Riemann's Functional Equation is Not Valid
and its Implication on the Riemann Hypothesis

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ABSTRACT

Riemann's functional equation was formulated by Riemann that supposedly extended the domain of the Riemann zeta function from the right half-plane into the entire complex plane except at $s = 1$. It also led him to obtained a real function that contains zeros for some *s*. Now, the real function was also related to the zeta function which in turn has something to do with the distribution of prime numbers. This led him to obtained a formula for relating the zeros of the zeta function to the number of primes given a certain number. Riemann then conjectured that all the zeros of the zeta function are at $s = ½ + \omega i$, which is now known as the **Riemann Hypothesis**. Hence Riemann's functional equation *is* the foundation upon which the Riemann Hypothesis is based. But the functional equation, as shall be shown here, is not valid such that the Riemann Hypothesis crumbles on its claim.

Riemann's Functional Equation

The Riemann zeta function (or zeta function) is shown below

(1) $$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s = \sigma + \omega i,$$

where s is a complex variable with real part σ and imaginary part ω, and i is the imaginary unit , $i = \sqrt{-1}$. It is known that the infinite series in (1) is undefined if $\sigma \le 0$, conditionally convergent if $0 < \sigma \le 1$, and absolutely converges if $\sigma > 1$. $\zeta(s)$ is related to the distribution of prime numbers for one obtains from (1) the infinite product,

(2) $$\zeta(s) = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})(1-7^{-s}) \cdots (1-p^{-s})} = \prod_{p}^{\infty} \frac{1}{1-p^{-s}}.$$

This had driven Riemann to obtained a formula for relating the supposedly *non-trivial zeros* of $\zeta(½+\omega i)$ with the number of primes given a certain number. A simple inspection of (2) and one can easily conclude that such zeros are nowhere to be found.

Analytic Continuation

If a function $f_1(s)$ is analytic in domain D_1 and a second function $f_2(s)$ is analytic in domain D_2:

(a) $f_2(s) \neq f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$.
(b) $f_2(s) = f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$.
(c) If $f_2(s) = f_1(s)$ for each s in the *intersection* $D_1 \cap D_2$ and if $D_1 \subset D_2$ or D_2 is the *union* $D_1 \cup D_2$, then, $f_2(s)$ is the *analytic continuation* of $f_1(s)$ into the second domain D_2.

(a) simply point to the fact that the two functions though seemingly equal, are not; while (b) shows that they are only equal at their common points. Analytic continuation is only recommended in (c) since $f_1(s)$'s domain will be extended from D_1 into domain D_2.

For example, consider the three functions shown below

$$f_1(s) = \frac{1}{1-s} = -\left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \cdots \frac{1}{s^n}\right) = -\sum_{n=1}^{\infty} \frac{1}{s^n}, \qquad |s| > 1,$$

$$f_2(s) = \frac{1}{1-s} = 1 + s + s^2 + s^3 + \cdots + s^n = \sum_{n=0}^{\infty} s^n, \qquad |s| < 1,$$

$$\text{and} \quad f_3(s) = \frac{1}{1-s}, \qquad s \neq 1,$$

that have similar closed form, $\frac{1}{1-s}$. But $f_2(s) \neq f_1(s)$ since they are on different domains while $f_1(s) = f_3(s)$ only at their intersection, that is, when their modulus are greater than one. The function $f_3(s)$ occupies the entire domain except at $s = 1$, hence $f_3(s)$ is the analytic continuation of either $f_1(s)$ or $f_2(s)$.

Table 1 Some Values for $f_1(s)$, $f_2(s)$, and $f_3(s)$

s	$f_1(s)$	$f_2(s)$	$f_3(s)$
-2	1/3	undefined	1/3
-1	undefined	undefined	1/2
0	undefined	1	1
1	undefined	undefined	undefined
2	-1	undefined	-1
3	-2	undefined	-2

As one can see from some of the values given on Table 1, at their intersection, $f_3(s) = f_1(s)$, as expected.

Now, consider the widely known Riemann's functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \tag{3}$$

where $\Gamma(1-s)$ and $\zeta(1-s)$ are the reflections of the gamma and the zeta functions,. Formula (3) is considered as the analytic continuation of (1) into the entire complex plane except at $s = 1$. The so-called *trivial zeros* of (3) are at each even integer $s = -2n$, $n \geq 1$.

