Lorentz Invariant -Et+px Vs. Nonrelativistic Oscillator Scaling and Quantum Wavelength

Francesco R. Ruggeri Hanwell, N.B. Aug. 27, 2022

In a previous note (1) we argued that the notion of a quantum wavelength follows from the comparison of the classical energy conservation equation for an oscillator with the time-independent Schrodinger equation. In particular for the classical oscillator pp/2m + k/2 xx = E transforms into cos(wt)cos(wt) + sin(wt)sin(wt) = 1 which shows the oscillatory nature of the system explicitly. The time independent Schrodinger equation must also fit this form in the sense that kinetic energy and potential energy must become functions of a variable y such that there are no coefficients. E on the RHS is multiplied by a factor and cannot depend on parameters such as k or m and so is proportional to sqrt(k/m) i.e. the angular frequency.
As a result momentum squared scales as 1/xx and momentum as 1/x. Thus momentum is of the form of 1/wavelength.These arguments seem to rely only on nonrelativistic physics.
In a previous note (2) we argued that the quantum wavelength follows from p scaling x in the Lorentz invariant -Et+px. This is strictly a relativistic argument so how can it be compatible with the nonrelativistic oscillator argument suggested above?
We use the result of (3) in which we argue that for a relativistic Klein-Gordon oscillator near the turning points velocity must be very low and then become 0. Thus a relativistic oscillator equation should become a nonrelativistic one in the neighbourhood of turning points. In other words the time-independent Schrodinger equation should hold in such a neighbourhood. Classically the form cos(wt)cos(wt)+sin(wt)sin(wt) should also hold. Thus p behaving as 1/x must apply both in the relativistic scenario because one has the Klein-Gordon equation, but also due to the classical oscillator scaling situation because the time-independent Schrodinger equation holds in the vicinity of the endpoints. Thus both arguments for p behaving as 1/x should apply and be consistent with each other.

Classical Oscillator and the Time-Independent Schrodinger Equation

The classical oscillator energy conservation equation: pp/2m + k/2 xx = E may be transformed into cos(wt)cos(wt) + sin(wt)sin(wt) = 1 which explicitly shows periodicity. This requires w=sqrt(k/m), x= Asin(wt), v= dx/dt and E=.5kAA. Any amplitude A may be chosen within reason.
The time-independent Schrodinger equation for any energy level and any potential mimics the form of classical conservation of energy i.e. KE(x) + V(x) = En. The difference with classical physics is that En is discrete and KE(x) = -1/2m d/dx dW/dx / W, (but this function must equal the classical kinetic energy at x). As a result, for the oscillator one expects scaling of x such that all coefficients disappear from the KE(x) and V(x) terms because the classical expression is of the form: cos(wt)cos(wt) + sin(wt)sin(wt) = 1. This occurs if one scales x→ y such that yy= .5sqrt(km) xx, but on the RHS one has E sqrt(m/k) so E should be proportional to sqrt(k/m) = w=classical angular frequency so that all parameters disappear. This approach “works” because the momentum operator is -id/dx i.e. proportional to 1/x. Thus momentum is of the form 1/wavelength which is a nonclassical result, namely quantum result, which emerges by

comparing an equation with an average kinetic energy and V(x) to a classical result. In particular if one uses:

KE(x) = {Sum over p pp/2m f(p,x) } / {Sum over p f(p,x)} ((1))

Then f(p,x) = f(px) and p scales as 1/x. These ideas have been presented in (1).

-Et+px as a Lorentz Invariant

The Lorentz invariant -Et+px suggests that p scales x and E scales t. -Et+px = A happens to be both the relativistic and nonrelativistic free particle classical action associated with a velocity v=x/t. If one considers x=1/p and t=1/E then -E/E + p/p =0. Thus there seems to be a region of uncertainty for x and t which is compatible with a given A corresponding to v=x/t. In other words the particle moves with v=x/t, but has a wavelength proportional to 1/p and a frequency proportional to 1/E. These arguments follow strictly from the relativistic invariant quantity -Et+px. No mention of nonrelativistic physics is made.

Dilemma

In the above two sections different arguments are presented for the existence of a quantum wavelength proportional to 1/p. The first uses nonrelativistic physics only and is even strongly based on classical mechanics. The second argument is strictly relativistic. How is it possible for these two to be consistent?

To attempt to resolve this issue we consider the relativistic Klein-Gordon equation for an oscillator. As in (3) we argue that at the classical turning points (endpoints) the velocity is 0 so it must be very low in a neigbourhood of these endpoints. As a result, there is no reason to use the relativistic Klein-Gordon equation in such a small neighbourhood. The time-independent Schrodinger equation should be fine and in fact a nonrelativistic classical conservation of energy would also apply (classically). Thus even in a relativistic problem, one may see the classical oscillator problem emerge.

The relativistic problem may be associated with a superposition of free particle wavefunctions which are linked to -Et+px. Thus the relativistic argument for p scaling x i.e. the existence of a wavelength proportional to 1/p holds. At the same the nonrelativistic oscillator emerges at the turning point and it requires p to be proportional to 1/x when compared to the classical oscillator equation. The relativistic result, however, already ensures that p goes as 1/x i.e the p operator is -id/dx. This holds both relativistically and nonrelativistically. Thus we argue that the two approaches, relativistic and the nonrelativistic oscillator, are closely linked and compatible. Both approaches consistently lead to p behaving as 1/x.

Conclusion

In conclusion we argue that two seemingly mutually exclusive arguments, namely -Et+px a Lorentz invariant implying p scales x and a nonrelativistic time-independent Schrodinger scaling of momentum as 1/x for an oscillator are compatible. We note that relativistically the momentum operator is -id/dx and this also holds nonrelativistically so both relativistic and nonrelativistic momentum scale as 1/x. In the oscillator case, however, one does not need to know that momentum behaves as 1/x a priori. One may define KE(x) = { Sum over p pp/2m f(p,x) } / {Sum over p f(p,x)} and conclude that p goes as 1/x and f(p,x) =f(px) by using scaling arguments (without any notion of a Lorentz invariant).

The two approaches are consistent, we argue, because given a Klein Gordon relativistic oscillator equation, a nonrelativistic type quantum equation should hold near the endpoints (i.e. the time-independent Schrodinger equation). This may be compared to the nonrelativistic classical conservation of energy equation in this neighbourhood and the conclusion that p behaves as 1/x deduced. This same result, however, is consistent with the relativistic result which holds for the general problem i.e. the Klein-Gordon equation whose solution should be a linear combination of exp(ipx) terms linked to -Et+px which is a relativistic invariant. Thus there is compatibility and yet one may use either relativistic or nonrelativstic physics to deduce that wavelength is proportional to 1/p.

References

1. Ruggeri, Francesco R. Energy-Classical Frequency in Quantum Oscillator Part IV (preprint, zenodo,2022)
2. Ruggeri, Francesco R. Quantum Momentum Scaling px and Special Relativity (preprint, zenodo,2022)
3. Ruggeri, Francesco R. Relativistic Harmonic Oscillator and the Klein Gordon Equation (preprint, zenodo,2019)