COMPl$_e$ib: *CO*nstrained *M*atrix–optimization *P*roblem *lib*rary – a collection of test examples for nonlinear semidefinite programs, control system design and related problems

F. LEIBFRITZ *

Abstract. The purpose of this paper is to describe a collection of test examples which can be used for testing and comparing algorithms for nonlinear semidefinite programs (NSDPs), bilinear matrix inequality (BMI) problems, (linear) control system design and related problems. *COMPl$_e$ib* consists of examples collected from the engineering literature and real–life sources for linear time–invariant (LTI) control systems. It contains models from real world engineering applications as well as idealized or pure academic versions of problems similar to those arising in practice. This collection may serve directly for test purposes in the construction of new tools for LTI control system design, i. e. algorithms for model reduction, Riccati (Lyapunov) equation solvers, tools for $\mathcal{H}_\infty$, $\mathcal{H}_2$ or mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis.

On the other hand, it is well–known, that a very wide variety of control synthesis problems can be reformulated as constrained matrix optimization problems. In example, the design of a low–order output feedback controller for a LTI system is an important instance of a difficult and in general non–convex, nonlinear control problem which can be transformed into a (non–convex) nonlinear semidefinite program. We present NSPD as well as BMI formulations of several fixed–order synthesis problems which can be formed by using the data of *COMPl$_e$ib* . Then, after forming the individual NSDP or BMI problem from *COMPl$_e$ib* , it is straightforward to use *COMPl$_e$ib* as a test set environment for (special) NSDP or BMI solvers like IPCTR [23], SSDP [12] or PENBMI [18]. Moreover, as a byproduct, we explain the usage of the test collection as a tool for testing and comparing linear SDP solvers like SeDuMi [27] or SDPT3 [29].

Finally, *COMPl$_e$ib* is also an interesting test environment for algorithms computing (optimal) ϵ–pseudospectral abscissas or complex stability radi of (non–symmetric) matrices of the form $A + BFC$, where A, B, C are given matrices and F denotes the unknown matrix variable (see, i. e. [9], [7], [10]).

Key Words. *collection of test examples; constrained matrix optimization problem; nonlinear semidefinite program; bilinear matrix inequality; output feedback control; control system design; pseudospectral abscissa*

AMS subject classification.

1. Data structure in *COMPl$_e$ib* and formulation of the (closed loop) feedback control problem . The current version of *COMPl$_e$ib* 1.0 consists of more than 120 examples collected from the engineering literature and real–life (engineering) applications for LTI control systems. A typical instance of such a control system can be stated as follows. Consider a LTI plant of order n_x with state space realization:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_1w(t) + Bu(t), \\ z(t) & = & C_1x(t) + D_{11}w(t) + D_{12}u(t), \\ y(t) & = & Cx(t) + D_{21}w(t), \end{array} \tag{1.1}$$

where $x \in \boldsymbol{R}^{n_x}$, $u \in \boldsymbol{R}^{n_u}$, $y \in \boldsymbol{R}^{n_y}$, $z \in \boldsymbol{R}^{n_z}$, $w \in \boldsymbol{R}^{n_w}$ denote the state, control input, measured output, regulated output, and noise input, respectively. The current version of *COMPl$_e$ib* consists simply of the data matrices $A \in \boldsymbol{R}^{n_x \times n_x}$, $B_1 \in \boldsymbol{R}^{n_x \times n_w}$, $B \in \boldsymbol{R}^{n_x \times n_u}$, $C_1 \in \boldsymbol{R}^{n_z \times n_x}$, $D_{11} \in \boldsymbol{R}^{n_z \times n_w}$, $D_{12} \in \boldsymbol{R}^{n_z \times n_u}$, $C \in \boldsymbol{R}^{n_y \times n_x}$ and $D_{21} \in \boldsymbol{R}^{n_x \times n_w}$. In particular, all 124 test problems in *COMPl$_e$ib* 1.0 are coded and stored in standard MATLAB matrix format. We have decided to use this format, since the main advantage of MATLAB is the platform independence. The heart of *COMPl$_e$ib* is the MATLAB function file *COMPleib.m*. In a MATLAB environment, the data of the individual test example of *COMPl$_e$ib* can be accessed by calling the MATLAB function therein. For example, in MATLAB, the command

>> [A,B1,B,C1,C,D11,D12,D21,nx,nw,nu,nz,ny] = COMPleib('AC1');

returns the real data matrices A, B_1, B, C_1, C, D_{11}, D_{12} and D_{21} of (1.1) as well as the integers (dimension parameters) n_x, n_w, n_u, n_z and n_y of the *COMPl$_e$ib* example *AC1*. Together with the MATLAB function file *COMPleib.m*, *COMPl$_e$ib* is provided with several binary MATLAB data files (MAT-files) which contains the data matrices of some individual (large)

*University of Trier, Department of Mathematics, D–54286 Trier, Germany. (leibfr@uni-trier.de)

test examples. In particular, release 1.0 of *COMPl$_e$ib* contains also the following MAT-files: *ac10.mat, ac13_14.mat, ac18.mat, bdt2.mat, cbm.mat, cdp.mat, cm1.mat – cm6.mat (6 files), dlr2_3.mat, he6.mat, he7.mat, hf2d1.mat – hf2d18.mat (18 files), ih.mat, iss1_2.mat, je1.mat, je2_3.mat, lah.mat, tl.mat*. Note, the name of the MAT-file corresponds to the example name in *COMPl$_e$ib* . For more details we refer to the *COMPl$_e$ib* user manual [22]. Moreover, more information on the individual examples in *COMPl$_e$ib* are given in the companion paper [21] (see also Section 4).

Depending on the control design goals, it is possible to derive particular constrained matrix–optimization problems from the specific control synthesis problem. Thus, the given data in *COMPl$_e$ib* can be used as a benchmark collection for a very wide variety of algorithms solving matrix–optimization problems; in example, it can be used for testing solvers for nonlinear semidefinite programs (NSDPs), bilinear matrix inequality (BMI) problems, linear SDPs, Riccati or Lyapunov equations, respectively. Moreover, as a byproduct, *COMPl$_e$ib* is also interesting for testing procedures for other (related) matrix problems. In example, the minimization of the ϵ–pseudospectral abscissa of $A + BFC$, defined as the largest real part of all elements of the pseudospectrum of $A + BFC$, for a fixed $\epsilon \geq 0$ and unknown $F \in \boldsymbol{R}^{n_u \times n_y}$. Another instance is the computation of complex stability radius of the matrix $A + BFC$ (see, i. e. [9], [7], [10] and [30]). A description of some of these constrained matrix optimization problems will be given in the paragraph below.

Typically, we assume that the triple (A, B, C) is stabilizable and detectable (see, i. e. [32, p. 51 and p. 52]). For a given integer $0 \leq n_c \leq n_x$ consider the n_c–th reduced order (output feedback) control (ROC) law:

$$\begin{array}{rcl} \dot{x_c}(t) & = & A_c x_c(t) + B_c y(t), \\ u(t) & = & C_c x_c(t) + D_c y(t), \end{array} \tag{1.2}$$

where $x_c \in \boldsymbol{R}^{n_c}$ denotes the state of the dynamic ROC law and the controller matrices $A_c \in \boldsymbol{R}^{n_c \times n_c}$, $B_c \in \boldsymbol{R}^{n_c \times n_y}$, $C_c \in \boldsymbol{R}^{n_u \times n_c}$, $D_c \in \boldsymbol{R}^{n_u \times n_y}$ are not known, respectively. We collect the unknown control variables in the gain matrix F, defined by

$$F = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \in \boldsymbol{R}^{(n_c+n_u)\times(n_c+n_y)} \tag{1.3}$$

Note, depending on the order n_c of the control law (1.2), we obtain the following classifications:

(i) if $n_c \equiv 0$ then $F \equiv D_c$ and (1.2) coincides with the so–called static output feedback (SOF) control law

$$u(t) = Fy(t), \quad F \in \boldsymbol{R}^{n_u \times n_y}. \tag{1.4}$$

If $0 < n_c < n_x$ (usually we have $n_c \ll n_x$) with n_c fixed, then F is given by (1.3) and (1.2) is the so–called fixed output feedback ROC law. In general, the corresponding control design problems are instances for which no convex formulation has been found. They can be only transformed to non–convex NSDPs, or, equivalently, to non–convex BMI problems.

(ii) if, in addition to (i) in case of $n_c \equiv 0$, we set $C = I_{n_x}$ and $D_{21} = 0$ in (1.1), i. e. $y = x$, then (1.2) coincides with the so–called static state feedback (SF) control law

$$u(t) = Fx(t), \quad F \in \boldsymbol{R}^{n_u \times n_x}. \tag{1.5}$$

In this case, it is always possible to reformulate the underlying control design problems to linear (and, thus convex) SDPs.

(iii) if $n_c \equiv n_x$, (1.2) is a so–called full–order dynamic output feedback control law, and, if, in addition $C = I_{n_x}$ and $D_{21} = 0$ in (1.1), (1.2) denotes a full–order dynamic state feedback controller. Typically, when $n_c \equiv n_x$, the usual optimal control design problems are also rewritable to convex matrix optimization problems, i. e. to linear SDPs.