Table 2 Some Values of $\zeta(s)$ for (1) and (3)

$\zeta(s)$	Formula (1)	Formula (3)
$\zeta(-4)$	undefined	0
$\zeta(-3)$	undefined	1/120
$\zeta(-2)$	undefined	0
$\zeta(-1)$	undefined	-1/12
$\zeta(1)$	$1 + 1/2 + 1/3 + 1/4 + \cdots = \infty$	undefined

$\zeta(2)$	$\pi^2 / 6$	I'm not sure
$\zeta(4)$	$\pi^4 / 90$	I'm not sure

As can be seen from Table 2, the two formulas have no common points and are unequal for all *s*. Hence (3) is not the analytic continuation of $\zeta(s)$ into the entire complex plane.

Simple analysis will reveal that (3) is false: $\zeta(s)$ as defined by (1) needs $\sigma > 1/2$ to converge while the right-side function in (3) needs $\sigma < 1/2$ due to the presence of $\zeta(1-s)$ [3]. The two functions, therefore, have no common points and are not equal.

Simple Guides for Analytic Continuation

1. Let a function $f_1(s)$ be analytic in domain D_1, check if $f_1(s) \neq \infty$ for values of *s* that are not in D_1. If there is a function $f_2(s)$ that is analytic in domain D_2 equal to $f_1(s)$ at D_1 and $D_1 \subset D_2$, then, $f_2(s)$ is the analytical continuation of $f_1(s)$.
2. If $f_1(s) = \infty$ for all values of *s* that are not in D_1, then, it is completely defined by its domain D_1.
3. *If* $f_1(s)$ is an integral, perform 1 and 2 *after* the integral has been evaluated, because analytic continuation can not be perform if the integral doesn't exist!

For example, consider the function below

$$f_1(s) = \frac{1}{s-1} \qquad \sigma > 1,$$

for values of $s < 1$, $f_1(s) \neq \infty$, hence one can analytically continue $f_1(s)$ into $f_2(s)$, if there is one and maybe valid for all *s* except at $s = 1$. That is,

$$f_2(s) = \frac{1}{s-1} \qquad s \neq 1.$$

Consider the integral

$$f_1(s) = \int_0^\infty e^{-(s-1)t} dt.$$

The integral exist if $\sigma > 1$ and its value being $1/(s-1)$

$$f_1(s) = \frac{1}{s-1} \qquad \sigma > 1.$$

The function

$$f_2(s) = \frac{1}{s-1} \qquad s \neq 1,$$

is the analytic continuation of $f_1(s)$ into the domain D_2, that is, analytic for all s except at $s = 1$.

But if one looks at the zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

one can easily verify that $\zeta(s) = \infty$ if $\sigma \leq 0$: thus, a valid function equal to $\zeta(s)$ at $\sigma \geq 0$ does not exist. Therefore, $\zeta(s)$ could not be extended on the left half-plane. The zeta function is completely defined by (1) on the right half-plane.

Now, consider the integral shown below

(4) $$I(s) = \int_C (-z)^{s-1} e^{-z} dz \qquad \Re(z) > 0 \text{ and } \sigma > 0,$$

formula (4) have to complex quantities z and s, where z is the complex variable, and s is a constant relative to z. The integral will converge if the real parts of z and s are greater than zero. Since s is a constant in (4), one can move $(-1)^{s-1}$ outside the integral sign, that is

$$I(s) = (-1)^{s-1} \int_C z^{s-1} e^{-z} dz,$$

$(-1)^{s-1}$ is a multivalued complex quantity with principal values $(-1)^{-1}(-1)^s = -e^{\pm(\pi i s)}$. Integrating (4) over the Hankel contour: it starts from $+\infty$ towards the circle with a very small radius ρ then goes around the circle counterclockwise once and goes to back $+\infty$.

$$I(s) = (-1)^{s-1}\left(\int_{\infty}^{0} x^{s-1} e^{-x} dx + \int_{o}^{2\pi} (\rho e^{i\theta})^{s-1} e^{-\rho e^{i\theta}} i \rho e^{i\theta} d\theta + \int_{0}^{\infty} (x e^{2\pi i})^{s-1} e^{-x e^{2\pi i}} dx \right),$$

the second integral above approaches zero as $\rho \to 0$

$$I(s) = (-1)^{s-1}\left(-\int_{0}^{\infty} x^{s-1} e^{-x} dx + + \int_{0}^{\infty} (x e^{2\pi i})^{s-1} e^{-x e^{2\pi i}} dx \right) = (-1)^{s-1}\left(\int_{0}^{\infty} x^{s-1} e^{-x} dx \right)(e^{2\pi i s} - 1),$$

and then choosing the principal value of $(-1)^{s-1} = -e^{-i\pi s}$,

(5) $$I(s) = -2i\sin(\pi s)\Gamma(s).$$

The integral in (4) is only valid if $\sigma > 0$, but (5) is now valid for negative values of σ except for the negative integers due to analytic continuation and has zeros at $s = n$ for integers $n \geq 1$.

