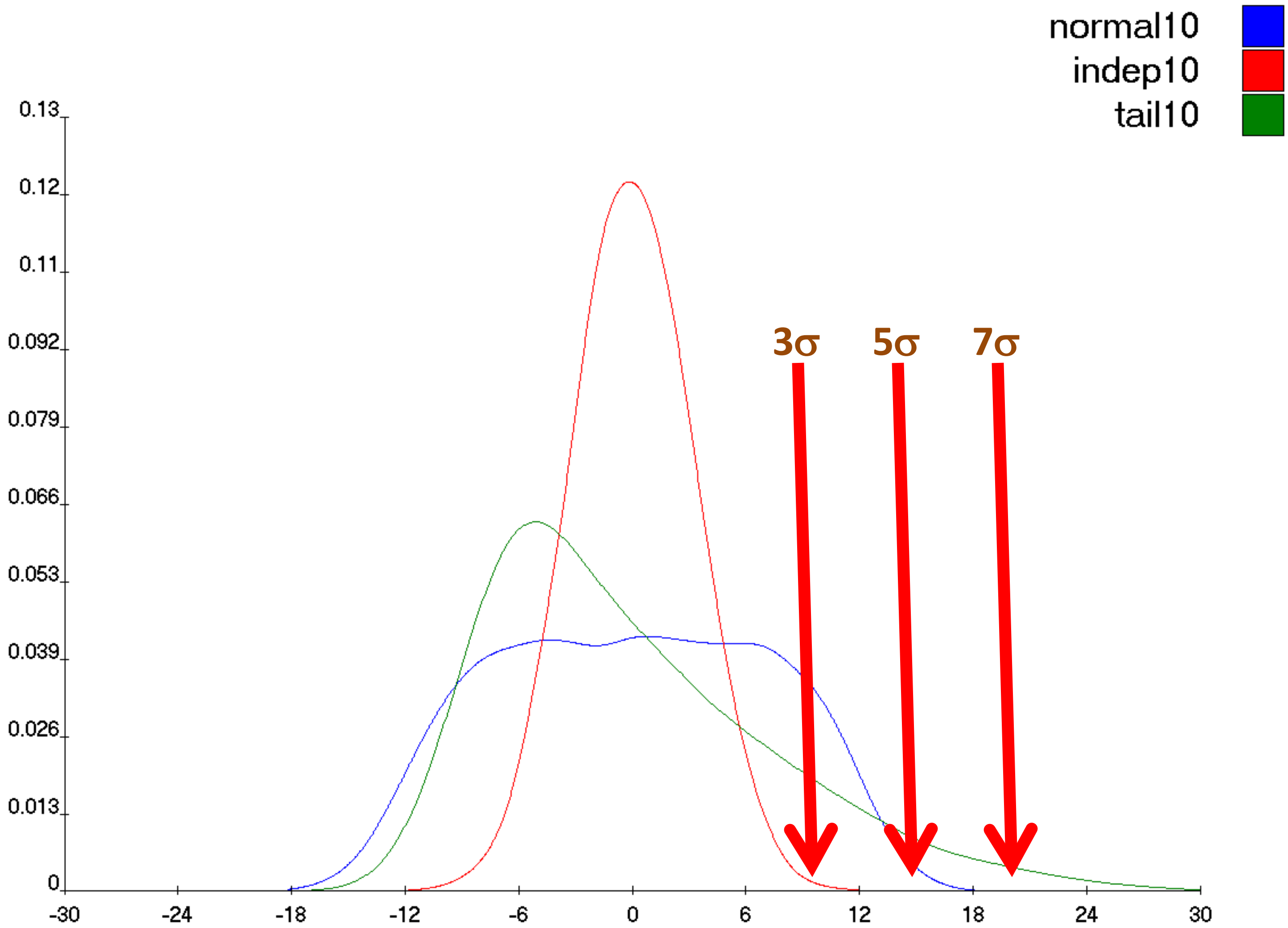


Sum 10 standard normals

Independent

pairwise corr=0.5

Pairwise tail dependent, corr=0.5



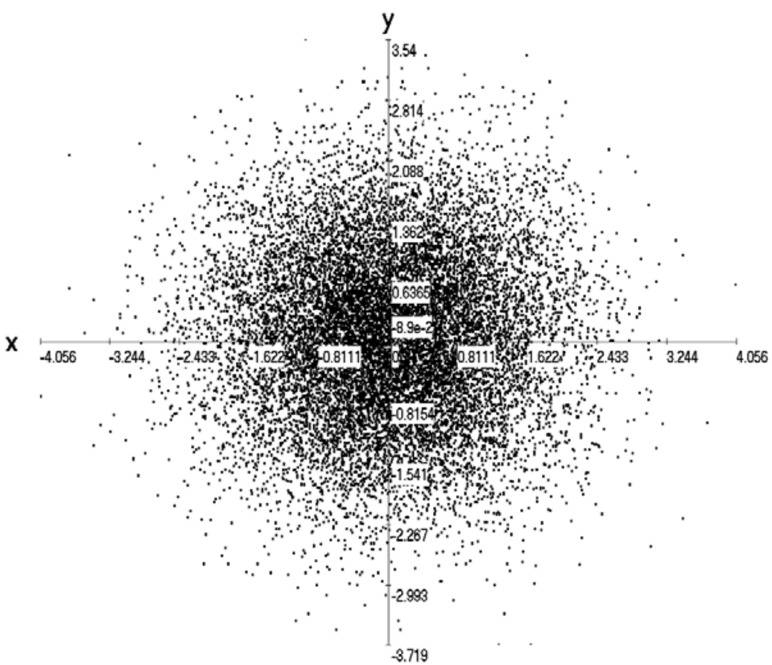
Representing Dependence: Copulae

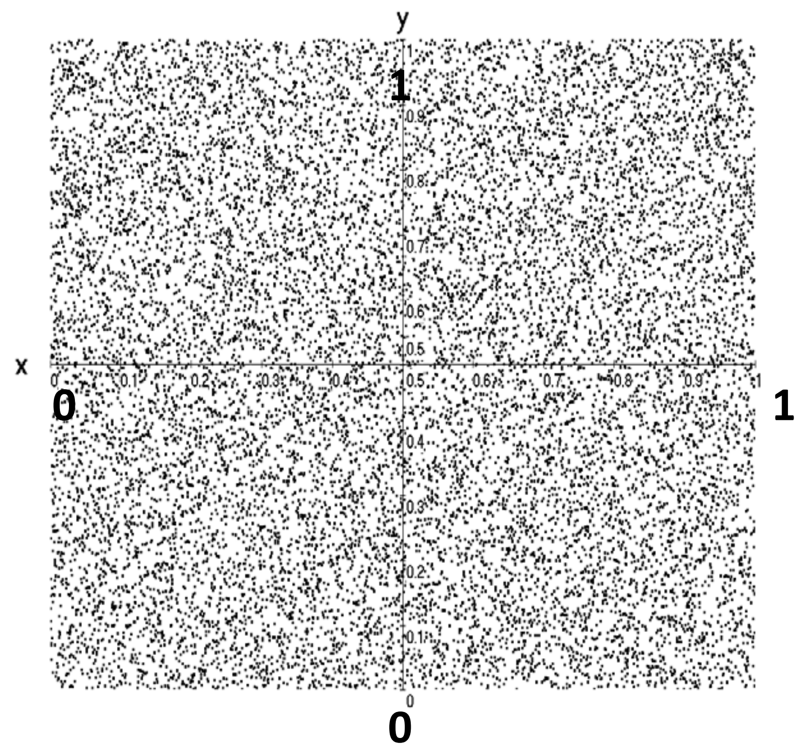
Variable View

Copula View

probability integral transformation

Independent



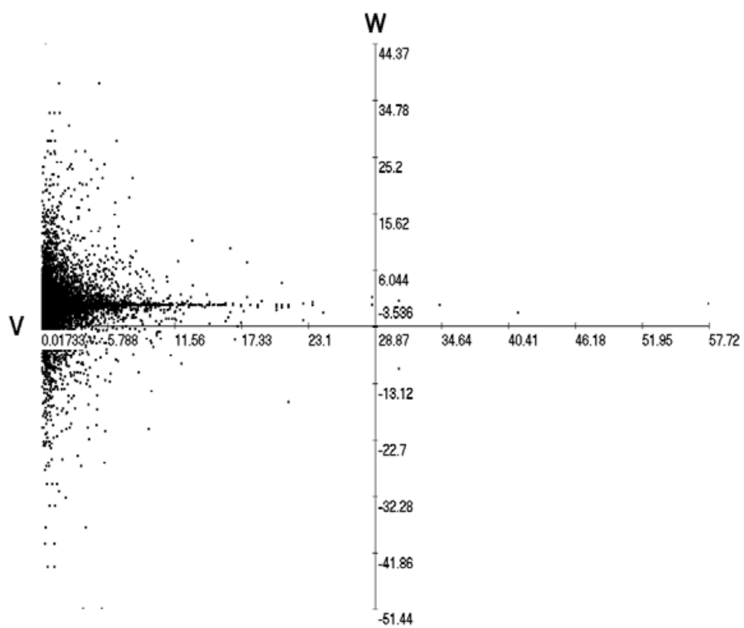


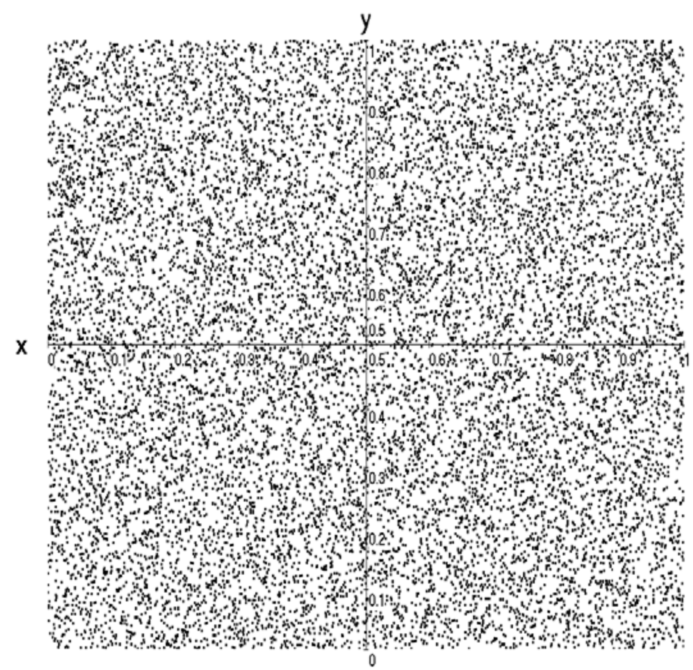
Representing Dependence: Copulae

Variable View