The reduced order controller synthesis problem can be always transformed to a static controller synthesis problem by augmentation of the plant state x with the controller state x_c (see, i. e. [28]). In particular, we redefine the state, control and measurement variables by their augmented counterparts

$$x(t) := \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix}, \quad y(t) := \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}, \tag{1.6}$$

and the corresponding dimensions by $n_x := n_x + n_c$, $n_y := n_y + n_c$, $n_u := n_u + n_c$, respectively. Moreover, we redefine the state space data by their augmented counterparts as follows:

$$\begin{gathered} A := \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \quad B := \begin{bmatrix} 0 & B \\ I_{n_c} & 0 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & I_{n_c} \\ C & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D_{21} := \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \\ C_1 := \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad D_{12} := \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad D_{11} := D_{11}. \end{gathered} \tag{1.7}$$

If we replace the system quantities in (1.1) by the augmented counterparts and substitute the ROC law (1.2) into the augmented state space plant, then we get the closed loop system in SOF form:

$$\Sigma_{cl} \quad \begin{cases} \dot{x}(t) &= A(F)x(t) + B(F)w(t), \\ z(t) &= C(F)x(t) + D(F)w(t), \end{cases} \tag{1.8}$$

where

$$A(F) = A + BFC, \quad B(F) = B_1 + BFD_{21}, \quad C(F) = C_1 + D_{12}FC, \quad D(F) = D_{11} + D_{12}FD_{21}$$

are the (augmented) closed loop matrices, respectively.

2. Constrained matrix–optimization problems . In this section we describe several constrained matrix–optimization problems which can be built by using the data matrices defined in *COMPl$_e$ib* . We describe several feedback control design problems and state some of the corresponding constrained matrix optimization problems. Depending on the particular design goals, it is possible to derive corresponding (non–convex) NSDPs or BMI–problems or linear SDPs, respectively. Moreover, from the data matrices in *COMPl$_e$ib* it is also possible to formulate algebraic Riccati or Lyapunov equations which arise in many feedback control design problems. In the following subsections we describe and collect several meaningful feedback design control problems that can be formulated as constrained matrix optimization problems. Note, however, there are much more contributions in the control literature of control problems leading to NSDPs or BMIs. Thus, our list is far from being exhaustive. In example, fixed order output feedback design problems are typical examples of (in general) non–convex and nonlinear matrix optimization problems. Generally, these problems can be formulated as nonlinear semidefinite programs, or, equivalently, as bilinear matrix inequality problems. Note, these problems are not only nonlinear, they are also non–convex and thus sometimes more difficult to solve than convex matrix optimization problems. For an interesting classification of the difficulty of such problems (in particular of BMIs), we refer the interested reader to the discussion stated in [14, Section 1].

On the other hand, in example, considering state feedback control design problems, it was noticed in [3] that a clever change of variables in the original bilinear matrix optimization problems leads to linear SDPs. The formulation of several control problems that can be formulated as linear matrix inequality (LMI) problems are stated in [5] and [13]. Finally, note, most of the linear (and therefore convex) SDPs collected in the last two references have at least a BMI counterpart. Some of them are stated below.

2.1. NSDP and BMI formulation of several output feedback reduced order design problems . In this paragraph, we describe several output feedback reduced order control design problems and state the corresponding NSDP as well as BMI–problem formulation. Particularly, we suppose that $0 \leq n_c < n_x$ is fixed. In this case, the corresponding constrained matrix–optimization problems are non convex and nonlinear. We focus our main discussion on the following instances: ROC–$\mathcal{H}_2$, ROC–$\mathcal{H}_\infty$ and ROC–$\mathcal{H}_2/\mathcal{H}_\infty$, respectively. The concepts of $\mathcal{H}_\infty$ norm and $\mathcal{H}_2$ norm are well known (see, i.e., [32]). Therefore, we will omit detailed discussion and content ourselves with starting the following definitions for reference. Assume the reduced order control law (1.2) is fixed such that the closed loop system (1.8) is internally stable, i.e., the real parts of the eigenvalues of $A(F)$ are all strictly negative. In this case, $A(F)$ is called *Hurwitz*. Due to the Lyapunov theorem (see, i.e., [26, Theorem 11, §4]) there is an elegant representation of the Hurwitz property. It reduces the stability problem to an algebraic problem.

Theorem 2.1 (Lyapunov Theorem). *Let $A \in \mathbf{R}^{n_x \times n_x}$, $B \in \mathbf{R}^{n_x \times n_u}$ and $C \in \mathbf{R}^{n_y \times n_x}$ be given, then the following are equivalent:*

(i) There exists $F \in \mathbf{R}^{n_u \times n_y}$ such that $A(F) = A + BFC$ is Hurwitz.

(ii) For each $W \in \mathbf{R}^{n_x \times n_x}$, there exists $F \in \mathbf{R}^{n_u \times n_y}$ such that the Lyapunov equation

$$A(F)^T X + XA(F) + W = 0 \tag{2.1}$$

has a unique solution $X \in \mathbf{R}^{n_x \times n_x}$. If $W \succ 0$ ($W \succeq 0$), then $X \succ 0$ ($X \succeq 0$).

(iii) There exist $F \in \mathbf{R}^{n_u \times n_y}$ and $V \in \mathcal{S}^{n_x}$ such that

$$\mathcal{F}_s := \{F \in \mathbf{R}^{n_u \times n_y} \mid \exists\, V \in \mathcal{S}^{n_x} : \ A(F)^T V + VA(F) = -I \prec 0, \quad V \succ 0\} \neq \emptyset, \tag{2.2}$$

where $\mathcal{F}_s$ denotes the set of stabilizing ROC gains F.

Obviously, there exist other equivalent definitions of $\mathcal{F}_s$. In particular, (2.2) is also equivalent to the matrix conditions:

$$\exists\, F \in \mathbf{R}^{n_u \times n_y},\ V \in \mathcal{S}^{n_x} : \quad A(F)^T V + VA(F) + I = 0, \quad V \succ 0. \tag{2.3}$$

2.1.1. Optimal fixed order $\mathcal{H}_2$ synthesis: NSDP and BMI–problem formulation . The most basic optimal control problem is the following $\mathcal{H}_2$ design problem (see i.e. [19], [28] and the references therein):

Optimal fixed order $\mathcal{H}_2$ synthesis: *Suppose that $D_{11} = 0$ and $D_{21} = 0$. Given real matrices $A, B, C, B_1, C_1, D_{12}$ and an integer $0 \leq n_c < n$, find a controller gain F of order n_c such that the closed loop matrix $A(F)$ is Hurwitz and the $\mathcal{H}_2$ norm of the closed loop system (1.8) is minimal.*

If $A(F)$ is Hurwitz, it is well known that the $\mathcal{H}_2$ norm of the closed loop system (1.8) is given by

$$||\Sigma_{cl}||^2_{\mathcal{H}_2} = \langle Q, C(F)^T C(F) \rangle = Tr(C(F)QC(F)^T), \tag{2.4}$$

where $Q \in \mathcal{S}^{n_x}$ satisfies the Lyapunov equation

$$A(F)Q + QA(F)^T + B_1 B_1^T = 0. \tag{2.5}$$

Note, due to the assumption $D_{11} = 0$ and $D_{21} = 0$, we have $D(F) = 0$ and $B(F) = B_1$. Hence, using Theorem 2.1, (2.3) and (2.4), the optimal fixed order output feedback $\mathcal{H}_2$ problem is equivalent to the following nonlinear semidefinite program:

$$\begin{array}{ll} \min_{F,Q,V} & Tr((C_1 + D_{12}FC)Q(C_1 + D_{12}FC)^T) \\ \text{s. t.} & (A + BFC)Q + Q(A + BFC)^T + B_1 B_1^T = 0, \\ & (A + BFC)V + V(A + BFC)^T + I = 0, \ V \succ 0. \end{array} \tag{2.6}$$

Note, it is essential to use different Lyapunov variables Q and V. Here, an optimal Q corresponds with the solution of the Lyapunov equation (2.5), while an optimal V together with an optimal F satisfies the stability constraints (2.3). At an optimal point (F, L, V) of (2.6), we can only guarantee that, in general, $B(F)B(F)^T$ is positive semidefinite and $Q \succeq 0$. Hence, it is very likely, that the set of optimal solutions of (2.6) is empty if we would use the same matrix variable for the stability constraint (2.3) and the Lyapunov matrix equation (2.5).

The bilinear dependence of the constraints on the free controller parameter F and the variables Q, V make the problem non–convex. The SDP–constraints $A(F)^T V + VA(F) = -I \prec 0$, $V \succ 0$ ensure the internal stability of the closed loop system (1.8), i. e. $A(F)$ is Hurwitz. Due to the cone constraint, (2.6) is a non convex and nonlinear semidefinite program, the so–called optimal $\mathcal{H}_2$–NSDP. If the solution set of (2.6) is not empty, then F solves the optimal fixed order $\mathcal{H}_2$ synthesis problem.