Now, consider the contour integral

(6) $$I_1(s) = \int_C \frac{(-x)^{s-1}}{e^x - 1}\,dx = (-1)^{s-1}\int_\infty^\infty \frac{x^{s-1}}{e^x-1}dx \qquad x > 0 \text{ and } \sigma > 0,$$

$$I_1(s) = (-1)^{s-1}\left(\int_\infty^0 \frac{x^{s-1}}{e^x-1}dx + \int_0^{2\pi} \frac{(\rho e^{i\theta})^{s-1}}{e^{\rho e^{i\theta}}-1}\rho i e^{i\theta}\,d\theta + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^{xe^{2\pi i}}-1}dx\right),$$

$$(-1)^{s-1}\left(\int_0^\infty \frac{x^{s-1}}{e^x-1}dx\right)(e^{2\pi i s}-1),$$

and since

(7) $$\int_0^\infty \frac{x^{s-1}}{e^x-1}dx = \zeta(s)\Gamma(s),$$

thus,

(8) $$I_1(s) = -2i\sin(\pi s)\,\zeta(s)\Gamma(s).$$

Due to the presence of $\zeta(s)$ on (8), $I_1(s)$ is only valid on the right half-plane.

By using the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

and formulas (6), (7), and (8), one obtains

(9) $$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_\infty^\infty \frac{(-x)^{s-1}}{e^x-1}\,dx.$$

Riemann argued that $\zeta(s)$ defined by (9) is now valid over the entire *s*-plane except at *s* = 1, by simply rearranging formulas (6), (7), and (8)! One must bear in mind that the integral must first be evaluated in order for analytical continuation to be applied. Hence (9) is simply an identity, that is, the zeta function is equal to itself

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-x)^{s-1}}{e^x-1}dx = -\frac{\Gamma(1-s)}{2\pi i}\{-2i\sin(\pi s)\zeta(s)\Gamma(s)\} = \zeta(s).$$

Moreover, $\int_{\infty}^{\infty}\frac{(-x)^{s-1}}{e^x-1}dx$ involves the integral

$$\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx,$$

which will only be valid for positive values of σ and undefined for negative values of σ and $\sigma = 0$.

PROOF :
$$\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx = \int_0^{\infty}x^{s-1}\sum_{n=1}^{\infty}e^{-nx}dx = \left(\sum_{n=1}^{\infty}n^{-s}\right)\left(\int_0^{\infty}x^{s-1}e^{-x}dx\right)$$

and the integral $\int_0^{\infty}x^{s-1}e^{-x}dx$ is only defined if $\sigma > 0$, that is,

$$\int_0^{\infty}x^{s-1}e^{-x}dx = \int_0^{\infty}x^{s-1}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^n}{n!}\right)dx$$

$$= \left(\frac{x^s}{s} - \frac{x^{s+1}}{(s+1)1!} + \frac{x^{s+2}}{(s+2)2!} - \frac{x^{s+3}}{(s+3)3!} + \frac{x^{s+4}}{(s+4)4!} - \cdots + \frac{(-1)^n x^{s+n}}{(s+n)n!}\right)\Bigg|_0^{\infty},$$

since if one substitute negative values of σ or $\sigma = 0$ on the lower limit of the last expression above, the integral won't exist! Remember that analytic continuation should be perform *after* the integral has been obtained and not *before*. Thus, the integral in (9) is not valid if $\sigma \leq 0$ and (8) is the valid integral.