Copula View

Independent



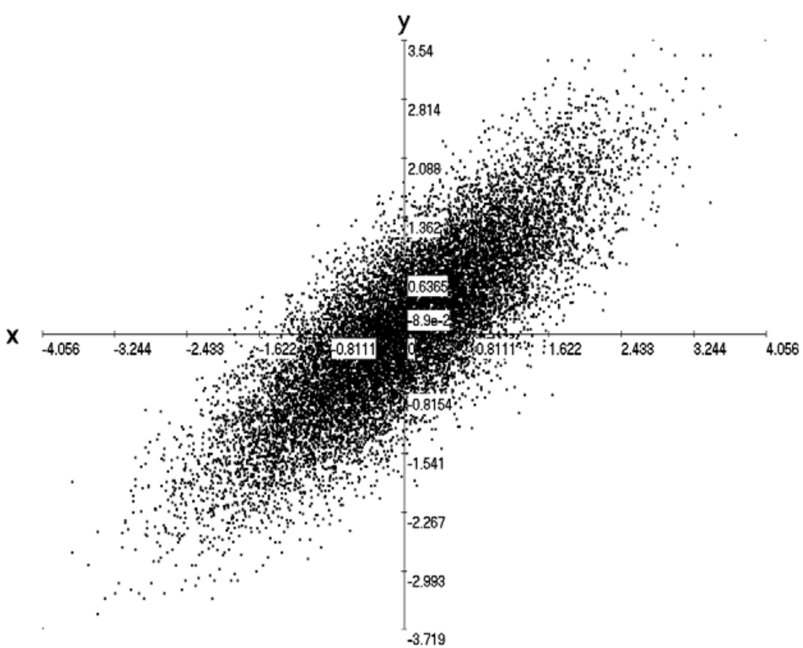


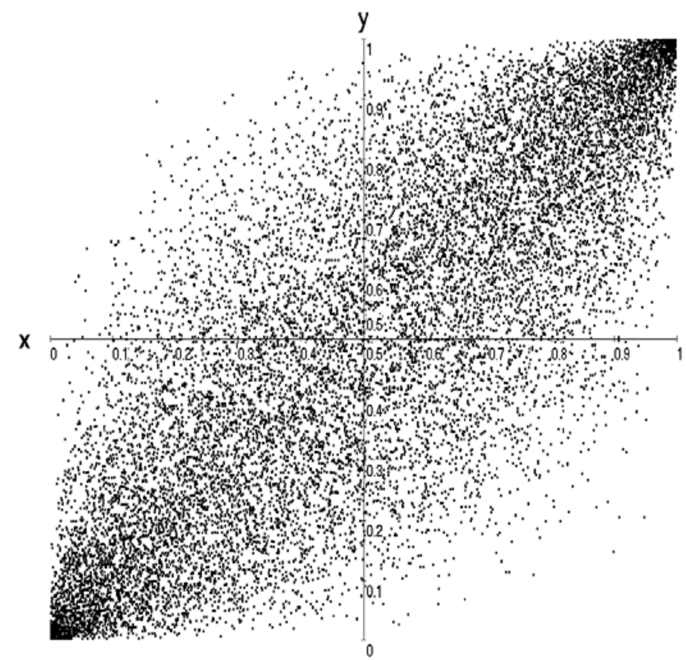
Representing Dependence: Copulae

Variable View

Copula View

Rank correlation 0.8



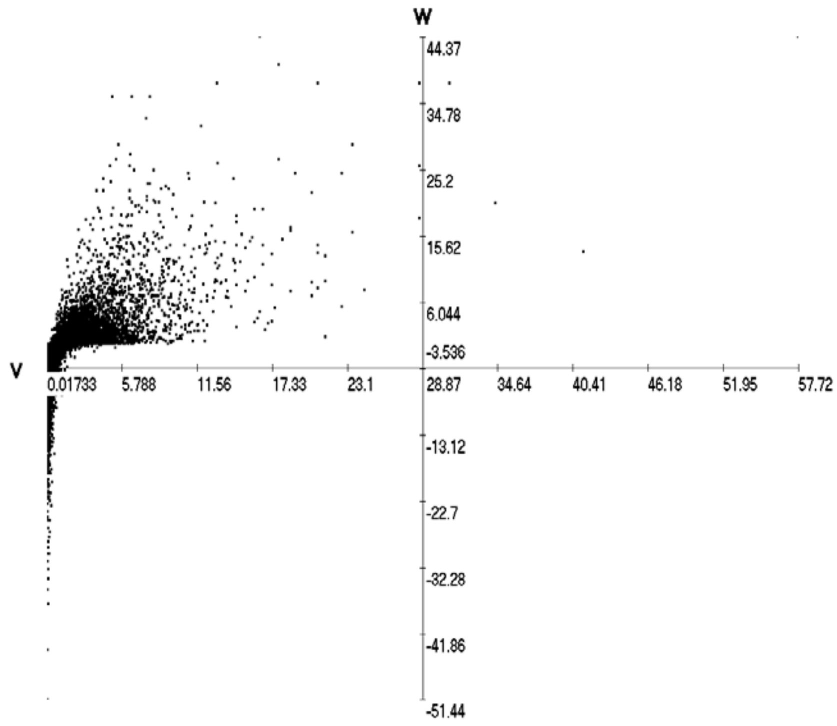


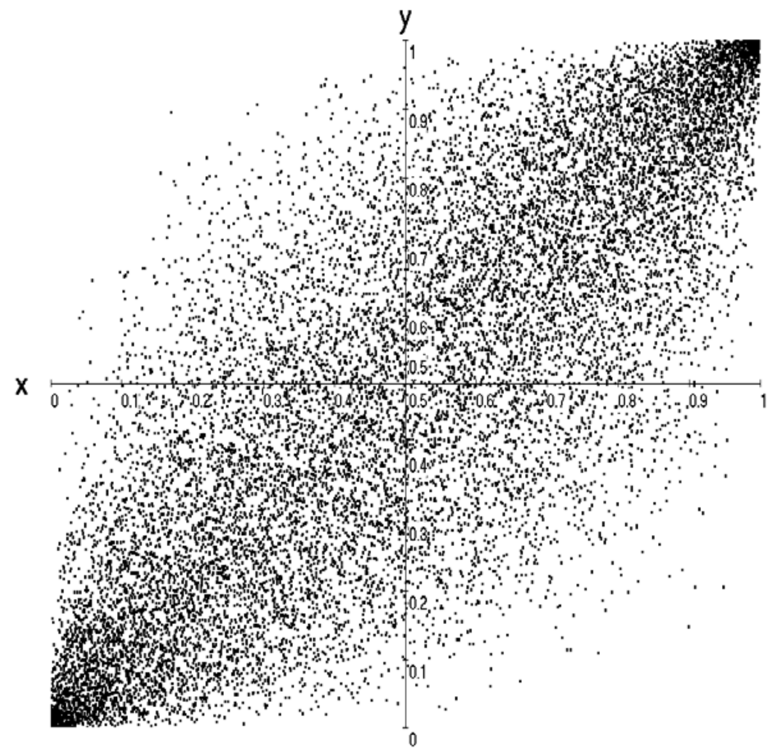
Representing Dependence: Copulae

Variable View

Copula View

Rank correlation 0.8

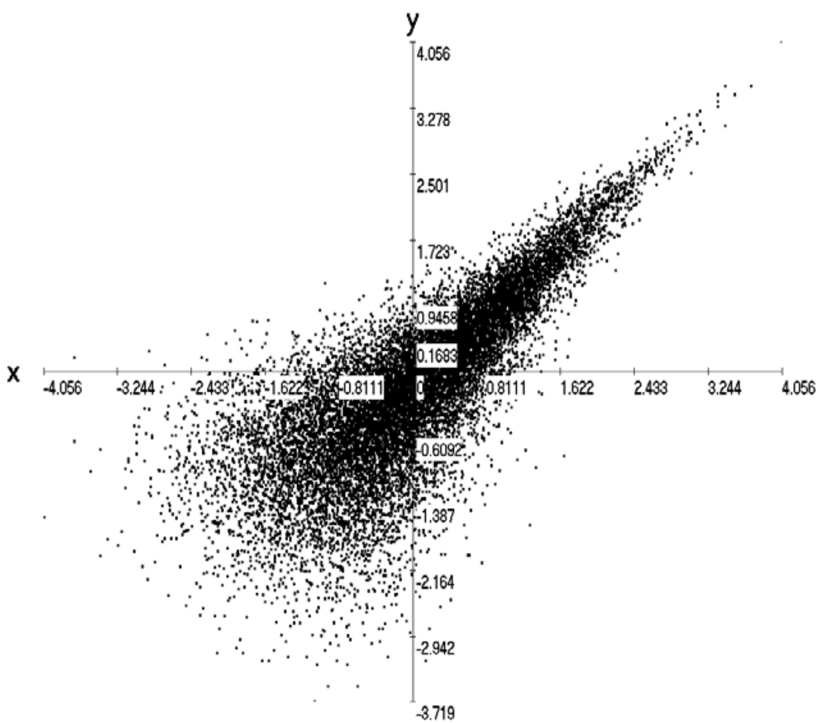


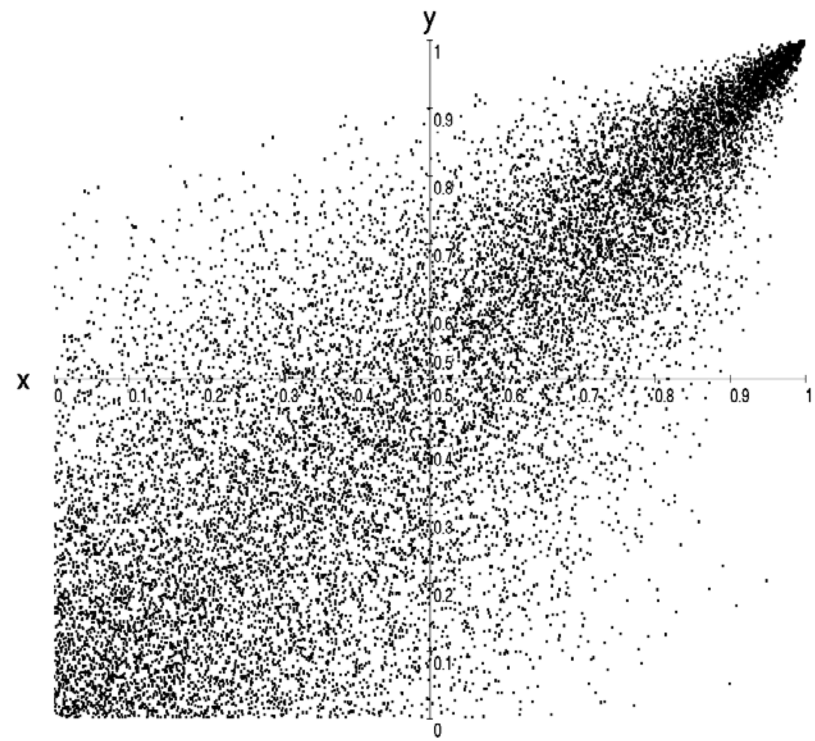


