Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA) by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be derived. This contradiction is the consequence of two properties of a specific set which we use to reformulate the conjecture.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold*.

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \ \vee \ x = mk = 4k.$

A few examples of the covering:

$x = 19$: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

$x = 36$: (**3·12**, 7·12, 11·12)

$x = 38$: (**19·2**, 21·2, 23·2)

$x = 42$: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

$x = 64$: (3·16, **4·16**, 5·16)

$x = 10000$: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall\, p, q \in \mathbb{P}_3,\ p < q \quad \forall\, k \in \mathbb{N} \qquad (pk, mk, qk) \in S_g$, where $m = (p + q) / 2$.

The proof is motivated by the following view.

There are two possibilities for S_g, exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (C), every n given by ¬SSGB as well as every multiple nk, $k \in \mathbb{N}$, equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples if n exists (¬SSGB) is equal to that if n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case ¬SSGB they don't.

First of all, we note that each of the two properties (C) and (M) is a condition sine qua non for the proof, for the following reasons.

¬(C) immediately implies ¬SSGB, since an $n \geq 4$ different from all S_g triple components pk, mk, qk is in particular different from all m in S_g.
The proof would no longer be possible if, for example, we omitted the factor k in the definition of S_g, because then the corresponding (C) could no longer be guaranteed.

Similarly, the property (M) rules out the possibility that there is an $n \geq 4$ different from all m (i.e. $\neg$SSGB) and n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that $\neg$SSGB applies due to a missing prime number pair. This means that the proof would no longer be possible here either if we left out any prime number pair in the formulation of SSGB and S_g.

We will now show that $((C) \wedge (M))$ leads to a contradiction.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \;\; pk = yk' \;\vee\; mk = yk' \;\vee\; qk = yk' \}$ and

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \;\; pk \neq yk' \;\wedge\; mk \neq yk' \;\wedge\; qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by $\neg$SSGB as described above. Then, we have

(*) $\neg$SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption $\neg$SSGB with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form (pk = nk', mk, qk) with k = k' in case n is prime, due to (C)

(ii) S_g triples of the form (pk = nk', mk, qk) with $k \neq k'$ in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k, 4k = nk', 5k) in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk = nk', mk, qk), (pk, mk = nk', qk) or (pk, mk, qk = nk')

and

(v) S_g triples of the form ($pk \neq nk'$, $mk \neq nk'$, $qk \neq nk'$), i.e. those S_g triples where none of the nk' equals a component.

So, $S_g+(n)$ is the union of the triples of the above types (i) to (iv) and $S_g-(n)$ is the union of the triples of type (v).

Now, we define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \ SSGB \text{ holds} \}$.

So, by (*) we obtain

(1) $\neg SSGB \Rightarrow S_1 = S_g+(n) \cup S_g-(n)$.

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we can write

(1') $\forall\, y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow S_1 = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So, we obtain

(2) $\forall\, y \in \mathbb{N}_3 \quad SSGB \Rightarrow S_2 = S_g+(y) \cup S_g-(y)$.

(1') and (2) yield

(3) $\forall\, y \in \mathbb{N}_3 \quad ((\neg SSGB \Rightarrow S_1 = S_g+(y) \cup S_g-(y)) \wedge (SSGB \Rightarrow S_2 = S_g+(y) \cup S_g-(y)))$.

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \in \mathbb{N}$ we define

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), (3) implies

(4) $\forall\, k \in \mathbb{N} \quad \forall\, y \in \mathbb{N}_3$

$$((\neg SSGB \;\Rightarrow\; M_1(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \})$$

$$\wedge$$

$$(\quad SSGB \;\Rightarrow\; M_2(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \})).$$

We set $M_1 := M_1(1)$ and $M_2 := M_2(1)$. So we get

(4') $\forall\, y \in \mathbb{N}_3$

$$((\neg SSGB \;\Rightarrow\; M_1 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \})$$

$$\wedge$$

$$(\quad SSGB \;\Rightarrow\; M_2 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \})).$$

Since by definition for every $y \in \mathbb{N}_3$ $S_g+(y) \cup S_g-(y)$ equals S_g regardless of whether or not SSGB holds, for every $y \in \mathbb{N}_3$ the set $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ either equals $\mathbb{N}_4$ or a proper subset of $\mathbb{N}_4$ regardless of whether or not SSGB holds. This means that

(5) Whatever the set $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ is equal to — $\mathbb{N}_4$ or a proper subset of $\mathbb{N}_4$ — it is the same in both implications of (4').

Under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg$SSGB they don’t. Therefore, we have

(6.1) $SSGB \;\Rightarrow\; M_2 = \mathbb{N}_4$

and

(6.2) $\neg SSGB \;\Rightarrow\; M_1 \neq \mathbb{N}_4$.

Then, applying (5), from (4') and (6.1) we get

$(\neg SSGB \Rightarrow M_1 = \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = \mathbb{N}_4)$,

and from (4') and (6.2) we get

$(\neg SSGB \Rightarrow M_1 \neq \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 \neq \mathbb{N}_4)$.

Thus, we obtain

$$(\neg SSGB \Rightarrow M_1 = \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = \mathbb{N}_4)$$

(7) $\wedge$

$$(\neg SSGB \Rightarrow M_1 \neq \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 \neq \mathbb{N}_4).$$

Also, we have $SSGB \Rightarrow M_1 = \{\ \}$ and $\neg SSGB \Rightarrow M_2 = \{\ \}$. Together with (6.1) and (6.2), this yields

(8) $SSGB \Leftrightarrow M_2 = \mathbb{N}_4$

and

(9) $M_1 \neq \mathbb{N}_4$.

Then, due to (8) and (9), (7) becomes

$(\neg SSGB \Rightarrow FALSE \quad \wedge \quad TRUE)$

$\wedge$

$(\qquad\qquad TRUE \quad \wedge \quad SSGB \Rightarrow \neg SSGB)$.

And this yields $(SSGB \wedge \neg SSGB)$. □