An equivalent formulation of (2.6) is the following NSDP:

$$\begin{array}{ll} \min_{F,P,W} & Tr(PB_1B_1^T) \\ \text{s. t.} & (A+BFC)^T P + P(A+BFC) + (C_1 + D_{12}FC)^T (C_1 + D_{12}FC) = 0, \\ & (A+BFC)^T W + W(A+BFC) \prec 0, \ W \succ 0. \end{array} \tag{2.7}$$

Due to a quadratic term in F, the NSDPs (2.6) and (2.7) are nonlinear and not only bilinear constrained matrix–optimization problems. For deriving an equivalent BMI formulation of (2.6), in addition, we should suppose that $B_1B_1^T \succ 0$. If this is not satisfied (i. e. in *COMPl$_e$ib*), one can use $B_1B_1^T + \epsilon I_{n_x}$ instead of $B_1B_1^T$ for a fixed small positive scalar ϵ. Note, this assumption is necessary for guaranteeing that $Q \succ 0$ makes sense. Then, introducing a further matrix variable $X \in \mathbf{R}^{n_z \times n_z}$ and applying the Schur–Complement–Lemma (see, i. e. [5], [16, Theorem 7.76], [19, Lemma 2.2.1]) to the nonlinear matrix inequality (NMI)

$$X - C(F)QC(F) = X - C(F)QQ^{-1}QC(F)^T \succeq 0,$$

we get the following equivalent BMI–problem formulation of (2.6):

$$\begin{array}{ll} \min_{F,Q,X} & Tr(X) \\ \text{s. t.} & (A+BFC)Q + Q(A+BFC)^T + B_1B_1^T \preceq 0, \ Q \succ 0 \\ & \begin{bmatrix} X & (C_1 + D_{12}FC)Q \\ Q(C_1 + D_{12}FC)^T & Q \end{bmatrix} \succeq 0. \end{array} \tag{2.8}$$

Note, (2.8) is bilinear in F and Q, but it is still non convex due to the bilinearity of the free matrix variables.

2.1.2. Optimal fixed order $\mathcal{H}_\infty$ synthesis: NSDP and BMI–problem formulation . $\mathcal{H}_\infty$ synthesis is an attractive model–based control design tool and it allows incorporation of model uncertainties in the control design (see i. e. the pioneering paper of Zames [31]). In this paragraph, we reformulate the optimal fixed order $\mathcal{H}_\infty$ problem (for the ROC system (1.8)) to a general NSDP and also state an equivalent BMI formulation of this problem class (see, i.e., [15], [19]).

Let a scalar $\gamma > 0$ be given. Assume that $A(F)$ is Hurwitz and the $\mathcal{H}_\infty$ norm of (1.8) is less than γ, i. e. $||\Sigma_{cl}||_{\mathcal{H}_\infty} < \gamma$. Then it is a standard fact (see i. e. [19, Corollary 2.1.8] or [33]) that this is equivalent to the existence of a unique matrix $P \in \mathcal{S}^{n_x}$ with $P \succeq 0$ and $F \in \mathbf{R}^{n_u \times n_y}$ satisfying the (continuous time) algebraic Riccati equation

$$A(F)^T P + PA(F) + \gamma^{-1} C(F)^T C(F) + \gamma^{-1} M(F,P,\gamma) R(F,\gamma)^{-1} M(F,P,\gamma)^T = 0, \tag{2.9}$$

and

$$R(F,\gamma) := I_{n_w} - \gamma^{-2} D(F)^T D(F) \succ 0, \tag{2.10}$$

where $R : \mathbf{R}^{n_u \times n_y} \times \mathbf{R} \to \mathcal{S}^{n_w}$ and the matrix function $M : \mathbf{R}^{n_u \times n_y} \times \mathcal{S}^{n_x} \times \mathbf{R} \to \mathbf{R}^{n_x \times n_w}$ is defined as

$$M(F,P,\gamma) := PB(F) + \gamma^{-1} C(F)^T D(F). \tag{2.11}$$

Moreover, the perturbed system matrix $\tilde{A}(F,\gamma)$ is Hurwitz, where the (in general) non–symmetric matrix function $\tilde{A} : \boldsymbol{R}^{n_u \times n_y} \times \mathcal{S}^{n_x} \times \boldsymbol{R} \to \boldsymbol{R}^{n_x \times n_x}$ is given by

$$\tilde{A}(F,P,\gamma) := A(F) + \gamma^{-1} B(F) R(F,\gamma)^{-1} M(F,P,\gamma)^T. \tag{2.12}$$

Therefore, we formulate the optimal fixed order $\mathcal{H}_\infty$ problem as follows:

Optimal fixed order $\mathcal{H}_\infty$ synthesis: Given real matrices $A, B, C, B_1, C_1, D_{11}, D_{12}, D_{21}$ and an integer $0 \le n_c < n_x$, find a controller gain F of order n_c, $P \in \mathcal{S}^{n_x}$ and $\gamma > 0$ such that for minimal γ, the triple (F,P,γ) satisfies the Riccati equation (2.9), $R(F,\gamma) \succ 0$, $P \succeq 0$ and the Hurwitz property of $\tilde{A}(F,P,\gamma)$.

Using Theorem 2.1, (2.9) and (2.10), the optimal fixed order $\mathcal{H}_\infty$ control problem is equivalent to the following non–convex and nonlinear semidefinite program:

$$\begin{array}{ll} \min_{F,P,W,\gamma} & \gamma \\ \text{s. t.} & A(F)^T P + P A(F) + \gamma^{-1} C(F)^T C(F) + \gamma^{-1} M(F,P,\gamma) R(F,\gamma)^{-1} M(F,P,\gamma)^T = 0, \\ & \tilde{A}(F,P,\gamma)^T W + W \tilde{A}(F,P,\gamma) + I = 0, \;\; R(F,\gamma) \succ 0 \;\; W \succ 0, \;\; P \succeq 0, \;\; \gamma > 0. \end{array} \tag{2.13}$$

This version of the so–called $\mathcal{H}_\infty$–NSDP is highly nonlinear in the free variables. The advantage of this formulation is the usage of nonlinear matrix equations (one Riccati and one Lyapunov equation) instead of difficult NMIs and "simple linear non–negativity" conditions on some of the variables plus the biquadratic matrix inequality $R(F,\gamma) \succ 0$. In example, the positive definite condition on W which models together with the Lyapunov matrix equation the Hurwitz property of the perturbed system matrix $\tilde{A}(F,P,\gamma)$ and the positive semidefinite condition on P. In particular, an interior point type solver can exploit the inherent structure of the $\mathcal{H}_\infty$–NSDP (2.13) (see, i. e. [23], [24], [25], [19]), which is not possible if we use the strict bounded real lemma (see, i. e. [5], [33]) for deriving an equivalent optimal BMI version of the $\mathcal{H}_\infty$–NSDP (see, i. e. [15]).

Using the strict bounded real lemma, one can show that the existence of F and $P \succeq 0$ satisfying (2.9), (2.10) and (2.12) is equivalent to the existence of $F \in \boldsymbol{R}^{n_u \times n_y}$ and $X \in \mathcal{S}^{n_x}$ with $X \succ P \succeq 0$ satisfying the strict Riccati inequality

$$A(F)^T X + X A(F) + \gamma^{-1} C(F)^T C(F) + \gamma^{-1} M(F,X,\gamma) R(F,\gamma)^{-1} M(F,X,\gamma)^T \prec 0, \tag{2.14}$$

and $R(F,\gamma) \succ 0$, where the matrix functions $R(F,\gamma)$ and $M(F,X,\gamma)$ are defined as in (2.10) and (2.11), respectively. Then, applying the Schur–Complement–Lemma on (2.14) and (2.10) twice, we obtain the following BMI version of the optimal fixed order $\mathcal{H}_\infty$ synthesis problem (see, i. e. [15, Problem 2]):

$$\min_{F,X,\gamma} \gamma \;\text{ s. t. }\;\; X \succ 0, \quad \gamma > 0, \quad \begin{bmatrix} A(F)^T X + X A(F) & X B(F) & C(F)^T \\ B(F)^T X & -\gamma\, I_{n_w} & D(F)^T \\ C(F) & D(F) & -\gamma\, I_{n_z} \end{bmatrix} \prec 0. \tag{2.15}$$

Due to the bilinearity of the free matrix variables F and X, the BMI–formulation of the ROC–$\mathcal{H}_\infty$ is also a non–convex and nonlinear constrained matrix–optimization problem. The major drawback of the BMI–formulation (2.15) is the bilinear matrix inequality. As far as we know, it is not possible to exploit the inherent structure of this BMI–problem in an optimization solver. Moreover, note, the left hand side of the matrix inequality lies in $\mathcal{S}^{n_x+n_w+n_z}$.

Assuming $D_{11} = 0$ and $D_{21} = 0$, i. e. there is no sensor noise and no noise acting on the regulated output, then the $\mathcal{H}_\infty$–NSDP (2.13) reduces to

$$\begin{array}{ll} \min_{F,P,W,\gamma} & \gamma \\ \text{s. t.} & A(F)^T P + P A(F) + \gamma^{-1} C(F)^T C(F) + \gamma^{-1} P B_1 B_1^T P = 0, \;\; P \succeq 0 \\ & (A(F) + \gamma^{-1} B_1 B_1^T P)^T W + W (A(F) + \gamma^{-1} B_1 B_1^T P) + I = 0, \;\; W \succ 0, \;\; \gamma > 0. \end{array} \tag{2.16}$$

Note, in this case, we have $D(F) = 0$ and $B(F) = B_1$. Thus, $R(F,\gamma) \equiv I_{n_w}$ and $M(F,P,\gamma) \equiv PB_1$, respectively.