Representing Dependence: Copulae

Variable View Copula View

Tail Dependent Rank correlation 0.8

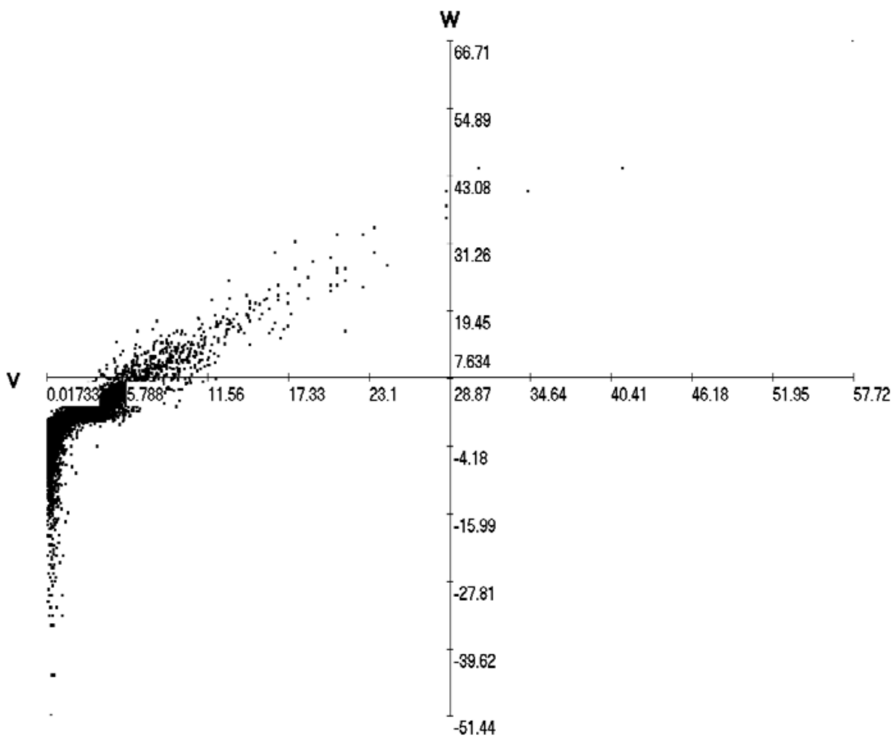


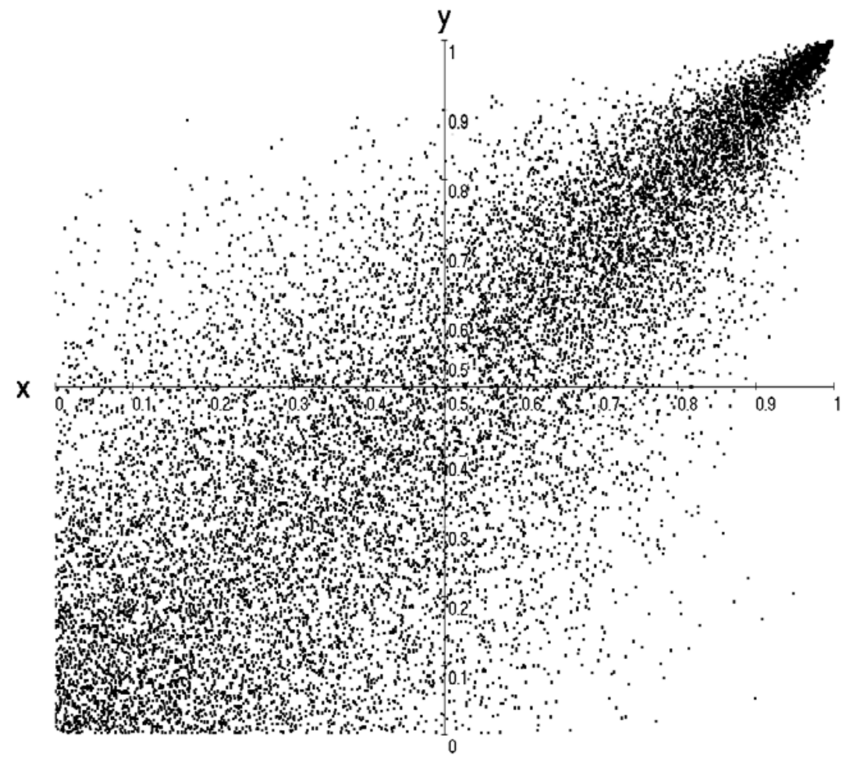


Representing Dependence: Copulae

Variable View Copula View

Tail Dependent Rank correlation 0.8

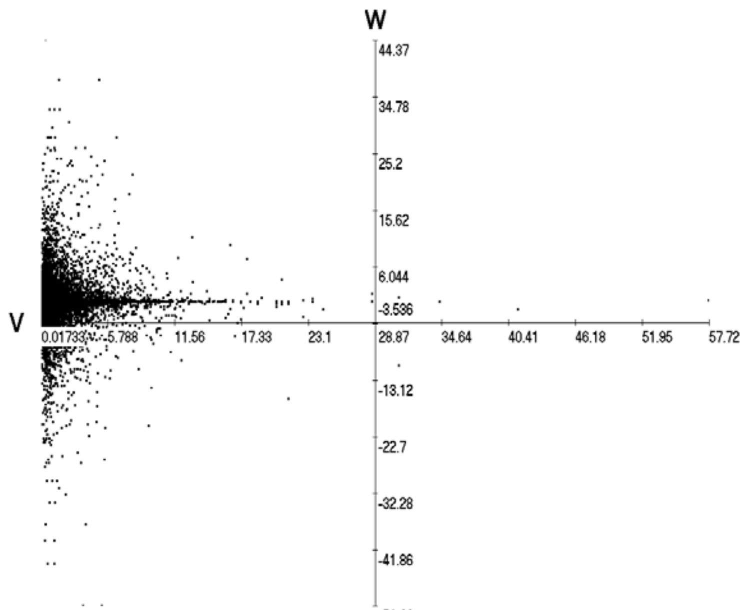


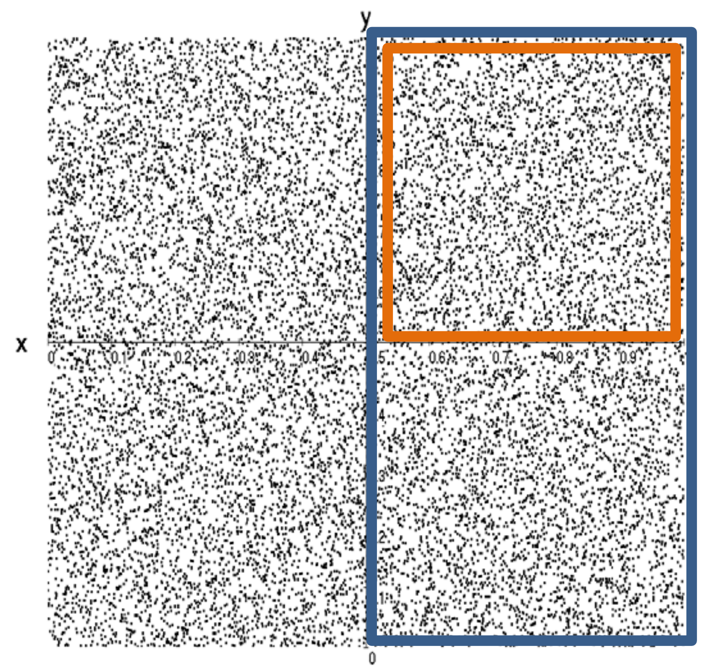


How to elicit dependence

Suppose X is above its median, what is the probability that Y is above its median?