The Misapplication of the Residue Theorem

Riemann applied the Residue Theorem on (6) by assuming the quantities $2\pi ni$ for $n = \pm1, \pm2, \pm3$, and so on, as the *poles* of (6). The function $q(z) = 1/(e^z - 1)$ in (6) is a periodic function of z,

$$q(z+2\pi i) = \frac{1}{e^{(z+2\pi i)} - 1} = \frac{1}{e^z - 1} \qquad \Re(z) > 0,$$

since the exponential function e^Z is a periodic function with period $2\pi i$. Hence the term $2\pi ni$ are the multiple of the *fundamental period* $2\pi i$. It is, therefore, not valid to treat them as the poles of (6). In fact, the only pole of $q(z)$ is at $z = 0$ with residue 1,

$$q(z) = \frac{1}{e^z - 1} = e^{-z} + e^{-2z} + e^{-3z} \cdots + e^{-nz} = \sum_{n=1}^{\infty} e^{-nz} \ , \qquad \Re(z) > 0,$$

so that its contour integral on a simple closed path is $2\pi i$, that is,

$$\oint \frac{dz}{e^z - 1} = 2\pi i.$$

Unfortunately, one can not use $z = 0$ on (6) since it will be undefined. Thus, Riemann's determination led him to apply the Residue Theorem on (6), that is,

$$\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz = -2\pi i \left(\sum_{n=1}^{\infty} (-2\pi ni)^{s-1} + (2\pi ni)^{s-1} \right) = -2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s),$$

and by equating the last expression to formula (8),

(9) $$-2i\sin(\pi s)\zeta(s)\Gamma(s) = -2\pi i \sin\left(\frac{\pi s}{2}\right) 2^s \pi^{s-1} \zeta(1-s),$$

one obtains the Riemann's functional equation (3),

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

And, by using equations

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \sqrt{\pi}\,\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

one arrives at the equality,

(10) $$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The equality in (10), if it is valid, will mean that the function to the left, say $\Phi(s)$, will be equal to its reflection $\Phi(1-s)$; which means that at *s* = ½ + *ωi*, $\Phi(1/2 + \omega i)$ is equal to $\Phi(1/2 - \omega i)$, and, in accordance with the Reflection Principle

$$\overline{\Phi(s)} = \Phi(\bar{s}),$$

$\Phi(1/2 + \omega i)$ will be a real function. And so , if $\Phi(1/2 + \omega i)$ is real and has zeros for some values of *ω*, then $\zeta(1/2 + \omega i)$ has also zeros from (10). Since, it is known, that $\Gamma\left(\frac{s}{2}\right)$ has no zeros for all *s*, Riemann conjectured that all the zeros of $\zeta(s)$ are at *s* = ½ + *ωi*, which is the famous Riemann Hypothesis.

The function $\Phi(s)$ has no zeros if it ain't real, because it will be just like $\zeta(s)$ that has no zeros since its modulus is always greater than zero on the right half-plane. The zeros that many are looking for are most likely the zeros of of the real part of $\Phi(1/2 + \omega i)$, that is,

$$\Re\{\Phi(1/2 + \omega i)\} = 0 \quad \text{for some values of } \omega,$$

but $\Phi(1/2 + \omega i)$ does not converge due to $\zeta(1/2 + \omega i)$ [3].

Thus, the Riemann Hypothesis relies upon the validity of $\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz$ being equal to $-2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s)$ in oder for (10) to be true so that $\Phi(1/2 + \omega i)$ will be a real function that has zeros which has the implication of $\zeta(1/2 + \omega i)$ having zeros. I've shown that's not the case, and Riemann's functional equation could not be valid and the Riemann hypothesis rests on a false premise.

Conclusions

(a) The Riemann zeta function is completely defined by (1) on the right half-plane.

(b) The analytic continuation of an integral function must be perform *after* the integral has been evaluated and *not before.*

(c) The function $-2i\sin(\pi s)\zeta(s)\Gamma(s)$ is a valid contour integral of $\int_C \frac{(-x)^{s-1}}{e^x-1}\,dx$ while $-2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s)$ is *not*.

(d) I have noticed recently that Riemann applied the Residue theorem on $\int_C \frac{(-x)^{s-1}}{e^x-1}\,dx$ in the clockwise direction; it doesn't matter, since the Residue theorem can not be applied to it.

(e) From (c), the functional equation $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$ is not true.

(f) From (d), $\Phi(1/2 + \omega i)$ is not a real function and has *no* zeros just like the zeta function on the right half-plane.

(g) From (e), the Riemann hypothesis is, therefore, false.

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[2] Brown, James Ward; Churchill, Ruel V. (1996). *Complex Variables and Applications* (6th ed.). Singapore: McGraw Hill International Editions.

[3] Evangelista, Armando M. (2018). *The Domain of the Riemann Zeta Function on the Complex Plane.* http://vixra.org/abs/1808.0684

LINKS:

- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis