Riemann's Functional Equation is Not Valid and its Implication on the Riemann Hypothesis

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ABSTRACT

Riemann's "functional" equation was formulated by Riemann that supposedly extended the domain of the Riemann zeta function from the right half-plane into the entire complex plane except at $z = 1$. It *is* also the foundation upon which the Riemann Hypothesis is based: that all the zeros of the zeta function are at $z = ½ + iy$. But the equation, as shall be shown here is not valid such that the Riemann Hypothesis crumbles on its claim.

INTRODUCTION

The Riemann zeta function (or zeta function) is shown below

(1) $$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad z = x + i\,y$$

where z is a complex variable with real part x and imaginary part y, and i is the imaginary unit , $i = \sqrt{-1}$. It is known that the infinite series in (1) is undefined if $x \leq 0$, conditionally convergent if $½ < x \leq 1$, and absolutely converges if $x > 1$ [3]. $\zeta(z)$ is related to the distribution of prime numbers for one obtains from (1) the infinite product

(2) $$\zeta(z) = \frac{1}{(1-2^{-z})(1-3^{-z})(1-5^{-z})(1-7^{-z})\ \cdots} = \prod_{p}^{\infty} \frac{1}{1-p^{-z}}$$

This had driven Riemann to obtained a formula for relating the supposedly *non-trivial zeros* of $\zeta(½ + i\,y)$ with the number of primes given a certain number. A simple inspection of (2) and one can easily conclude that such zeros are nowhere to be found.

ANALYTIC CONTINUATION

If a function $f_1(z)$ is analytic in domain D_1 and a second function $f_2(z)$ is analytic in domain D_2:

(a) $f_2(z) \neq f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$.
(b) $f_2(z) = f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$.
(c) If $f_2(z) = f_1(z)$ for each z in the *intersection* $D_1 \cap D_2$ and if $D_1 \subset D_2$ or D_2 is the *union* $D_1 \cup D_2$, then, $f_2(z)$ is the *analytic continuation* of $f_1(z)$ into the second domain D_2.

(a) Simply point to the fact that the two functions, though seemingly equal, are not; while (b) shows that they are only equal at their common points. Analytic continuation is only recommended in (c) since $f_1(z)$'s domain will be extended from D_1 into domain D_2.

For example, consider the three functions shown below

$$f_1(z) = \frac{1}{1-z} = -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{1}{z^n} \qquad |z| > 1$$

$$f_2(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

and $$f_3(z) = \frac{1}{1-z} \qquad z \neq 1$$

that have similar closed form, $1/(1 - z)$. But $f_2(z) \neq f_1(z)$ since they are on different domains while $f_1(z) = f_3(z)$ only at their intersection, that is, when their modulus are greater than one. The function $f_3(z)$ occupies the entire domain except at $z = 1$, hence $f_3(z)$ is the analytic continuation of either $f_1(z)$ or $f_2(z)$.

Table 1 Some Values for $f_1(z)$, $f_2(z)$, and $f_3(z)$

z	$f_1(z)$	$f_2(z)$	$f_3(z)$
-2	1/3	undefined	1/3
-1	undefined	undefined	1/2
0	undefined	1	1
1	undefined	undefined	undefined
2	-1	undefined	-1
3	-2	undefined	-2

As one can see from some of the values given on Table 1, at their intersection, $f_3(z) = f_1(z)$, as expected.

Simple Guides for Analytic Continuation

1. Let a function $f_1(z)$ be analytic in domain D_1, check if $f_1(z) \neq \infty$ for values of z that are not in D_1. If there is a function $f_2(z)$ that is analytic in domain D_2 equal to $f_1(z)$ at D_1 and $D_1 \subset D_2$, then, $f_2(z)$ is the analytical continuation of $f_1(z)$.

2. If $f_1(z) = \infty$ for all values of z that are not in D_1, then it is completely defined by its domain D_1.

For example, consider the function below

$$f_1(z) = \frac{1}{z-1} \qquad x > 1$$

for values of $z < 1$, $f_1(z) \neq \infty$, hence one can analytically continue $f_1(z)$ into $f_2(z)$, if there is one and maybe valid for all z except at $z = 1$. That is,

$$f_2(z) = \frac{1}{z-1} \qquad z \neq 1$$

But if one looks at the zeta function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

one can easily verify that $\zeta(z) = \infty$ if $x \leq 0$: thus, a valid function equal to $\zeta(z)$ at $x \geq 0$ does not exist. Therefore, $\zeta(z)$ could not be extended on the left half-plane. The zeta function is completely defined by (1) on the right half-plane.

RIEMANN'S FUNCTIONAL EQUATION

Now, consider the widely known Riemann's "functional" equation

(3) $$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\,\zeta(1-s)$$

Formula (3) is considered as the analytic continuation of (1) into the entire complex plane except at $s = 1$:

> "This equation now gives the value of the function $\zeta(s)$ for all complex numbers s and shows that this function is one-valued and finite for all finite values of s with the exception of 1, and also that it is zero if s is equal to a negative even integer".

Bernhard Riemann *On the Number of Prime Numbers less than a Given Quantity* (page 3).

Table 2 Some Values of $\zeta(s)$ for (1) and (3)

$\zeta(s)$	Formula (1)	Formula (3)
$\zeta(-4)$	undefined	0
$\zeta(-3)$	undefined	1/120
$\zeta(-2)$	undefined	0

$\zeta(-1)$	undefined	-1/12
$\zeta(1)$	$1 + 1/2 + 1/3 + 1/4 + \cdots = \infty$	undefined
$\zeta(2)$	$\pi^2/6$	undefined
$\zeta(4)$	$\pi^4/90$	undefined

Since s is constant [4] and so is $\zeta(s)$, that is, the same value of s should give us the same value of $\zeta(s)$. As can be seen from Table 2, the two formulas have no common points and are unequal. Hence (3) is an invalid eqution.

Now, consider the integral shown below

$$I(s) = \int_C (-z)^{s-1} e^{-z} dz \tag{4a}$$

formula (4a) have to complex quantities z and s, where z is the complex variable, and s is a complex constant. One can also express (4a) as

$$I(s) = (-1)^{s-1} \int_{z_1}^{z_2} z^{s-1} e^{-z} dz$$

where z_1 and z_2 are the endpoints of path C and since $e^{-z} = 1 - \frac{z}{1!} + \frac{z}{2!} - \frac{z}{3!} + \cdots$, we have

$$(-1)^{s-1} \int_{z_1}^{z_2} z^{s-1} e^{-z} dz = (-1)^{s-1} \int_{z_1}^{z_2} \left(z^{s-1} - z^s + \frac{z^{s+1}}{2!} - \frac{z^{s+2}}{3!} + \cdots \right) dz$$

$$(-1)^{s-1} \int_{z_1}^{z_2} z^{s-1} e^{-z} dz = (-1)^{s-1} \left(\frac{z^s}{s} - \frac{z^{s+1}}{s+1} + \frac{z^{s+2}}{(s+2)2!} - \frac{z^{s+3}}{(s+3)3!} + \cdots \right) \Bigg|_{z_1}^{z_2} \tag{4b}$$

It is clear from (4b) that the contour integral is dependent on: the value of s, the endpoints of path C, and the chosen path itself. $(-1)^{s-1}$ is a multivalued complex quantity with principal values $(-1)^{-1}(-1)^s = -e^{\pm(\pi i s)}$.

For example, if the path C_1 is the circle: $z = Re^{i\theta}$ $0 \le \theta \le 2\pi$ and $R < \infty$:

$$(-1)^{s-1} \int_{z_1}^{z_2} z^{s-1} e^{-z} dz = (-1)^{s-1} \left(\frac{z^s}{s} - \frac{z^{s+1}}{s+1} + \frac{z^{s+2}}{(s+2)2!} - \frac{z^{s+3}}{(s+3)3!} + \cdots \right) \Bigg|_{R}^{Re^{2\pi i}}$$

$$= (-1)^{s-1}\left\{\left(\frac{(Re^{2\pi i})^{s}}{s} - \frac{(Re^{2\pi i})^{s+1}}{s+1} + \frac{(Re^{2\pi i})^{s+2}}{(s+2)2!} - \cdots\right) - \left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)\right\}$$

$$= -e^{-\pi i s}\left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)(e^{2\pi i s} - 1) = \left(\frac{R^s}{s} - \frac{R^{s+1}}{s+1} + \frac{R^{s+2}}{(s+2)2!} - \cdots\right)(-e^{\pi i s} + e^{-\pi i s})$$

$$I_1(s) = -2i\sin(\pi s)R^s\left(\frac{1}{s} - \frac{R}{s+1} + \frac{R^2}{(s+2)2!} - \frac{R^3}{(s+3)3!} + \cdots\right)$$

As can be seen from $I_1(s)$, it depends on the appropriate value of s, the endpoints of path C_1, and the chosen path itself.

Thus, getting the contour integral of (4a) over the Hankel contour will give us a different result. The Hankel contour starts from $+\infty$ towards the circle with a very small radius ρ then goes around the circle counterclockwise once and goes to back $+\infty$.

$$I_2(s) = (-1)^{s-1}\left(\int_{\infty}^{0} x^{s-1}e^{-x}dx + \int_{o}^{2\pi} (\rho e^{i\theta})^{s-1}e^{-\rho e^{i\theta}} i\rho e^{i\theta}d\theta + \int_{0}^{\infty} (xe^{2\pi i})^{s-1}e^{-xe^{2\pi i}}dx\right)$$

the second integral above approaches zero as $\rho \to 0$

$$I_2(s) = (-1)^{s-1}\left(-\int_0^{\infty} x^{s-1}e^{-x}dx + + \int_0^{\infty} (xe^{2\pi i})^{s-1}e^{-xe^{2\pi i}}dx\right) = (-1)^{s-1}\left(\int_0^{\infty} x^{s-1}e^{-x}dx\right)(e^{2\pi i s} - 1)$$

and then choosing the principal value of $(-1)^{s-1} = -e^{-i\pi s}$

(5) $$I_2(s) = -2i\sin(\pi s)\Gamma(s)$$

Thus, $I_1(s) \neq I_2(s)$.

Now, consider the contour integral

(6a) $$\int_C \frac{(-z)^{s-1}}{e^z - 1}dz$$

which can also be expressed as

(6b) $$(-1)^{s-1}\zeta(s)\int_{z_1}^{z_2} z^{s-1}e^{-z}dz = (-1)^{s-1}\zeta(s)\left(\frac{z^s}{s} - \frac{z^{s+1}}{s+1} + \frac{z^{s+2}}{(s+2)2!} - \frac{z^{s+3}}{(s+3)3!} + \cdots\right)\Bigg|_{z_1}^{z_2}$$

It is clear from (6b) that the contour integral is dependent on: the value of *s*, the endpoints of path *C*, and the chosen path itself. Hence getting the contour integral of (6a) using the Hankel contour will give us

$$I_1(s) = \int_{+\infty}^{+\infty} \frac{(-z)^{s-1}}{e^z-1}\,dz$$

$$I_1(s) = (-1)^{s-1}\left(\int_{\infty}^{0} \frac{x^{s-1}}{e^x-1}dx + \int_{0}^{2\pi} \frac{(\rho e^{i\theta})^{s-1}}{e^{\rho e^{i\theta}}-1}\rho i e^{i\theta}\,d\theta + \int_{0}^{\infty} \frac{(xe^{2\pi i})^{s-1}}{e^{xe^{2\pi i}}-1}dx\right)$$

$$(-1)^{s-1}\left(\int_{0}^{\infty} \frac{x^{s-1}}{e^x-1}dx\right)(e^{2\pi i s}-1)$$

and since

(7) $$\int_{0}^{\infty} \frac{x^{s-1}}{e^x-1}dx = \zeta(s)\Gamma(s)$$

thus,

(8) $$I_1(s) = -2i\sin(\pi s)\,\zeta(s)\Gamma(s) \qquad \Re(s)=\sigma>1$$

By using the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and formulas (6), (7), and (8), one obtains