Independent ⇒ Prob = ½

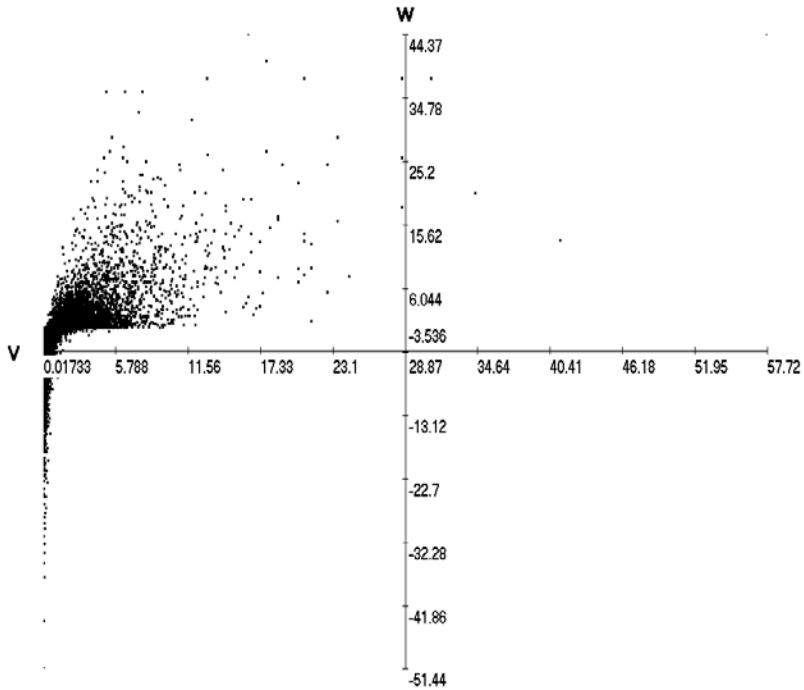


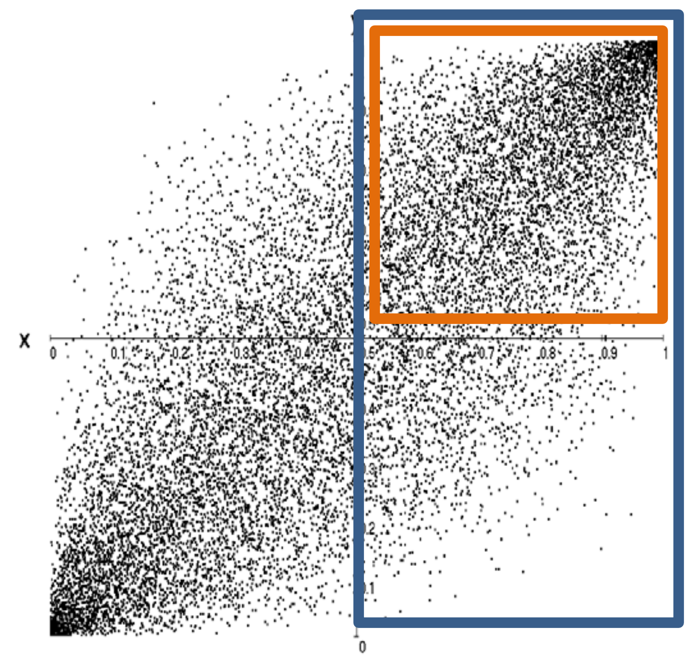


How to elicit dependence

Suppose X is above its median, what is the probability that Y is above its median?

$\text{Rank correlation} = 0.75 \Rightarrow$ Prob = 0.7

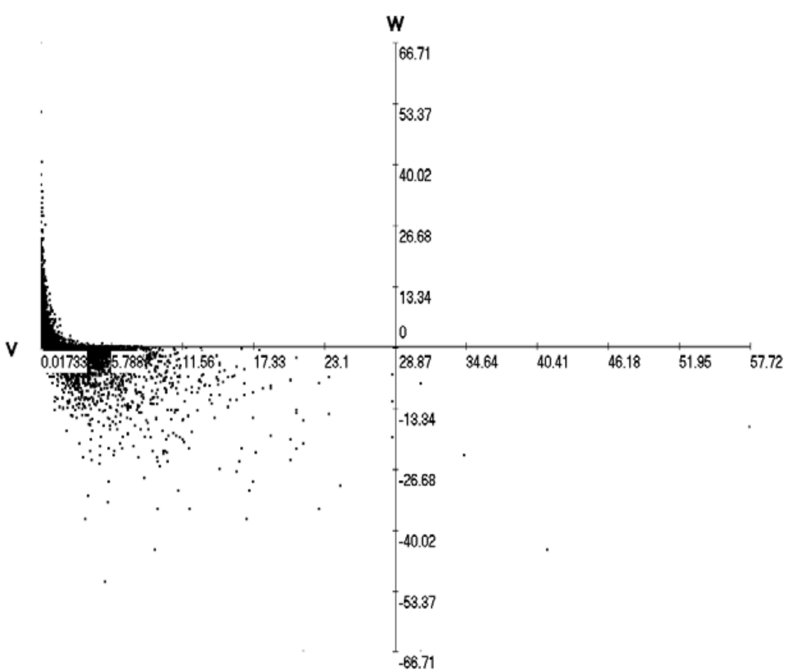


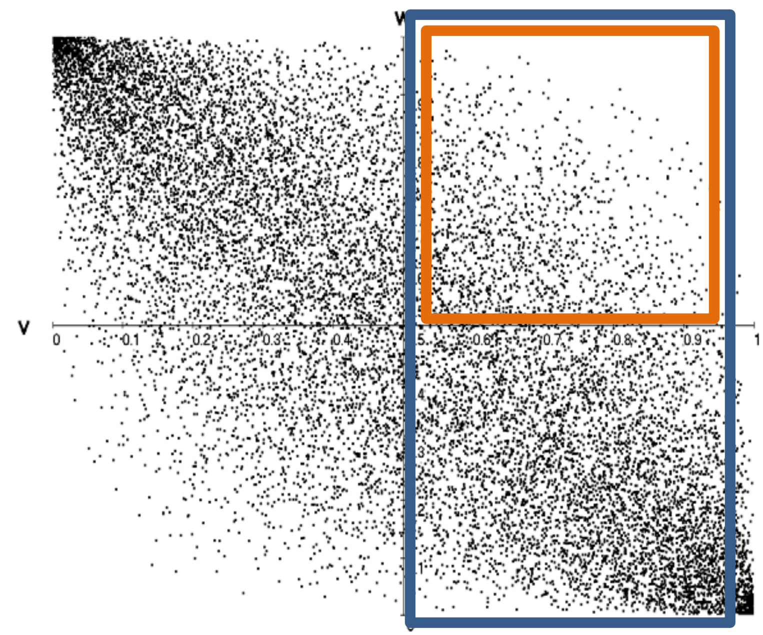


How to elicit dependence

Suppose X is above its median, what is the probability that Y is above its median?