In the last part of this paragraph, we state the following equivalent (dual) versions of the $\mathcal{H}_\infty$–NSDPs (2.13), (2.16) and the $\mathcal{H}_\infty$–BMI formulation (2.15). An alternative formulation of (2.13) is the following NSDP:

$$
\begin{array}{ll}
\min_{F,Q,V,\gamma} & \gamma \\
\text{s. t.} & A(F)Q + QA(F)^T + \gamma^{-1}B(F)B(F)^T + \gamma^{-1}\hat{M}(F,Q,\gamma)^T\hat{R}(F,\gamma)^{-1}\hat{M}(F,Q,\gamma) = 0, \\
& \hat{A}(F,Q,\gamma)V + V\hat{A}(F,Q,\gamma)^T + I = 0, \quad \hat{R}(F,\gamma) \succ 0 \quad V \succ 0, \quad Q \succeq 0, \quad \gamma > 0,
\end{array}
\tag{2.17}
$$

where

$$
\hat{R}(F,\gamma) := I_{n_z} - \gamma^{-2}D(F)D(F)^T, \quad \hat{M}(F,Q,\gamma) := C(F)Q + \gamma^{-1}D(F)B(F)^T
$$

and

$$
\hat{A}(F,Q,\gamma) := A(F) + \gamma^{-1}\hat{M}(F,Q,\gamma)^T\hat{R}(F,\gamma)^{-1}C(F).
$$

If $D_{11} = 0$ and $D_{21} = 0$, (2.17) reduces to

$$
\begin{array}{ll}
\min_{F,Q,V,\gamma} & \gamma \\
\text{s. t.} & A(F)Q + QA(F)^T + \gamma^{-1}B_1B_1^T + \gamma^{-1}QC(F)^TC(F)Q = 0, \quad Q \succeq 0 \\
& (A(F) + \gamma^{-1}QC(F)^TC(F))V + V(A(F) + \gamma^{-1}QC(F)^TC(F))^T + I = 0, \quad V \succ 0, \quad \gamma > 0,
\end{array}
\tag{2.18}
$$

which in turn is equivalent to (2.16). Finally, the following BMI–problem

$$
\min_{F,Y,\gamma} \gamma \ \text{ s. t. } \ Y \succ 0, \quad \gamma > 0, \quad \begin{bmatrix} A(F)Y + YA(F)^T & YC(F)^T & B(F) \\ C(F)Y & -\gamma\, I_{n_z} & D(F) \\ B(F)^T & D(F)^T & -\gamma\, I_{n_w} \end{bmatrix} \prec 0. \tag{2.19}
$$

is equivalent to (2.15).

2.1.3. Optimal fixed order $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis: NSDP and BMI–problem formulation . A combination of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design objectives leads to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis (see, i.e., [2], [20], [17]). For simplifying the representation of the corresponding NSDP and BMI formulation, respectively, during the whole subparagraph, we assume that $D_{11} = 0$ and $D_{21} = 0$. Due to this simplifying assumption, the regulated output of the closed loop system (1.8) is not driven (directly) by a noise input signal. Therefore, the $\mathcal{H}_2$ norm of Σ_{cl} is finite and given by (2.4), if $A(F)$ is Hurwitz. Note, for simplifying our presentation, we have assumed (implicitly) that the representation of z and y is noise free. For the more general case, one alternative is to consider two different noise signals, i.e., w_0, w_1, and transfer functions, i. e. T_0, T_1, of the closed loop system. In example, T_0 maps w_0 into z and defines the $\mathcal{H}_2$ norm. Similarly, T_1 maps w_1 into z and is used for describing the $\mathcal{H}_\infty$ norm bound (see, i.e., [2], [20], [17]).

The design goals are the following: For a given positive scalar γ, the $\mathcal{H}_\infty$ norm of the closed loop system Σ_{cl} is less than γ, $A(F)$ is Hurwitz and the $\mathcal{H}_2$ norm of Σ_{cl} is minimal. Formally, we have:

Optimal fixed order $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis: *Given real matrices $A, B, C, B_1, C_1, D_{11}, D_{12}, D_{21}$ with $D_{11} = 0, D_{21} = 0$, a scalar $\gamma > 0$ and an integer $0 \le n_c < n$, find a controller gain F of order n_c, such that $A(F)$ is Hurwitz, $||\Sigma_{cl}||_{\mathcal{H}_\infty} < \gamma$ and $||\Sigma_{cl}||_{\mathcal{H}_2}$ is minimal.*

By the discussion of the $\mathcal{H}_\infty$ problem, we already know that $A(F)$ is Hurwitz and $||\Sigma_{cl}||_{\mathcal{H}_\infty} < \gamma$ if and only if there exist F and $Q \succeq 0$ satisfying the (simplified) Riccati equation

$$
A(F)Q + QA(F)^T + \gamma^{-1}B_1B_1^T + \gamma^{-1}QC(F)^TC(F)Q = 0 \tag{2.20}
$$

such that $A(F) + \gamma^{-1}QC(F)^TC(F)$ is Hurwitz (compare, i. e. the definition of the $\mathcal{H}_\infty$–NSDP (2.18)). On the other hand, the $\mathcal{H}_2$ norm of Σ_{cl} is given by (2.4) and (2.5), if $A(F)$ is Hurwitz. Obviously, a common solution which fulfills both equations may not exists. One way out of this dilemma is to introduce a further matrix variable Q_0 which satisfies the Lyapunov equation and enter the objective function of the corresponding NSDP. Another possibility is due to the following Lemma, which is similar to [19, Lemma 4.1.1] (see also the discussion in [20, Section 2] and the references therein).

LEMMA 2.2. *Consider the closed loop system Σ_{cl} and let $\gamma > 0$ be given. Suppose $A(F)$ is Hurwitz. If there exists a pair (F, Q) satisfying (2.20) such that $A(F) + \gamma^{-1}QC(F)^TC(F)$ is Hurwitz, then*

a) $||\Sigma_{cl}||_{\mathcal{H}_\infty} < \gamma$

b) $0 \preceq Q_0 \preceq Q$, where $Q_0 \succeq 0$ satisfies (2.5). Consequently, we have

$$||\Sigma_{cl}||^2_{\mathcal{H}_2} = Tr(C(F)Q_0C(F)^T) \leq Tr(C(F)QC(F)^T). \tag{2.21}$$

By Lemma 2.2, the $\mathcal{H}_\infty$ constraint is automatically enforced when a solution to the Riccati equation (2.20) exists and it yields an upper bound to the $\mathcal{H}_2$ norm of Σ_{cl}. This motivates the following (simplified) version of the optimal fixed order $\mathcal{H}_2/\mathcal{H}_\infty$–NSDP (see also (2.6) and (2.18)):

$$\begin{array}{ll} \min_{F,Q,V} & Tr((C_1 + D_{12}FC)Q(C_1 + D_{12}FC)^T) \\ \text{s. t.} & A(F)Q + QA(F)^T + \gamma^{-1}B_1B_1^T + \gamma^{-1}QC(F)^TC(F)Q = 0, \;\; Q \succeq 0 \\ & (A(F) + \gamma^{-1}QC(F)^TC(F))V + V(A(F) + \gamma^{-1}QC(F)^TC(F))^T + I = 0, \;\; V \succ 0. \end{array} \tag{2.22}$$

By using a dualization argument, the $\mathcal{H}_2/\mathcal{H}_\infty$–NSDP (2.22) is equivalent to the following nonlinear semidefinite program (see also (2.7) and (2.16)):

$$\begin{array}{ll} \min_{F,P,W} & Tr(PB_1B_1^T) \\ \text{s. t.} & A(F)^TP + PA(F) + \gamma^{-1}C(F)^TC(F) + \gamma^{-1}PB_1B_1^TP = 0, \;\; P \succeq 0 \\ & (A(F) + \gamma^{-1}B_1B_1^TP)^TW + W(A(F) + \gamma^{-1}B_1B_1^TP) + I = 0, \;\; W \succ 0. \end{array} \tag{2.23}$$

By replacing the Riccati equations in the NSDPs (2.22) and (2.23), respectively, and using the same arguments as in the previous two paragraphs, it is straightforward to derive the following BMI formulations of the $\mathcal{H}_2/\mathcal{H}_\infty$–NSDPs. In particular, the NSPD (2.22) is equivalent to the fixed order $\mathcal{H}_2/\mathcal{H}_\infty$–BMI problem (cf. (2.8) and (2.19)):

$$\begin{array}{ll} \min_{F,Q,X} & Tr(X) \\ \text{s. t.} & \begin{bmatrix} A(F)Y + YA(F)^T + \gamma^{-1}B_1B_1^T & YC(F)^T \\ C(F)Y & -\gamma\, I_{n_z} \end{bmatrix} \prec 0, \\ & \begin{bmatrix} X & C(F)Y \\ YC(F)^T & Y \end{bmatrix} \succeq 0, \quad Y \succ 0. \end{array} \tag{2.24}$$

Moreover, the NSDP (2.23) can be transformed to the following equivalent BMI problem (see also (2.15):

$$\begin{array}{lll} \min_{F,X} & Tr(XB_1B_1^T) & \\ \text{s. t.} & X \succ 0, & \begin{bmatrix} A(F)^TX + XA(F) & XB_1 & C(F)^T \\ B_1^TX & -\gamma\, I_{n_w} & 0 \\ C(F) & 0 & -\gamma\, I_{n_z} \end{bmatrix} \prec 0. \end{array} \tag{2.25}$$

2.2. SDP formulation of state feedback control design instances . For some special cases (i. e. state feedback control design), it is well known that most of the problems above has a linear SDP counterpart. In particular, setting $C := I_{n_x}$ and $D_{21} := 0$ (i. e. $y = x$, $n_y = n_x$), a clever change of variable leads to linear SDP formulations of the underlying (special) control problems. In this subsection, we state only a very uncomplete list of such SDPs. For more SDP instances, we refer the interested reader to [5] and the references therein.

