Riemann's Functional Equation is Not Valid and its Implication on the Riemann Hypothesis

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ABSTRACT

Riemann's functional equation was formulated by Riemann that supposedly extended the domain of the Riemann zeta function from the right half-plane into the entire complex plane except at $z = 1$. It also led him to obtained a real function that contains zeros for some z. Now, the real function was also related to the zeta function which in turn has something to do with the distribution of prime numbers. This led him to obtained a formula for relating the zeros of the zeta function to the number of primes given a certain number. Riemann then conjectured that all the zeros of the zeta function are at $z = ½ + iy$, which is now known as the **Riemann Hypothesis**. Hence the Riemann's functional equation *is* the foundation upon which the Riemann Hypothesis is based. But the functional equation, as shall be shown here, is not valid such that the Riemann Hypothesis crumbles on its claim.

Introduction

The Riemann zeta function (or zeta function) is shown below

$$\text{(1)} \qquad \zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad z = x + iy.$$

where z is a complex variable with real part x and imaginary part y, and i is the imaginary unit $i = \sqrt{-1}$. It is known that the infinite series in (1) is undefined if $x \le 0$, conditionally convergent if $0 < x \le 1$, and absolutely convergent if $x > 1$. $\zeta(z)$ is related to the distribution of prime numbers for one obtains from (1) the infinite product,

$$\text{(2)} \qquad \zeta(z) = \frac{1}{(1-2^{-z})(1-3^{-z})(1-5^{-z})(1-7^{-z}) \cdots} = \prod_{p}^{\infty} \frac{1}{1-p^{-z}}$$

This had driven Riemann to obtained a formula for relating the supposedly *non-trivial zeros* of $\zeta(½ + iy)$ with the number of primes given a certain number. A simple inspection of (2) and one can easily conclude that such zeros are nowhere to be found.

Analytic Continuation

If a function $f_1(z)$ is analytic in domain D_1 and a second function $f_2(z)$ is analytic in domain D_2:

(a) $f_2(z) \neq f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$.
(b) $f_2(z) = f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$.
(c) If $f_2(z) = f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$ and if $D_1 \subset D_2$ or D_2 is the *union* $D_1 \cup D_2$, then, $f_2(z)$ is the *analytic continuation* of $f_1(z)$ into the second domain D_2.

(a) Simply point to the fact that the two functions though seemingly equal, are not; while (b) shows that they are only equal at their common points. Analytic continuation is only recommended in (c) since $f_1(z)$'s domain will be extended from D_1 into domain D_2.

For example, consider the three functions shown below

$$f_1(z) = \frac{1}{1-z} = -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{1}{z^n} \qquad |z| > 1$$

$$f_2(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$\text{and}\quad f_3(z) = \frac{1}{1-z} \qquad z \neq 1$$

that have similar closed form, $1/(1 - z)$. But $f_2(z) \neq f_1(z)$ since they are on different domains while $f_1(z) = f_3(z)$ only at their intersection, that is, when their moduli are greater than one. The function $f_3(z)$ occupies the entire domain except at $z = 1$, hence $f_3(z)$ is the analytic continuation of either $f_1(z)$ or $f_2(z)$.

Table 1 Some Values for $f_1(z)$, $f_2(z)$, and $f_3(z)$

z	$f_1(z)$	$f_2(z)$	$f_3(z)$
-2	1/3	undefined	1/3
-1	undefined	undefined	1/2
0	undefined	1	1
1	undefined	undefined	undefined
2	-1	undefined	-1
3	-2	undefined	-2

As one can see from some of the values given on Table 1, at their intersection, $f_3(z) = f_1(z)$, as expected.

Simple Guides for Analytic Continuation

1. Let a function $f_1(z)$ be analytic in domain D_1, check if $f_1(z) \neq \infty$ for values of s that are not in D_1. If there is a function $f_2(z)$ that is analytic in domain D_2 equal to $f_1(z)$ at D_1 and $D_1 \subset D_2$, then, $f_2(z)$ is the analytical continuation of $f_1(z)$.
2. If $f_1(z) = \infty$ for all values of z that are not in D_1, then it is completely defined by its domain D_1.

For example, consider the function below

$$f_1(z) = \frac{1}{z-1} \qquad x > 1$$

for values of $z < 1$, $f_1(z) \neq \infty$, hence one can analytically continue $f_1(z)$ into $f_2(z)$, if there is one and maybe valid for all z except at $z = 1$. That is,

$$f_2(z) = \frac{1}{z-1} \qquad z \neq 1$$

But if one looks at the zeta function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

one can easily verify that $\zeta(z) = \infty$ if $x \le 0$: thus, a valid function equal to $\zeta(z)$ at $x \le 0$ does not exist. Therefore, $\zeta(z)$ could not be extended on the left half-plane. The zeta function is completely defined by (1) on the right half-plane.