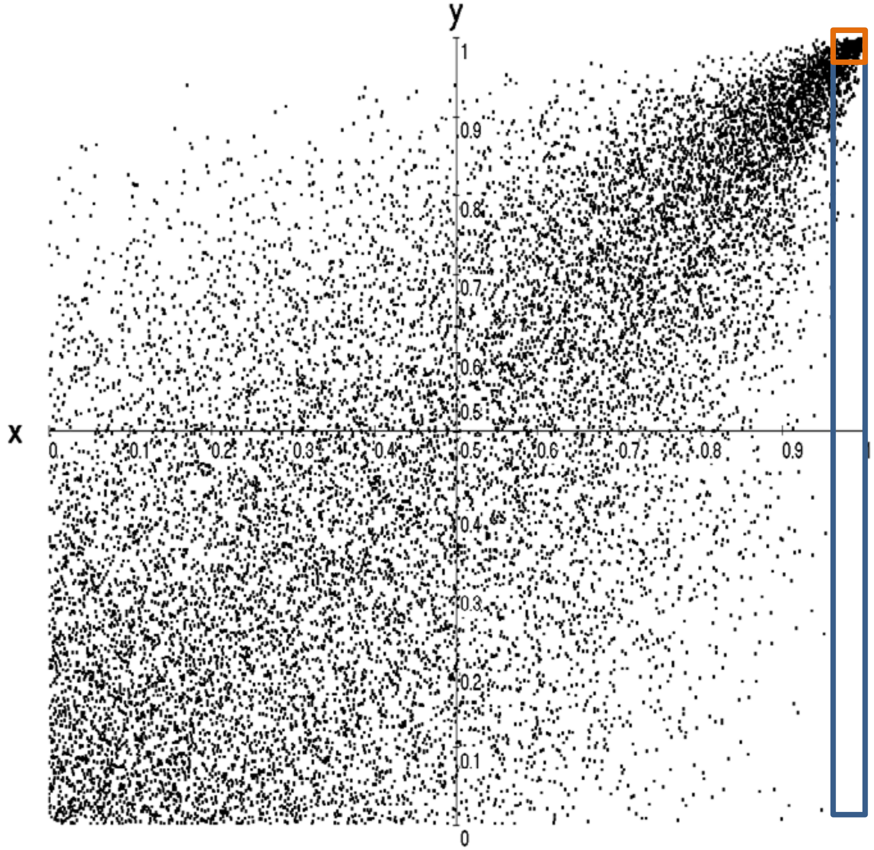
$\text{Rank correlation} = -0.75 \Rightarrow$ Prob = 0.3





How to elicit TAIL dependence

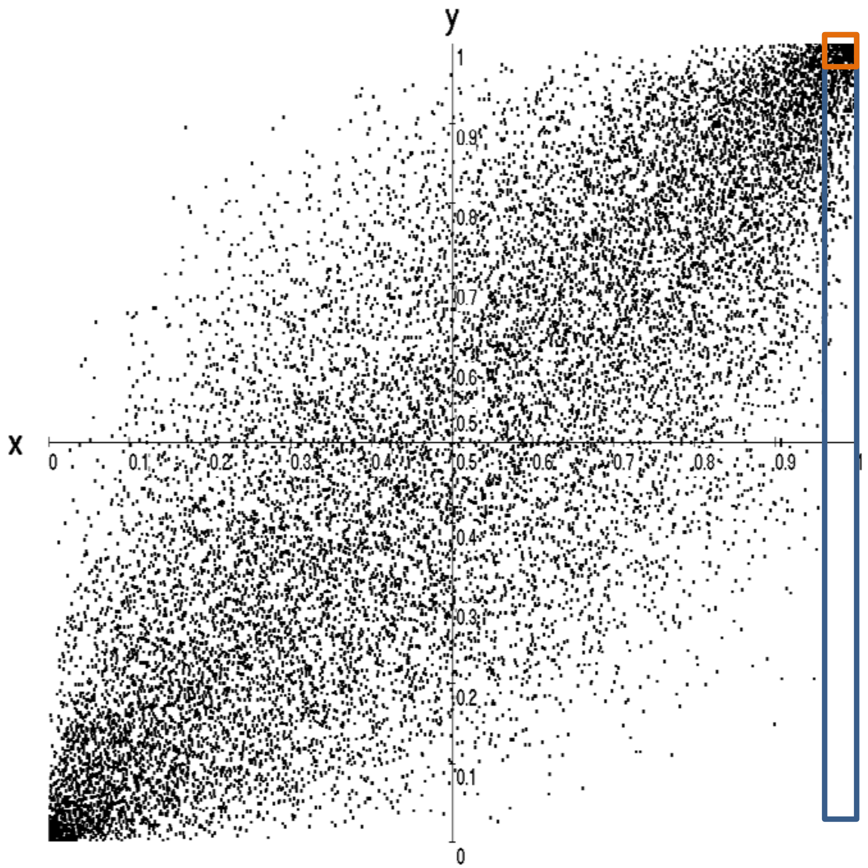
Suppose X is above its 95%-tile, what is the probability that Y is above its 95%-tile?



Prob = 0.75

How to elicit TAIL dependence

Suppose X is above its 95%-tile, what is the probability that Y is above its 95%-tile?