2.2.1. $\mathcal{H}_2$–SDP problem. Assuming $B_1 B_1^T \succ 0$, the linear $\mathcal{H}_2$–SDP problem is defined by:

$$\begin{array}{rc} & \min_{X,Y,Q} \ \mathrm{Tr}\,(X) \\ \text{s.t.} & \begin{bmatrix} X & C_1 Q + D_{12} Y \\ (C_1 Q + D_{12} Y)^T & Q \end{bmatrix} \succeq 0, \\ & AQ + QA^T + BY + Y^T B^T + B_1 B_1^T \prec 0, \quad Q \succ 0, \end{array} \tag{2.26}$$

where $Q = Q^T \in \boldsymbol{R}^{n_x \times n_x}$, $X = X^T \in \boldsymbol{R}^{n_z \times n_z}$ and $Y \in \boldsymbol{R}^{n_u \times n_x}$ are the optimization variable and in an optimal solution of (2.26), one can define $F = YQ^{-1} \in \boldsymbol{R}^{n_u \times n_x}$. Note, if $B_1 B_1^T \succ 0$ is not satisfied in *COMPl$_e$ib* , in the definition of (2.26) we replace $B_1 B_1^T$ by $B_1 B_1^T + \epsilon I_{n_x}$, where ϵ denotes a fixed small positive scalar.

2.2.2. $\mathcal{H}_\infty$–SDP problem. Similarly as in the subsection above, we get again a linear SDP version of the $\mathcal{H}_\infty$ problem, i. e. we have

$$\begin{array}{rc} & \min_{Q,Y,\gamma} \ \gamma \\ \text{s.t.} \quad \gamma > 0, \ \ Q \succ 0, & \begin{bmatrix} AQ + QA^T + BY + Y^T B^T & QC_1^T + Y^T D_{12}^T & B_1 \\ C_1 Q + D_{12} Y & -\gamma I_{n_z} & D_{11} \\ B_1^T & D_{11}^T & -\gamma I_{n_w} \end{bmatrix} \prec 0. \end{array} \tag{2.27}$$

where $Q = Q^T \in \boldsymbol{R}^{n_x \times n_x}$, $\gamma \in \boldsymbol{R}$ and $Y \in \boldsymbol{R}^{n_u \times n_x}$ are the free variable and in an optimal solution of (2.27), we set $F = YQ^{-1} \in \boldsymbol{R}^{n_u \times n_x}$.

2.2.3. $\mathcal{H}_2/\mathcal{H}_\infty$–SDP problem. The linear $\mathcal{H}_2/\mathcal{H}_\infty$–SDP is a combination of the SDPs defined in the last two paragraphs. For a given scalar $\gamma > 0$, the following linear problem represents one version of the $\mathcal{H}_2/\mathcal{H}_\infty$–SDP:

$$\begin{array}{rc} & \min_{Q,X,Y} \ \mathrm{Tr}\,(X) \\ \text{s.t.} & \begin{bmatrix} AQ + QA^T + BY + Y^T B^T & QC_1^T + Y^T D_{12}^T & B_1 \\ C_1 Q + D_{12} Y & -\gamma I_{n_z} & 0 \\ B_1^T & 0 & -\gamma I_{n_w} \end{bmatrix} \prec 0, \\ & Q \succ 0, \ \begin{bmatrix} X & C_1 Q + D_{12} Y \\ (C_1 Q + D_{12} Y)^T & Q \end{bmatrix} \succeq 0, \end{array} \tag{2.28}$$

where $Q = Q^T \in \boldsymbol{R}^{n_x \times n_x}$, $X = X^T \in \boldsymbol{R}^{n_z \times n_z}$ and $Y \in \boldsymbol{R}^{n_u \times n_x}$ are the free variable and in an optimal solution of (2.28), we define $F = YQ^{-1} \in \boldsymbol{R}^{n_u \times n_x}$.

2.3. Other related matrix design problems . Optimization problems involving the eigenvalues of a matrix variable have broad interest and application. One particular instance is the stability of the dynamical (closed loop control) system $\dot{x}(t) = (A + BFC)x(t)$, where F denotes an unknown matrix variable. It is well known that the (pure) stability analysis problem of this system is an eigenvalue problem of the following form: find $F \in \boldsymbol{R}^{n_u \times n_y}$ such that the real part of the maximal eigenvalue of the (in general nonsymmetric) closed loop matrix $A(F) = A + BFC$ is strictly negative. Using the Lyapunov theorem, this problem can be formulated as a nonlinear semidefinite program in the unknowns $P \in \mathcal{S}^{n_x}$, $F \in \boldsymbol{R}^{n_u \times n_y}$ and $\beta \in \boldsymbol{R}$ (cf. [5], [9]):

$$\min_{\beta,F,P} \beta \quad \text{s. t.} \quad P \succ 0, \quad (A + BFC)^T P + P(A + BFC) \preceq 2\beta P. \tag{2.29}$$

Obviously, if the optimal value of (2.29) is strictly negative, then there exists a feedback control law $u(t) = Fy(t)$ such that the closed loop system is (asymptotically) stable. Equivalently, an optimal value of (2.29) can be found by solving the following nonsmooth and nonconvex optimization problem (see, i. e. [7], [8] and the references therein):

$$\min_{F \in \mathbb{R}^{n_u \times n_y}} \alpha_0(A + BFC), \tag{2.30}$$

where

$$\alpha_0(A + BFC) := \sup\{Re(\lambda) \mid \lambda \in \Lambda(A + BFC)\} \tag{2.31}$$

is the so–called (pure) spectral abscissa of $A + BFC$, $Re(\lambda)$ denotes the real part of $\lambda \in \mathbb{C}$ and $\Lambda(A + BFC)$ defines the spectrum of $A + BFC$.

However, a simple optimization of the decay rate of the LTI system (which is the stability degree of A(F), i. e. , the negative of the maximum real part of the eigenvalues of A(F)) alone has serious drawbacks in the stability analysis of systems. More general, considering only the pure spectrum of a matrix has also many drawbacks in other areas (see, [30] and the references therein). Therefore, as Trefethen and others have pointed out (see, [8], [30] and the references therein), the ϵ–pseudospectra of a matrix are more informative and much more robust in many modelling frameworks, particularly in the analysis of robust stability of LTI systems. The ϵ–pseudospectral abscissa of $A + BFC$, defined as the largest real part of all elements of the pseudospectrum of $A + BFC$, for a fixed $\epsilon \geq 0$ and $F \in \mathbb{R}^{n_u \times n_y}$, is closely related to the $\mathcal{H}_\infty$ norm or the complex stability radius of a LTI system (see [8], [7], [10] and the references therein). For fixed $\epsilon \geq 0$, the ϵ–pseudospectral abscissa $\alpha_\epsilon(A + BFC)$ of $A + BFC$ is defined as the largest real part of the ϵ–pseudospectrum $\Lambda_\epsilon(A + BFC)$, i. e.

$$\alpha_\epsilon(A + BFC) := \sup\{Re(\lambda) \mid \lambda \in \Lambda_\epsilon(A + BFC)\}, \tag{2.32}$$

where $\Lambda_\epsilon(A+BFC) := \{\lambda \in \mathbb{C} \mid \lambda \in \Lambda(X), ||X-(A+BFC)||_2 \leq \epsilon\}$ denotes the ϵ–pseudospectrum of the matrix $A + BFC$. Note, when $\epsilon = 0$, the ϵ–pseudospectrum reduces to the spectrum and, thus, the ϵ–pseudospectral abscissa coincides with the pure spectral abscissa. With the definition of the ϵ–pseudospectral abscissa of a (in general nonsymmetric) matrix at hand, we are now in a position to state the robust counterpart of the nonsmooth and nonconvex matrix optimization problem (2.30) (see [7]):

$$\min_{F \in \mathbb{R}^{n_u \times n_y}} \alpha_\epsilon(A + BFC), \tag{2.33}$$

where $\epsilon \geq 0$ fixed. Just as with the pure spectral abscissa minimization problem (2.30), the ϵ–pseudospectral abscissa optimization problem (2.33) can be characterized via a nonlinear semidefinite program. In particular, by using the $\mathcal{S}$–procedure as in [5, page 67] or [1, Proposition 4.4.2], (2.33) is equivalent to the following NSDP in the unknowns $P \in \mathcal{S}^{n_x}$, $F \in \mathbb{R}^{n_u \times n_y}$ and $\beta, \mu, \omega \in \mathbb{R}$ (see, also [8]):

$$\min_{\beta,\mu,\omega,F,P} \beta \text{ s. t. } \begin{pmatrix} 2\beta P - A(F)^T P - PA(F) + \mu I - \omega C_z^T C_z & -\epsilon P B_w \\ -\epsilon B_w^T P & \omega I \end{pmatrix} \succeq 0, \; P \succ 0, \; \mu < 0, \tag{2.34}$$

and $\omega \geq 0$ if and only if we define $B_w := I$ and $C_z := I$. Note, for real β, the condition $\alpha_\epsilon(A + BFC) < \beta$ is equivalent to the condition that the $\mathcal{H}_\infty$ norm of the closed loop system $\dot{x}(t) = (A(F)-\beta I)x(t)+w(t), z(t) = x(t)$ is less than ϵ^{-1}, i. e. $||(sI-(A+BFC-\beta I))^{-1}||_{\mathcal{H}_\infty} < \epsilon^{-1}$ for $s \in \mathbb{C}$.

