Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). For a strengthened form of the strong Goldbach conjecture and its negation, we show that both the conjecture and the negation can be derived. This contradiction is the consequence of two properties of a specific set which we use to reformulate the conjecture.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \ \vee \ x = mk = 4k.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 36: (**3·12**, 7·12, 11·12)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 64: (3·16, **4·16**, 5·16)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\forall$ k $\in \mathbb{N}$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The proof is motivated by the following view.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (C), every n given by ¬SSGB as well as every multiple nk, k $\in \mathbb{N}$, equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples if n exists (¬SSGB) is equal to that if n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.

First of all, we note that each of the two properties (C) and (M) is a condition sine qua non for the proof, for the following reasons.

¬(C) immediately implies ¬SSGB, since an n ≥ 4 different from all S_g triple components pk, mk, qk is in particular different from all m in S_g.

The proof would no longer be possible if, for example, we omitted the factor k in the definition of S_g, because then the corresponding (C) could no longer be guaranteed.

Similarly, the property (M) rules out the possibility that there is an $n \geq 4$ different from all m (i.e. ¬SSGB) and n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that ¬SSGB applies due to a missing prime number pair. This means that the proof would no longer be possible here either if we left out any prime number pair in the formulation of SSGB and S_g.

We will now show that ((C) ∧ (M)) leads to a contradiction.

The following proof is independent of the choice of n if there is more than one in the case of ¬SSGB. For example, the minimal such n works.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \;\; pk = yk' \vee mk = yk' \vee qk = yk' \}$ and

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \;\; pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by ¬SSGB as described above. Then, we have

(*) ¬SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption ¬SSGB with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form (pk = nk', mk, qk) with k = k' in case n is prime, due to (C)

(ii) S_g triples of the form (pk = nk', mk, qk) with k ≠ k' in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k, 4k = nk', 5k) in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk = nk', mk, qk), (pk, mk = nk', qk) or (pk, mk, qk = nk')

and

(v) S_g triples of the form (pk ≠ nk', mk ≠ nk', qk ≠ nk'), i.e. those S_g triples where none of the nk' equals a component.

So, $S_g+(n)$ is the union of the triples of the above types (i) to (iv) and $S_g-(n)$ is the union of the triples of type (v).

Now, we define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \ SSGB \text{ holds} \}$.

So, by (*) we obtain

(1) $\neg SSGB \Rightarrow S_1 = S_g = S_g+(n) \cup S_g-(n)$.

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we can write

(1') $\forall y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow S_1 = S_g = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So, we obtain

(2) $\forall y \in \mathbb{N}_3 \quad SSGB \Rightarrow S_2 = S_g = S_g+(y) \cup S_g-(y)$.

(1') and (2) yield

(3) $\forall y \in \mathbb{N}_3$

$$((\neg SSGB \Rightarrow S_1 = S_g = S_g+(y) \cup S_g-(y)) \quad \wedge \quad (SSGB \Rightarrow S_2 = S_g = S_g+(y) \cup S_g-(y))).$$

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \in \mathbb{N}$ we define

$M(k) := \{ mk \mid (pk, mk, qk) \in S_g \}$

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), (3) implies

(4) $\forall$ k $\in \mathbb{N}$ $\forall$ y $\in \mathbb{N}_3$

(($\neg$SSGB => M_1(k) = M(k) = { mk | (pk, mk, qk) $\in$ S_g+(y) $\cup$ S_g-(y) })

$\wedge$

(SSGB => M_2(k) = M(k) = { mk | (pk, mk, qk) $\in$ S_g+(y) $\cup$ S_g-(y) })).

We set M := M(1), M_1 := M_1(1) and M_2 := M_2(1). So we get

(4') $\forall$ y $\in \mathbb{N}_3$

(($\neg$SSGB => M_1 = M = { m | (p, m, q) $\in$ S_g+(y) $\cup$ S_g-(y) })

$\wedge$

(SSGB => M_2 = M = { m | (p, m, q) $\in$ S_g+(y) $\cup$ S_g-(y) })).

Since for every y $\in \mathbb{N}_3$ S_g+(y) $\cup$ S_g-(y) equals S_g, there is a set X such that for every y $\in \mathbb{N}_3$ { m | (p, m, q) $\in$ S_g+(y) $\cup$ S_g-(y) } equals X. So, we obtain

(5) we have proved ($\neg$SSGB => M_1 = M = X $\wedge$ SSGB => M_2 = M = X).

As by (5) we have shown that under both assumptions SSGB and $\neg$SSGB M = X holds, we have a proof of M = X.

The set X is either equal to $\mathbb{N}_4$ or a non-empty proper subset of $\mathbb{N}_4$. Then, we apply the rule of cases (see "*proof by exhaustion*") to (5) by using the two exhaustive cases X = $\mathbb{N}_4$ and X ≠ $\mathbb{N}_4$.

Thus, (5) splits into

(5.1) $X = \mathbb{N}_4$ => we have proved ($\neg$SSGB => $M_1 = M = X$ $\wedge$ SSGB => $M_2 = M = X$)

$\wedge$

(5.2) $X \neq \mathbb{N}_4$ => we have proved ($\neg$SSGB => $M_1 = M = X$ $\wedge$ SSGB => $M_2 = M = X$).

Now we replace the set X by what is defined in each case, so that by transitivity we get

(6.1) $X = \mathbb{N}_4$ => we have proved ($\neg$SSGB => $M_1 = M = \mathbb{N}_4$ $\wedge$ SSGB => $M_2 = M = \mathbb{N}_4$)

$\wedge$

(6.2) $X \neq \mathbb{N}_4$ => we have proved ($\neg$SSGB => $M_1 = M \neq \mathbb{N}_4$ $\wedge$ SSGB => $M_2 = M \neq \mathbb{N}_4$).

So, we obtain

(7.1) $X = \mathbb{N}_4$ => we have a proof of $M = \mathbb{N}_4$

$\wedge$

(7.2) $X \neq \mathbb{N}_4$ => we have a proof of $M \neq \mathbb{N}_4$.

And so,

(8.1) $M = \mathbb{N}_4$ => we have a proof of $M = \mathbb{N}_4$

$\wedge$

(8.2) $M \neq \mathbb{N}_4$ => we have a proof of $M \neq \mathbb{N}_4$.

As we have not proved either $M = \mathbb{N}_4$ or $M \neq \mathbb{N}_4$, (8.1) is equivalent to $M \neq \mathbb{N}_4$ and (8.2) is equivalent to $M = \mathbb{N}_4$.

So, since SSGB <=> $M = \mathbb{N}_4$, ((8.1) $\wedge$ (8.2)) is equivalent to (SSGB $\wedge$ $\neg$SSGB).

□