Prob = 0.45

(Upper) Tail Dependence

$UTD(X,Y) = \lim_{u \to 1} P(X>X_u \mid Y>Y_u)$

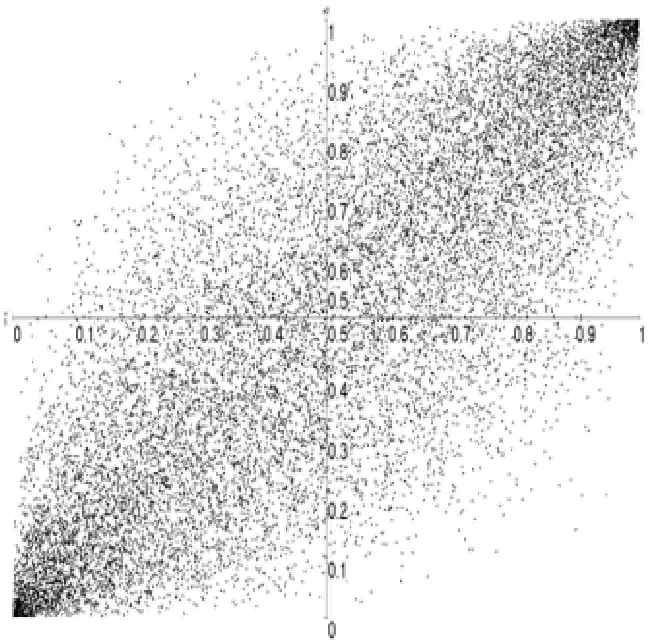
$X_u =$ u-th quantile of X, idem Y

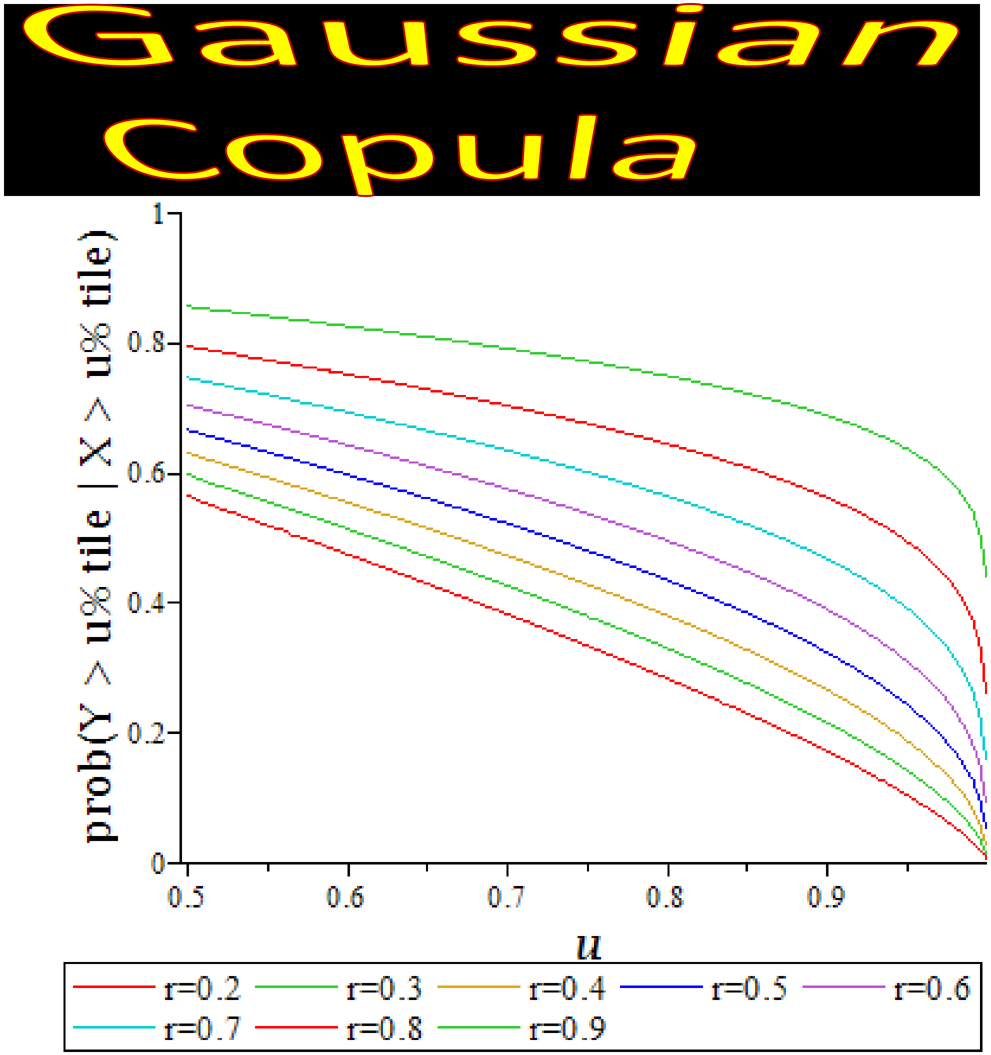
$UTD(X,Y) = 0 \Leftrightarrow$ X, Y **Tail Independent**