Another, more general interpretation of the NSDP (2.34) can be given in terms of norm–bounded perturbations $\Delta \in \mathbb{R}^{n_u \times n_y}$ with $||\Delta||_2 \leq \epsilon$ for the following ROC or SOF control system:

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t), \; u(t) = (F + \Delta)y(t),$$

i. e. here we assume that the (unknown) feedback gain F is drifting around a certain nominal feedback matrix. As a result, the closed loop matrix $A(F)$ is affected by perturbation of the form $B\Delta C$ with $||\Delta||_2 \leq \epsilon$. Hence, we get a perturbed closed loop system of the form

$$\dot{x}(t) = (A + BFC)x(t) + B\Delta Cx(t).$$

Then, defining $B_w := B$, $C_z := C$ (instead of $B_w := I$ and $C_z := I$), $w := \Delta Cx$, $z := y = Cx$, the optimal value of the general eigenvalue problem (2.34) is equal to the largest β such that the $\mathcal{H}_\infty$ norm of the closed loop system $\dot{x}(t) = (A(F) - \beta I)x(t) + B_w w(t), z(t) = C_z x(t)$ is less than ϵ^{-1}, i. e. $||C_z(sI - (A + BFC - \beta I))^{-1}B_w||_{\mathcal{H}_\infty} < \epsilon^{-1}$ for $s \in \mathbb{C}$. In this case, F is a robust output feedback gain.

An alternate equivalent formulation to the NSDP (2.34) in the unknowns $Q := P^{-1} \in \mathcal{S}^{n_x}$, $F \in \mathbb{R}^{n_u \times n_y}$ and $\beta, \mu, \omega \in \mathbb{R}$ is the following BMI problem:

$$\min_{\beta,\mu,\omega,F,P} \beta \text{ s. t. } \begin{pmatrix} 2\beta Q - A(F)Q - QA(F)^T + \mu I - \omega B_w B_w^T & -\epsilon Q C_z^T \\ -\epsilon C_z Q & \omega I \end{pmatrix} \succeq 0,\ Q \succ 0,\ \mu < 0, \tag{2.35}$$

and $\omega \geq 0$. Note, if we define $B_w := I$ and $C_z := I$ then the BMI problem (2.35) is equivalent to the ϵ–pseudospectral abscissa optimization problem (2.33).

Another interesting measure of robust stability is the complex stability radius $\xi(A + BFC)$ of a matrix $A(F) = A + BFC$, also known as the distance to the unstable matrices (see [7] and the refernces therein). It is well known that the complex stability radius of $A(F)$, defined by

$$\xi(A + BFC) := \min_{\lambda,X}\{||X - (A + BFC)||_2 \ \mid\ \sigma_{\min}(X - \lambda I) = 0,\ Re(\lambda) \geq 0\}, \tag{2.36}$$

where $\lambda \in \mathbb{C}$, $X \in \mathbb{C}^{n_x \times n_x}$ and $\sigma_{\min}(\cdot)$ denotes the smallest singular value of a matrix, is closely related to the $\mathcal{H}_\infty$ norm of the associated transfer function matrix $(sI - (A + BFC))^{-1}$. In particular, $\xi(A + BFC)^{-1}$ is the $\mathcal{H}_\infty$ norm of $(sI - (A + BFC))^{-1}$. Moreover, for any real β, we have the following relation (see [8])

$$\alpha_\epsilon(A + BFC) \geq \beta \quad \Longleftrightarrow \quad \xi(A + BFC - \beta I) \geq \epsilon.$$

Efficient algorithms to compute the pseudospectral abscissa and the complex stability radius of a matrix are available. Note, the latter is a special case of more general $\mathcal{H}_\infty$ norm computations. In example, the test examples collected in *COMPl$_e$ib* may serve as a benchmark collection for the well known $\mathcal{H}_\infty$ norm computation solvers [4], [6] and for ϵ–pseudospectral abscissa and related (nonconvex) matrix optimization problems, the algorithms developed in [9], [11] and [10] are good candidates which can be benchmarked by *COMPl$_e$ib* . Note, Burke, Lewis and Overton used some of our *COMPl$_e$ib* test examples during the development of their solvers described in [9], [11] and [10].

3. Output feedback control of 2D heat flow models . In this section we describe examples coming from output feedback control problems of two dimensional heat flow models. We present different instances of such parabolic control problems and formulate the corresponding control problem in infinite dimensional spaces. Using standard finite difference schemes we obtain finite dimensional approximations to the infinite dimensional control problems. Typically, the approximation is a large scale control system and the corresponding system matrix A is very sparse compared to the overall dimension, i. e. it is a sparse matrix containing only five diagonals with nonzero elements. In our benchmark collection, we state the data matrices of the corresponding discretized control systems. In particular, the discretization yields a control system of the following general form:

$$\begin{array}{rcl} E\dot{x}(t) & = & (A + \delta A)x(t) + G(x(t)) + B_1 w(t) + Bu(t), \quad x(0) = x_0, \\ z(t) & = & C_1 x(t) + D_{12} u(t), \\ y(t) & = & Cx(t), \\ u(t) & = & Fy(t), \end{array} \tag{3.1}$$

where $x \in \mathbb{R}^{n_x}$ is the approximation of the temperature, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ denotes the measurements, $w \in \mathbb{R}^{n_w}$ is a disturbance input, $z \in \mathbb{R}^{n_z}$ the regulated output, $E \in \mathbb{R}^{n_x \times n_x}$ is a regular diagonal matrix and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $B_1 \in \mathbb{R}^{n_x \times n_w}$, $C \in \mathbb{R}^{n_y \times n_x}$, $F \in \mathbb{R}^{n_u \times n_y}$, $C_1 = \sqrt{0.5c_1}\,[I_{n_x} \;\; 0_{n_u \times n_x}]^T$, $D_{12} = \sqrt{0.5d_1}\,[0_{n_x \times n_u} \;\; I_{n_u}]^T$ with $c_1, d_1 \in \mathbb{R}$ are given positive scalars. If $\delta A \equiv 0$ the system matrix A is not affected by a perturbation, and, if $G(x(t)) \equiv 0$, the system is linear. Depending on the corresponding heat flow model, we get linear or nonlinear control systems which we want to control by a linear output feedback control law of the form $u(t) = Fy(t)$.

3.1. Perturbed nonlinear heat equation with boundary control input . This example presents the model of a two dimensional nonlinear heat equation on an U–shaped domain as considered in [25]. In particular, the model consists of a nonlinear radiation term on one part of the boundary of the domain. Moreover, there are only two control variables available on some parts of the boundary and three fixed sensor locations in the domain which can be used for the output feedback control loop. A further difficulty is the assumption that a perturbation operator affects the heat flow of the model.

Let $v(\xi, \eta; t)$ denote the temperature at $(\xi, \eta) \in \Omega \subset \mathbb{R}^2$ and time $t > 0$. The boundary $\partial\Omega = \bigcup_{j=1}^{8} \Gamma_j$ of the U–shaped domain Ω is defined by

$$\begin{array}{lcllcl}
\Gamma_1 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi = a_2, \eta \in [b_2, b_3]\}, & \Gamma_2 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in [0, a_2], \eta = b_3\}, \\
\Gamma_3 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi = 0, \eta \in [0, b_3]\}, & \Gamma_4 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in [0, a_2], \eta = 0\}, \\
\Gamma_5 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi = a_2, \eta \in [0, b_1]\}, & \Gamma_6 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in [a_1, a_2], \eta = b_1\}, \\
\Gamma_7 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi = a_1, \eta \in [b_1, b_2]\}, & \Gamma_8 & = & \{(\xi,\eta) \in \mathbb{R}^2 \mid \xi \in [a_1, a_2], \eta = b_2\},
\end{array}$$

where $0 < a_1 < a_2$, $0 < b_1 < b_2 < b_3$ and $a_1 = \frac{1}{2}, a_2 = 1, b_1 = \frac{1}{3}, b_2 = \frac{2}{3}, b_3 = 1$. On this two dimensional domain we consider the following nonlinear heat flow control problem: $(v(\xi, \eta; t), u_4(t), \hat{u}(t))$ satisfy the perturbed diffusion equation

$$v_t(\xi, \eta; t) = \kappa(\Delta + \delta)v(\xi, \eta; t), \qquad \text{in } \Omega, t > 0 \tag{3.2}$$

with boundary and initial conditions

$$\begin{array}{rcl}
-\lambda \frac{\partial v}{\partial n}(\xi, \eta; t) & = & 0, \qquad \text{on } \Gamma_j, j = 1, 2, 3, 5, t > 0, \\
-\lambda \frac{\partial v}{\partial n}(\xi, \eta; t) & = & \alpha_4(v(\xi, \eta; t) - v_4^a + u_4(t)) + \varepsilon_4 \sigma(v(\xi, \eta; t)^4 - (v_4^a)^4), \quad \text{on } \Gamma_4, t > 0, \\
-\lambda \frac{\partial v}{\partial n}(\xi, \eta; t) & = & \hat{\alpha}(v(\xi, \eta; t) - \hat{v}^a + \hat{u}(t)), \qquad \text{on } \Gamma_j, j = 6, 7, 8, t > 0, \\
v(\xi, \eta, 0) & = & v_0(\xi, \eta), \qquad \text{in } \Omega,
\end{array} \tag{3.3}$$

where Δ denotes the Laplace operator and δ is a perturbation operator. Moreover, we use the following notation and thermal quantities of the material:

C :	heat capacity,	λ :	heat conduction,
ρ :	density,	$\kappa = \frac{\lambda}{C\rho}$:	diffusion coefficient,
α_4 :	heat exchange factor on Γ_4,	$\hat{\alpha}$:	heat exchange factor on $\Gamma_j, j = 6, 7, 8$,
v_4^a :	ambient temperature on Γ_4,	$\hat{v}^a$:	ambient temperature on $\Gamma_j, j = 6, 7, 8$,
u_4 :	boundary control on Γ_4,	$\hat{u}$:	boundary control on $\Gamma_j, j = 6, 7, 8$,
ε_4 :	radiation coefficient on Γ_4,	$\sigma = 5.6697 \cdot 10^{-8}$:	Stefan–Boltzmann constant
v :	temperature in $^\circ K$ on Ω,	v_0 :	initial temperature on Ω.