Riemann's Functional Equation

Now, consider the widely known Riemann's "functional" equation

(3) $$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

Formula (3) is considered as the analytic continuation of (1) into the entire complex plane except at $s = 1$:

> "This equation now gives the value of the function $\zeta(s)$ for all complex numbers *s* and shows that this function is one-valued and finite for all finite values of *s* with the exception of 1, and also that it is zero if *s* is equal to a negative even integer".
>
> Bernhard Riemann *On the Number of Prime Numbers less than a Given Quantity* (page 3).

Table 2 Some Values of $\zeta(s)$ for (1) and (3)

$\zeta(s)$	Formula (1)	Formula (3)
$\zeta(-4)$	undefined	0
$\zeta(-3)$	undefined	1/120
$\zeta(-2)$	undefined	0
$\zeta(-1)$	undefined	-1/12
$\zeta(1)$	$1 + 1/2 + 1/3 + 1/4 + \cdots = \infty$	undefined
$\zeta(2)$	$\pi^2/6$	undefined
$\zeta(4)$	$\pi^4/90$	undefined

Since *s* is constant [4] so is $\zeta(s)$, that is, the same value of *s* should give us the same value of $\zeta(s)$. As can be seen from Table 2, the two formulas have no common points and are unequal for all *s*. Hence (3) is an invalid equation.

Now, consider the integral shown below

(4a) $$I(s) = \int_C (-z)^{s-1} e^{-z} dz$$

formula (4a) have to complex quantities *z* and *s*, where *z* is the complex variable, and *s* is a complex constant. One can also express (4a) as

$$I(s) = (-1)^{s-1}\int_{z_1}^{z_2} z^{s-1}e^{-z}dz$$

where z_1 and z_2 are the endpoints of path C and since $e^{-z} = 1 - \frac{z}{1!} + \frac{z}{2!} - \frac{z}{3!} + \cdots,$ we have

$$(-1)^{s-1}\int_{z_1}^{z_2} z^{s-1}e^{-z}dz = (-1)^{s-1}\int_{z_1}^{z_2}\left(z^{s-1} - z^s + \frac{z^{s+1}}{2!} - \frac{z^{s+2}}{3!} + \cdots\right)dz$$

(4b) $$(-1)^{s-1}\int_{z_1}^{z_2} z^{s-1}e^{-z}dz = (-1)^{s-1}\left(\frac{z^s}{s} - \frac{z^{s+1}}{s+1} + \frac{z^{s+2}}{(s+2)2!} - \frac{z^{s+3}}{(s+3)3!} + \cdots\right)\Bigg|_{z_1}^{z_2}$$

It is clear from (4b) that the contour integral is dependent on: the value of s, the endpoints of path C, and the chosen path itself. $(-1)^{s-1}$ is a multi-valued complex quantity with principal values $(-1)^{-1}(-1)^s = -e^{\pm(\pi is)}$.

For example, if the path C_1 is the circle: $z = Re^{i\theta}$ $0 \le \theta \le 2\pi$ and $0 < R < \infty$:

$$(-1)^{s-1}\int_{z_1}^{z_2} z^{s-1}e^{-z}dz = (-1)^{s-1}\left(\frac{z^s}{s} - \frac{z^{s+1}}{s+1} + \frac{z^{s+2}}{(s+2)2!} - \frac{z^{s+3}}{(s+3)3!} + \cdots\right)\Bigg|_{R}^{Re^{2\pi i}}$$

$$= (-1)^{s-1}\left\{\left(\frac{(Re^{2\pi i})^s}{s} - \frac{(Re^{2\pi i})^{s+1}}{s+1} + \frac{(Re^{2\pi i})^{s+2}}{(s+2)2!} - \cdots\right) - \left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)\right\}$$

$$= -e^{-\pi is}\left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)(e^{2\pi is} - 1) = \left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)(-e^{\pi is} + e^{-\pi is})$$

$$I_1(s) = -2i\sin(\pi s)R^s\left(\frac{1}{s} - \frac{R}{s+1} + \frac{R^2}{(s+2)2!} - \frac{R^3}{(s+3)3!} + \cdots\right)$$

As can be seen from $I_1(s)$, it depends on the appropriate value of s, the endpoints of path C_1, and the chosen path itself.

Now, if z = $x + i0$ and we take the contour integral of (4a) from $x = +\infty$ to $x = +\infty$, that is

$$I_2(s) = (-1)^{s-1}\left(\int_{+\infty}^{+\infty} x^{s-1}e^{-x}dx\right)$$

$$I_2(s) = (-1)^{s-1}\left(\int_{\infty}^{0} x^{s-1}e^{-x}dx + \int_{0}^{\infty} (x)^{s-1}e^{-x}dx\right)$$

$$I_2(s) = (-1)^{s-1}\left(-\int_0^\infty x^{s-1}e^{-x}dx + \int_0^\infty (x)^{s-1}e^{-x}dx\right)$$

$$I_2(s) = 0$$

Thus, $I_1(s) \neq I_2(s)$.