Look at $P(X>X_u \mid Y>Y_u)$ as function of u

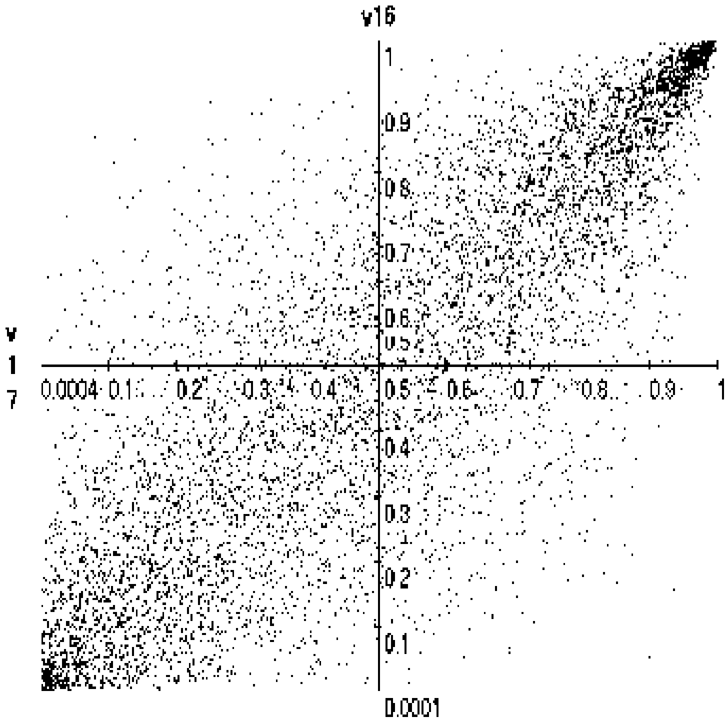
Gaussian copula, 0.8

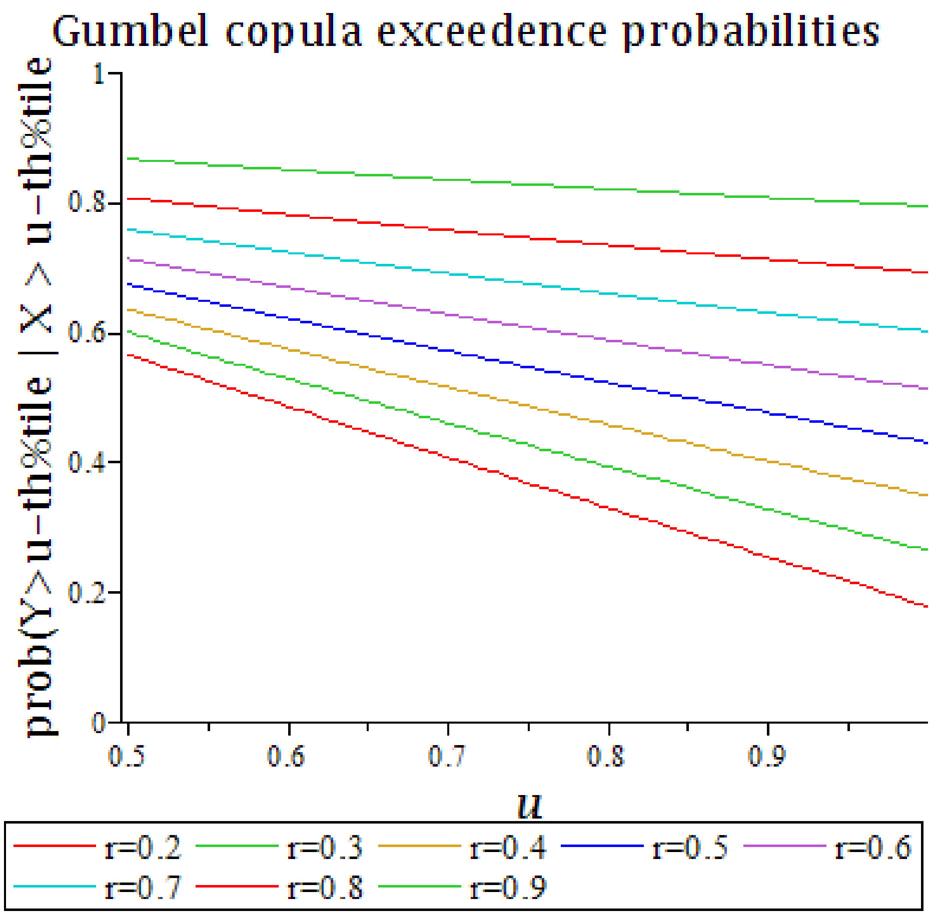
Tail Independent



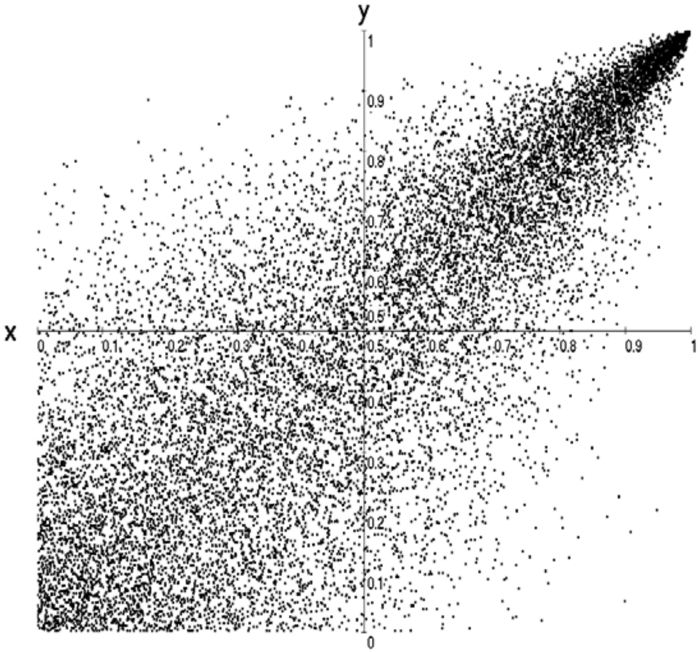


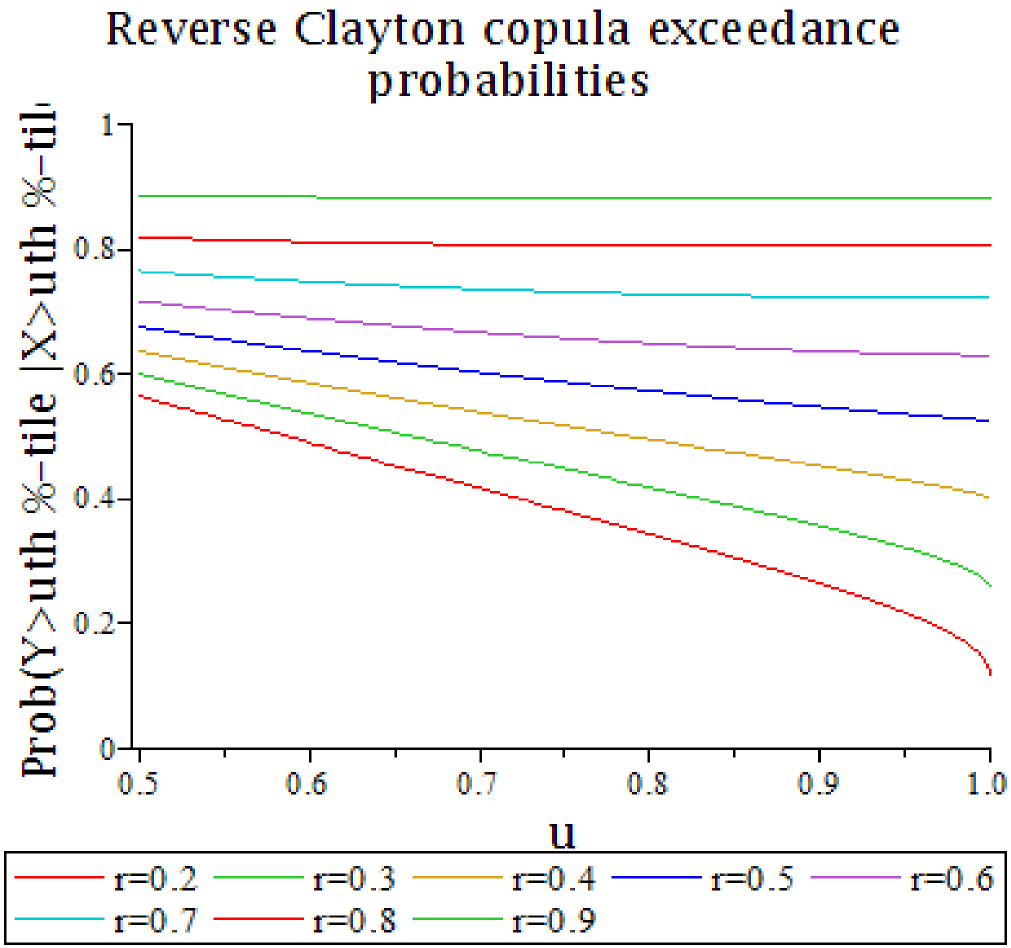
Gumbel copula 0.8
Upper tail Dependent



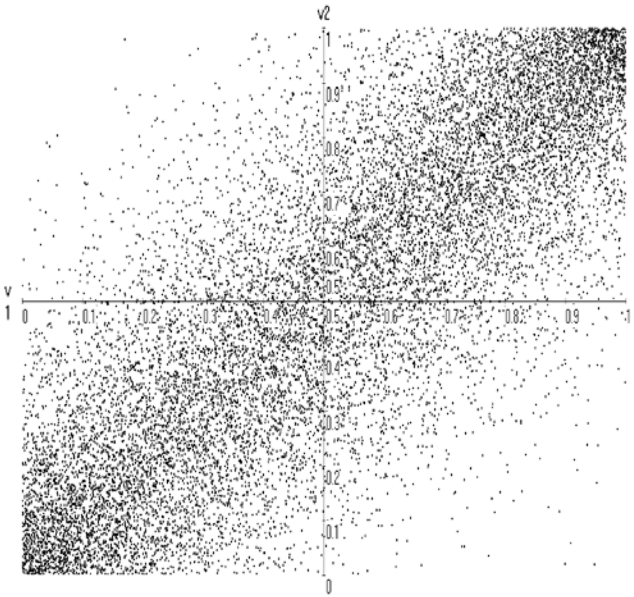


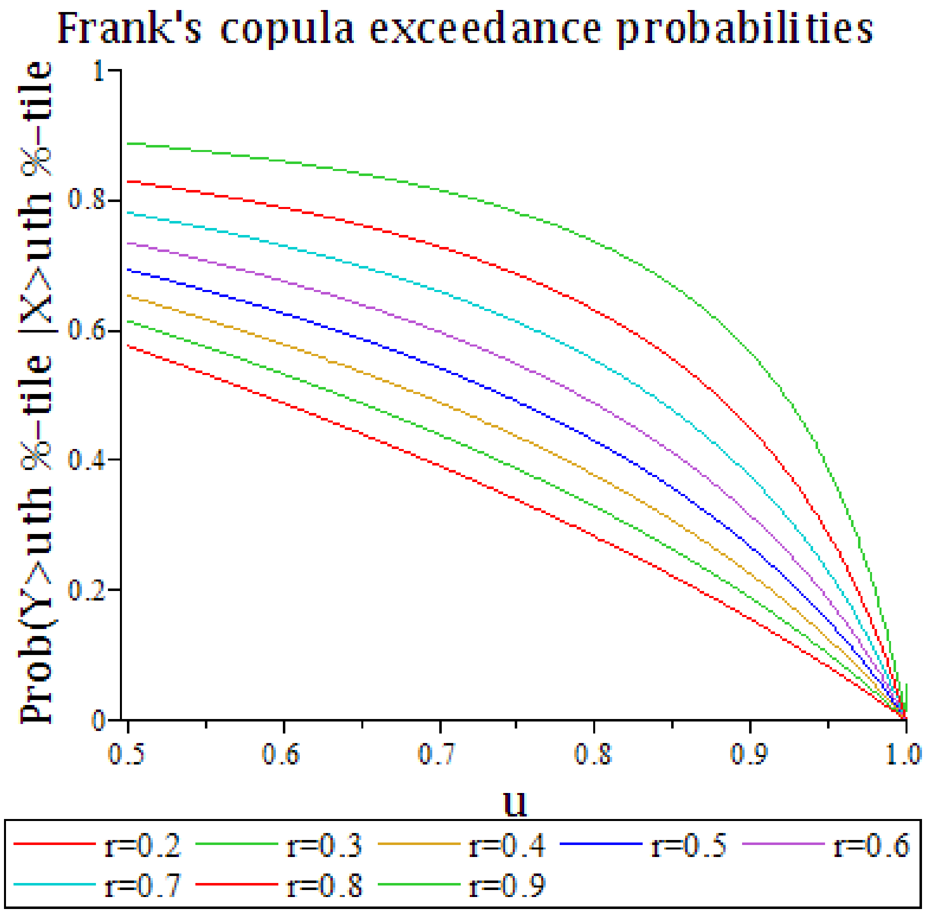
(reverse) Clayton 0.8
Upper Tail Dependent