The system is described by a linear perturbed partial differential equation (the heat equation) coupled, through the boundary conditions to a nonlinear radiation term. We are interested in using sensed information to design a low–order output feedback control law. Note, we have only two control inputs acting on Γ_4 and $\Gamma_6 \cup \Gamma_7 \cup \Gamma_8$. We assume that the only measured information available to these controls is the temperature at time t at three fixed sensor locations in the domain Ω. In example, the three observation variables $y_i, i = 1, 2, 3$, are given by

$$y_1(t) = v(0, b_3; t), \quad y_2(t) = v(0, 0; t), \quad y_3(t) = v(a_1, \tfrac{1}{2}b_3; t). \tag{3.4}$$

The nonlinear control system governed by equations (3.2) – (3.4) can be written as a dynamical system in an appropriate (infinite dimensional) state space. For computing a linear output feedback control law of the form $u(t) = Fy(t)$, we apply standard finite difference approximation techniques to the system described above. Although it is not essential to use finite differences, we restrict our attention to this approximation approach because it is easy to implement. Alternatively, a finite element approximation can be used for building the finite dimensional system corresponding to (3.2) – (3.4). We discretize the domain Ω by a uniform grid, where $h = \frac{a_2}{66} \approx 0.015$ is the spatial step size in $\xi-$ and η–direction. The resulting number of grid points is $n_x = 3796$. Thus, we obtain a large scale finite dimensional nonlinear control system of the form (3.1) with only two control inputs ($n_u = 2$) and three measured output variables ($n_y = 3$). Note, it is an approximation of the infinite dimensional nonlinear control system. Due to fixed ambient temperature $v^a = [v_4^a, \hat{v}^a]^T$ on some parts of the boundary, the term $B_1 w(t)$ with $w(t) \equiv [1, \ldots, 1]^T \in \boldsymbol{R}^{n_w}$ represents this constant part in our discrete model. The matrix $\delta A \in \boldsymbol{R}^{n_x \times n_x}$ approximates the perturbation operator δ and

$$G : \boldsymbol{R}^{n_x} \to \boldsymbol{R}^{n_x}, \quad G(x(t)) := N x(t)^4, \quad N \in \boldsymbol{R}^{n_x \times n_x}$$

models the approximation of the nonlinear boundary part $\varepsilon_4 \sigma v(\cdot; t)^4$ on Γ_4.

The authors in [25] combine a proper orthogonal decomposition (POD) approach with an interior point constrained trust region (IPCTR) algorithm for computing the static output feedback controller gain F. Instead of computing F directly from the large scale (nonlinear) control system (3.1), Leibfritz and Volkwein [25] first reduce the dimension of (3.1) to a very low dimensional approximation of (3.1) by POD. In example, they compute only five POD basis functions from the snapshots of the uncontrolled nonlinear system (for $u = 0$) which results in a very low dimensional POD system of order $n_{pod} = 5$. Then in a second step, they compute the SOF control gain F for the linearized POD system (i. e. they delete the nonlinear part). Finally, they plug in the POD controller gain F into the large dimensional (unstable) nonlinear system (3.1) of order $n_x = 3796$. Particularly, they have chosen the perturbation operator δA such that at least one real part of the eigenvalues of $(A + \delta A)$ is positive. For more details, we refer the interested reader to [23], [25] and the references therein.

In the following two case studies, we use the procedure proposed by [25] for computing a stabilizable SOF controller for the unstable nonlinear control system (3.1).

3.1.1. Case study: Copper . Choosing $\delta A = 0.3825 I_{n_x}, c_1 = 0.5$, $d_1 = 100$ and setting the following parameters,

$$C = 0.0914,\ \lambda = 0.05,\ \rho = 8.94,\ \alpha_4 = 0.1,\ \varepsilon_4 = 0.00023,\ \hat{\alpha} = 0.2,\ v_4^a = 1700^\circ K,\ \hat{v}^a = 400^\circ K,\ v_0 = 850^\circ K,$$

for the thermal properties of copper, the plots in Figure 3.1 and 3.3 illustrate pretty good that

FIG. 3.1. *Copper: with control at $T = 20$.*

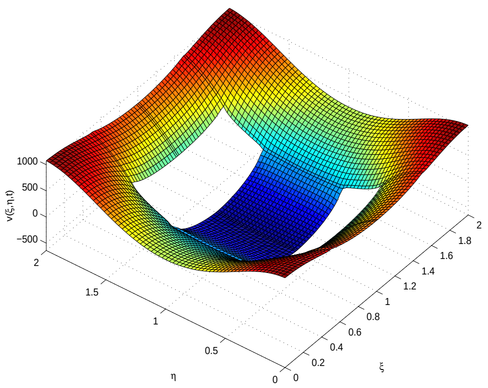
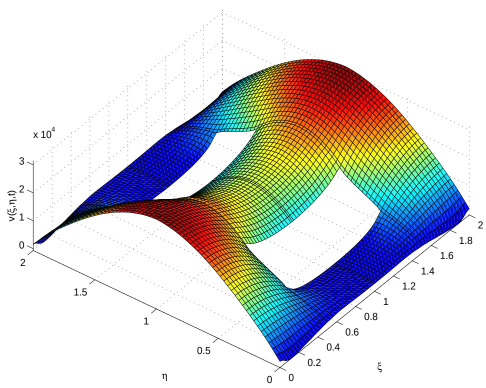


FIG. 3.2. *Copper: no control at $T = 20$.*



the SOF control law protect the material (here: copper) from overheating (melting temperature: 1356.2 Kelvin). The hot spot of the material is $300^\circ K$ below the melting temperature of copper.

Fig. 3.3. *Copper: with control at* $t = 1.5, 3, 10, 20$.

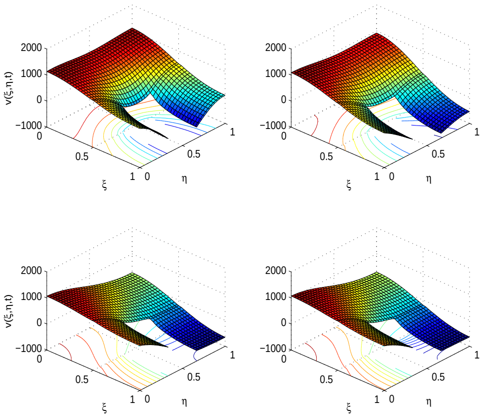
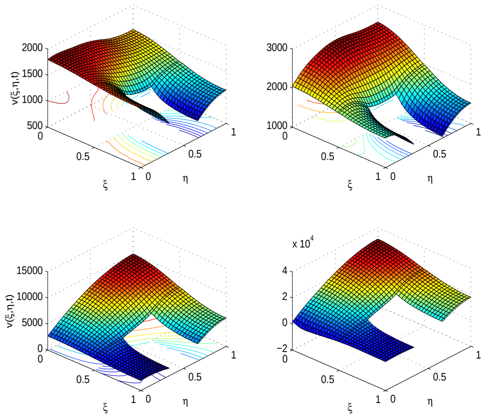


Fig. 3.4. *Copper: no control at* $t = 1.5, 3, 10, 20$.



Note, the inner holes in the domain of Ω are cooling channels which are embedded in the material. On the other hand, we see in Figure 3.2 and 3.4 the instability of the uncontrolled perturbed

Fig. 3.5. *Copper: optimal SOF control.*

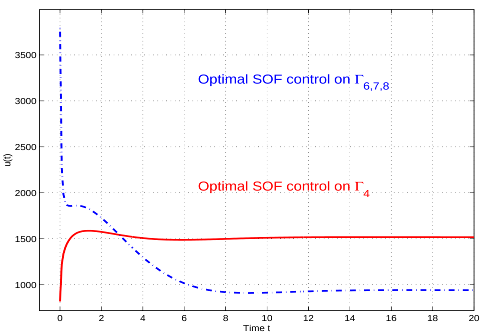
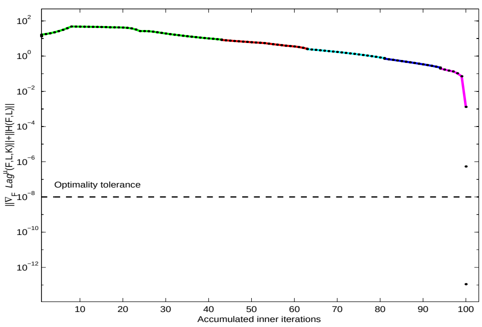


Fig. 3.6. *Convergence: IPCTR.*



nonlinear system. In the uncontrolled case, the temperature increases rapidly. After 1.5 seconds of the heating process, the temperature is above the melting temperature of copper. Roughly speaking, without any control, the system burns out completely.

The optimal boundary control input computed by IPCTR can be found in Figure 3.5. The blue dash dotted curve represents the control $\hat{u}$ acting on $\Gamma_6 \cup \Gamma_7 \cup \Gamma_8$. Or, in other words, $\hat{u}$ controls the cooling action of the system in the cooling channels. The red solid line represents the feedback control u_4 on Γ_4. This control acts on the outer boundary of the domain, where the ambient temperature $v_4^a = 1700°K$ is larger then the melting temperature of copper.