(9) $$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x-1}\,dx$$

Riemann reasoned that $\zeta(s)$ as defined by (9) is now valid over the entire *s*-plane except at *s* = 1, by simply rearranging formulas (6), (7), and (8)! Hence (9) is simply an identity, that is, $\zeta(s)$ is equal to itself

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x-1}dx = -\frac{\Gamma(1-s)}{2\pi i}\{-2i\sin(\pi s)\zeta(s)\Gamma(s)\} = \zeta(s)$$

Moreover, $\int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x-1}dx$ involves the integral

$$\int_{0}^{\infty} \frac{x^{s-1}}{e^x-1}dx$$

which will only be valid for positive values of σ and undefined for negative values of σ and $\sigma = 0$.

PROOF :

$$\int_0^\infty \frac{x^{s-1}}{e^x-1}dx = \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx = \left(\sum_{n=1}^\infty n^{-s}\right)\left(\int_0^\infty x^{s-1}e^{-x}dx\right)$$

and the integral $\int_0^\infty x^{s-1}e^{-x}\,dx$ is only defined if $\sigma > 0$, that is,

$$\int_0^\infty x^{s-1}e^{-x}dx = \int_0^\infty x^{s-1}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)dx$$

$$= \left(\frac{x^s}{s} - \frac{x^{s+1}}{(s+1)1!} + \frac{x^{s+2}}{(s+2)2!} - \frac{x^{s+3}}{(s+3)3!} + \frac{x^{s+4}}{(s+4)4!} - \cdots\right)\Bigg|_0^\infty$$

since if one substitute negative values of σ or $\sigma = 0$ on the lower limit of the last expression above, the integral won't exist! Thus, the integral in (9) is not valid if $\sigma \le 0$ and (8) is a valid integral.

THE MISAPPLICATION OF THE RESIDUE THEOREM

Riemann applied the Residue Theorem on (6) by *excluding* the pole at $z = 0$ and by *changing* the value of $s(\sigma<0)$. Unfortunately, one can not use $z = 0$ on (6) since it will be undefined. Thus, Riemann's determination led him to apply the Residue Theorem on (6), that is,

$$\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz = -2\pi i\left(\sum_{n=1}^\infty (-2\pi ni)^{s-1} + (2\pi ni)^{s-1}\right) = -2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s)$$

and by equating the last expression to formula (8),

$$-2i\sin(\pi s)\zeta(s)\Gamma(s) = -2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s)$$

one obtains equation (3)

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

And, by using equations

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \qquad \sqrt{\pi}\,\Gamma(s) = 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),$$

one arrives at the equality,

(10) $$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\Phi(s) = \Phi(1-s)$$

Again, if *s* is constant and so is $\Phi(s)$ such that $\Phi(s) \neq \Phi(1-s)$. Therefore, (10) is an invaild equation and that the Riemann Hypothesis has no basis for the conjectured zeros of $\zeta(z)$.

CONCLUSIONS

(a) The Riemann zeta function is completely defined by (1) on the right half-plane of the *z*-domain.

(b) The quantity $-2i\sin(\pi s)\zeta(s)\Gamma(s)$ is a valid contour integral of $\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz$ while $-2\pi i\sin\left(\frac{\pi s}{2}\right)2^s\pi^{s-1}\zeta(1-s)$ is *not*.

(c) From (b), the "functional" equation $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$ is invalid.

(d) From (c), the Riemann hypothesis is, therefore, false.

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LINKS

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- https://en.wikipedia.org/wiki/Riemann_hypothesis