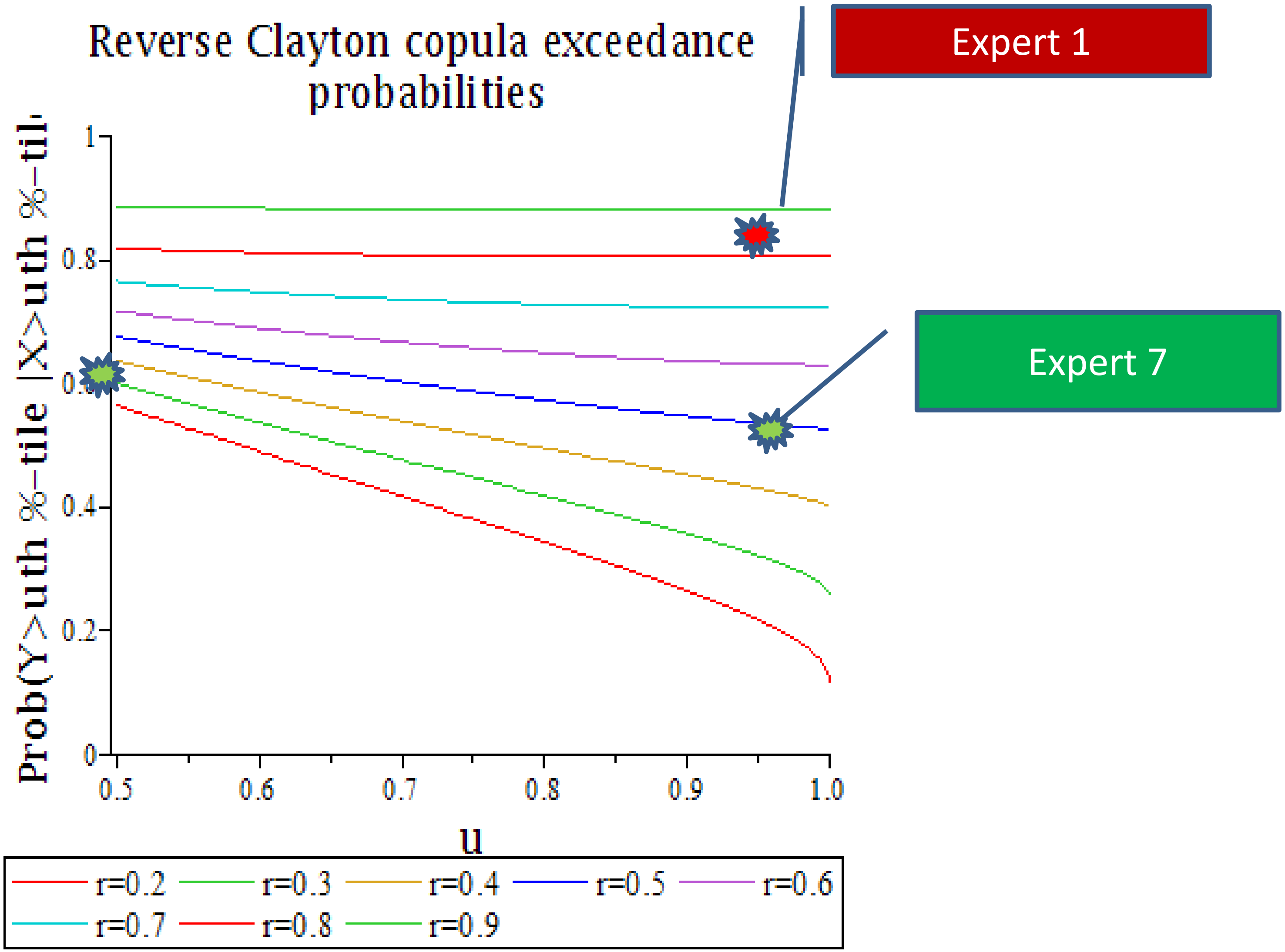


Frank Copula 0.8
Tail Independent





Greenland Ice Sheet
P(Discharge > D_{50} | Runoff > R_{50})
P(Discharge > D_{95} | Runoff > R_{95})



central correlation				
	Probability			Probability
Strong positive	0.9		Independent	0.5
medium positive	0.75		weak negative	0.4
weak positive	0.6		strong negative	0.1

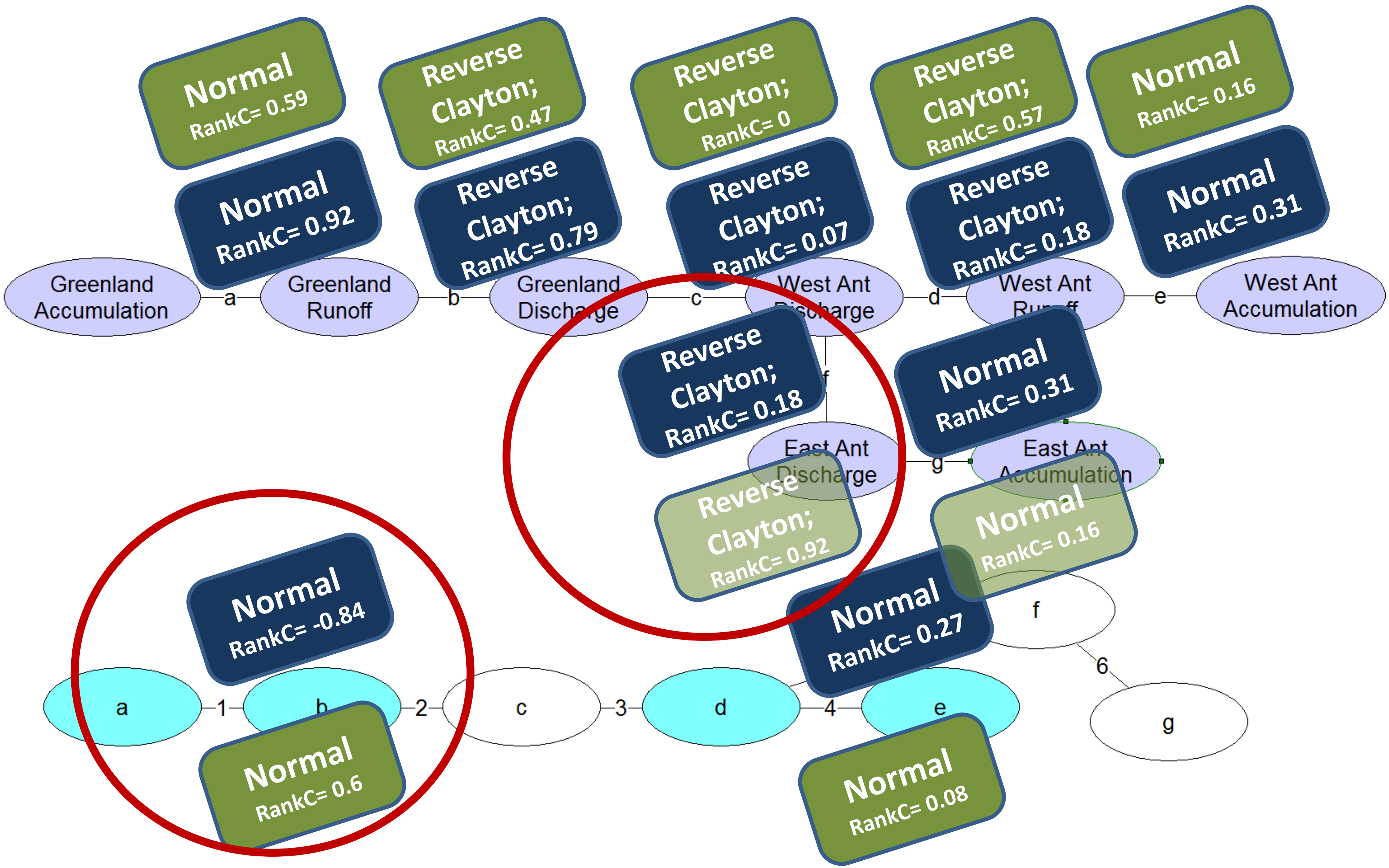
upper tail dependence (positive)	
strong	
moderate dependence	
moderate independence	
Strong independence	

Qualitative option:

Greenland Ice Sheet, 2100 5°C Warming	
Given ***runoff*** *≥ your 50% value, probability that* ***accumulation*** *also ≥ your 50% =*	**weak positive**
Given ***discharge*** *≥ your 50% value, probability that* ***accumulation*** *also ≥ your 50% =*	**independent**
Given ***runoff*** *≥ your 50% value, probability that* ***discharge*** *also ≥ your 50% =*	**strong positive**
Given ***runoff*** *≥ your 95% value, probability that* ***accumulation*** *also ≥ your 95% =*	**weak UTD**
Given ***discharge*** *≥ your 95% value, probability that* ***accumulation*** *also ≥ your 95% =*	**moderate Indep**
Given ***runoff*** *≥ your 95% value, probability that* ***discharge*** *also ≥ your 95% =*	**Strong UTD**

Does it Matter?

Ice sheet contribution to SLR by 2100CE with +3°C warming [mm]						
		mean	stdev	5%	50%	95%
Expert combination method	EW indep	615	270	238	581	1120
	PW Indep	335	200	71	307	719
	PW tail indep	337	216	64	305	749
	PW tail dep	338	229	71	292	785



Normal RankC= 0.59
Reverse Clayton; RankC= 0.47
Reverse Clayton; RankC= 0
Reverse Clayton; RankC= 0.57
Normal RankC= 0.16
Normal RankC= 0.92
Reverse Clayton; RankC= 0.79
Reverse Clayton; RankC= 0.07
Reverse Clayton; RankC= 0.18
Normal RankC= 0.31
Greenland Accumulation
a
Greenland Runoff
b
Greenland Discharge
c
West Ant Discharge
d
West Ant Runoff
e
West Ant Accumulation
Reverse Clayton; RankC= 0.18
f
Normal RankC= 0.31
East Ant Discharge
g
East Ant Accumulation
Reverse Clayton; RankC= 0.92
Normal RankC= 0.16
Normal RankC= -0.84
Normal RankC= 0.27
f
6
g
a
1
b
2
c
3
d
4
e
Normal RankC= 0.6
Normal RankC= 0.08