Now, consider the contour integral

(5) $$\int_C \frac{(-z)^{s-1}}{e^z-1}dz \quad .$$

The contour integral of (5) if z = $x + i0$ from x =+∞ to x = + ∞

$$I_3(s) = \int_{+\infty}^{+\infty} \frac{(-x)^{s-1}}{e^x-1}dx$$

$$I_3(s) = (-1)^{s-1}\left(\int_\infty^0 \frac{x^{s-1}}{e^x-1}dx + \int_0^\infty \frac{x^{s-1}}{e^x-1}dx\right)$$

$$I_3(s) = (-1)^{s-1}\left(-\int_0^\infty \frac{x^{s-1}}{e^x-1}dx + \int_0^\infty \frac{x^{s-1}}{e^x-1}dx\right)$$

$$I_3(s) = 0 \quad .$$

Thus,

$$I_3(s) = 0 \neq -2i\sin(\pi s)\zeta(s)\Gamma(s)$$

and

$$\zeta(s) \neq -\frac{\Gamma(1-s)}{2\pi i}\int_\infty^\infty \frac{(-x)^{s-1}}{e^x-1}dx$$

The Misapplication of the Residue Theorem

Riemann applied the Residue Theorem on (6) by assuming the quantities $2\pi ni$ for n = ±1, ±2, ±3, and so on, as the *poles* of (6). The function $q(z) = 1/(e^z - 1)$ in (6) is a periodic function of z,

$$q(z+2\pi i) = \frac{1}{e^{(z+2\pi i)} - 1} = \frac{1}{e^z - 1} \qquad \Re(z) > 0,$$

since the exponential function e^z is a periodic function with period $2\pi i$. Hence the term $2\pi ni$ are the multiple of the *fundamental period* $2\pi i$. It is, therefore, not valid to treat them as the poles of (6). In fact, the only pole of $q(z)$ is at $z = 0$ with residue 1,

$$q(z) = \frac{1}{e^z - 1} = e^{-z} + e^{-2z} + e^{-3z} \cdots + e^{-nz} = \sum_{n=1}^{\infty} e^{-nz}, \qquad \Re(z)>0,$$

so that its contour integral on a simple closed path is $2\pi i$, that is,

$$\oint \frac{dz}{e^z-1} = 2\pi i.$$

Unfortunately, one can not use $z = 0$ on (5) since it will be undefined. Thus, Riemann's determination led him to apply the Residue Theorem on (5)

$$\int_C \frac{(-z)^{s-1}}{e^z-1}\, dz = -2\pi i\left(\sum_{n=1}^{\infty}(-2\pi ni)^{s-1} + (2\pi ni)^{s-1}\right) = -2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s),$$

and by equating the last expression to $-2i\sin(\pi s)\zeta(s)\Gamma(s)$, we will have

$$-2i\sin(\pi s)\zeta(s)\Gamma(s) = -2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s).$$

One obtains the Riemann's functional equation (3),

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

And, by using equations

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \sqrt{\pi}\,\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

one arrives at the equality,

(7) $$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The equality in (7) is not valid or that the two equations are not equal. If they are equal, then the function to the left $\Phi(s)$ will be equal to its reflection $\Phi(1-s)$, $\Phi(1/2 + \omega i = \Phi(1/2 - \omega i))$. Thus, in accordance with the Reflection Principle

$$\overline{\Phi(s)} = \Phi(\bar{s}),$$

$\Phi(1/2 + \omega i)$ is real and will have zeros for some *ω*. It is known that $\Gamma\left(\frac{s}{2}\right)$ has no zeros for all *s* and this led Riemann to conjecture that all the zeros of $\zeta(s)$ are at *s* = ½ + *ωi,* which is the famous Riemann Hypothesis.

But the function $\Phi(s)$ has no zeros because it will be just like $\zeta(s)$ that has no zeros since its modulus is always greater than zero on the right half-plane.

The zeros that many are looking for are most likely the zeros of of the real part of $\Phi(1/2 + \omega i)$, that is,

$$\Re\{\Phi(1/2 + \omega i)\} = 0 \qquad \text{for some } \omega,$$

but $\Phi(1/2 + \omega i)$ does not converge due to $\zeta(1/2 + \omega i)$ [3].

Hence, the Riemann Hypothesis relies upon the validity of Riemann's functional equation and I've shown that to be not the case.

Conclusions

(a) The Riemann zeta function is completely defined by (1) on the right half-plane.
(b) The analytic continuation of an integral function must be performed *after* the integral has been evaluated and *not before.*
(c) $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \neq \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$.
(d) $\Phi(1/2 + \omega i)$ is not a real function and has *no* zeros just like the zeta function on the right half-plane.
(e) The Riemann hypothesis is false.

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LINKS

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- https://en.wikipedia.org/wiki/Riemann_hypothesis