The convergence behavior of IPCTR for computing the SOF gain F for the low dimensional POD control system is visualized in Figure 3.6. IPCTR determines the solution of the corresponding NSDP within 8 outer and totally 100 inner iterations. Moreover, IPCTR consumes 1.476 CPU seconds on a DELL notebook with a pentium III, 1.0 GHz, processor. The figure also demonstrates the global linear as well as the quadratic local convergence rates of the algorithm. In particular, for sufficiently small barrier parameters, the inner loop IPCTR is never active and the local rate of convergence is quadratic (see the last two iteration dots in Figure 3.6).

Summing up, this very simple output feedback control law with only two control inputs and three observation points is able to stabilize the nonlinear and unstable large dimensional control system. This example demonstrates that the combination of POD model reduction and nonlinear semidefinite programming can be considered as a useful tool for the design of simple output feedback control laws for large nonlinear and unstable PDE models. In the test collection, we denote by (HF2D1) the large scale approximation of the nonlinear heat flow model for copper, while (HF2D10) refers to the corresponding POD approximation.

3.1.2. Case study: Platinum . In this second example we study the behaviour of the perturbed nonlinear heat flow model if we replace copper by platinum. Note, the melting temperature of platinum is about $700°K$ higher than the metling temperature of copper. Choosing $\delta A = 0.5325 I_{n_x}, c_1 = 0.5$, $d_1 = 100$ and setting

$$C = 0.1297,\ \lambda = 0.7264,\ \rho = 21.4519,\ \alpha_4 = 0.51,\ \varepsilon_4 = 0.04,\ \hat{\alpha} = 0.32,\ v_4^a = 2500°K,\ \hat{v}^a = 1500°K,\ v_0 = 1250°K.$$

for thermal properties of platinum, we are able to draw similar observations as in the case study of copper. The temperature distribution of the controlled process can be found in Figures 3.7

Fig. 3.7. *Platinum: with control at* $T = 20$.

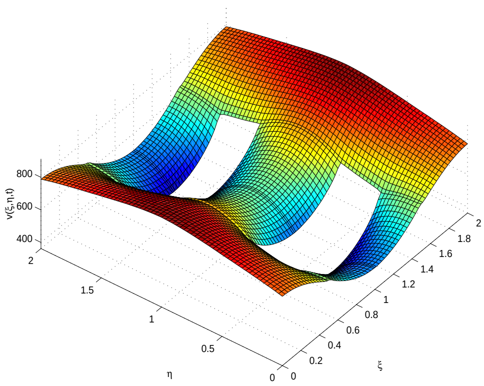


Fig. 3.8. *Platinum: no control at* $T = 20$.

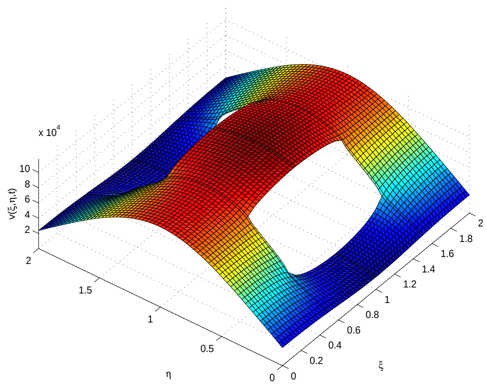


Fig. 3.9. *Platinum: with control at* $t = 1.5, 3, 10, 20$.

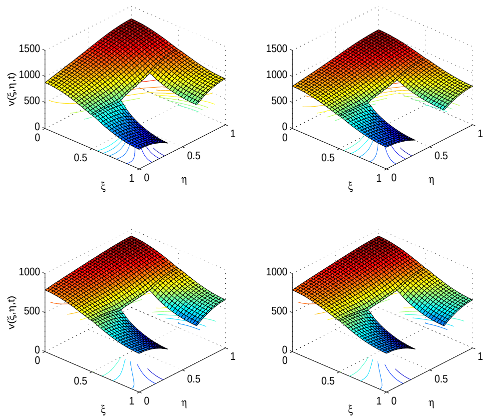
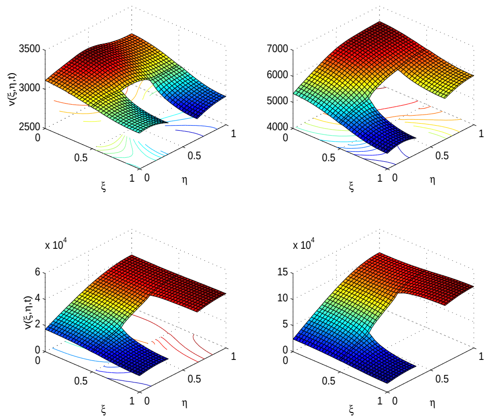


Fig. 3.10. *Platinum: no control at* $t = 1.5, 3, 10, 20$.



and 3.9. Again, we observe that the very simple SOF controller is able to protect platinum from overheating. Moreover, we see therein that the simple controller stabilizes the unstable perturbed nonlinear control system pretty good. The behaviour of the uncontrolled system is visualized in Figures 3.8 and 3.10, respectively. Obviously, the uncontrolled nonlinear perturbed system is not stable. After 1.5 seconds the temperature is above the melting temperature and after 3 seconds it is even above the boiling temperature ($4098°K$) of platinum. Thus, the material vaporizes within a very short time period.

In the test set, (HF2D2) denotes the large scale approximation of this nonlinear heat flow example, while the corresponding POD approximation is in (HF2D11).

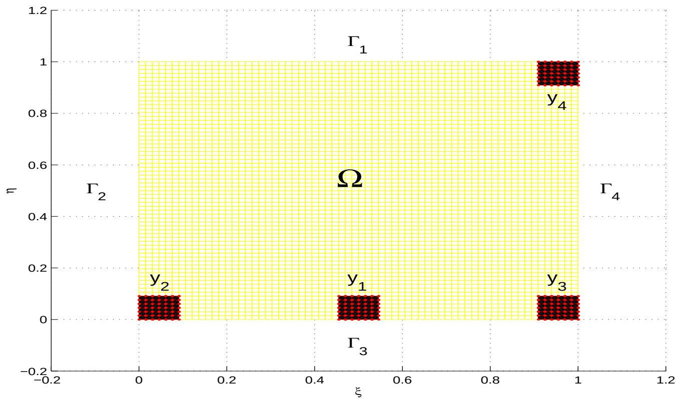
3.2. Boundary output feedback control examples of 2D heat flow models on a rectangular domain . In this subsection we describe some examples arising in two dimensional heat flow models on rectangular domains. We assume that the control variables act only on the (outer) boundary parts (denoted by Γ_3, Γ_4) of the domain. Moreover, there are only four (fixed) sensor locations which can be used for the output feedback control loop.

The boundary $\partial\Omega = \bigcup_{j=1}^{4} \Gamma_j$ of the rectangular domain $\Omega = [0, a] \times [0, b]$ is given by

$$\begin{array}{rcl@{\qquad}rcl} \Gamma_1 & = & \{(\xi,\eta) \in R^2 \mid \xi \in [0,a], \eta = b\}, & \Gamma_2 & = & \{(\xi,\eta) \in R^2 \mid \xi = 0, \eta \in [0,b]\}, \\ \Gamma_3 & = & \{(\xi,\eta) \in R^2 \mid \xi \in [0,a], \eta = 0\}, & \Gamma_4 & = & \{(\xi,\eta) \in R^2 \mid \xi = a, \eta \in [0,b]\}, \end{array}$$

where $0 < a$, $0 < b$ and, with no loss of generality, we set $a = 1, b = 1$. In the hole part of this subsection, we suppose that the boundary control variable $u_3(t), u_4(t)$ act on Γ_3, Γ_4, respectively. Thus, $n_u = 2$. Moreover, we assume that the only measured information available to the boundary

FIG. 3.11. *Sensor location in the domain* Ω.



control inputs is the average temperature at time t within fixed sensor domains located in Ω. Particularly, Figure 3.11 visualizes the location of the four sensors in Ω. The resulting four ($n_y = 4$) measurement variables $y_i, i = 1, 2, 3, 4$, are defined by the average temperature measured at the grid points within the i–th sensor domain. As we can see in Figure 3.11, the four sensors are located at the outer boundary of the domain, i. e. we measure the temperature on parts on Γ_3 and Γ_4.

3.2.1. Linear heat flow with boundary control . On the rectangular domain Ω we consider here the following two dimensional example of linear heat flow model:

$$\begin{array}{rcll} v_t(\xi,\eta;t) & = & \kappa\Delta v(\xi,\eta;t), & \text{in } \Omega, t > 0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & 0, & \text{on } \Gamma_j, j = 1,2, t > 0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & \alpha_j(v(\xi,\eta;t) - v_j^a + u_j(t)), & \text{on } \Gamma_j, j = 3,4, t > 0, \\ v(\xi,\eta,0) & = & v_0(\xi,\eta), & \text{in } \Omega, \end{array} \tag{3.5}$$

where Δ denotes the Laplace operator and

C :	heat capacity in $\frac{g}{cm^3}$,	λ :	heat conduction in $\frac{W}{Kcm}$,
ρ :	density in $\frac{J}{gK}$,	$\kappa = \frac{\lambda}{C\rho}$:	diffusion coefficient,
α_3 :	heat exchange factor on Γ_3,	α_4 :	heat exchange factor on Γ_4,
v_3^a :	ambient temperature in °K on Γ_3,	v_4^a :	ambient temperature in °K on Γ_4,
u_3 :	boundary control on Γ_3,	u_4 :	boundary control on Γ_4,
v :	temperature in $^\circ K$ on Ω,	v_0 :	initial temperature on Ω.

Applying a finite difference approximation scheme, we obtain a large scale linear output feedback control system of the form (3.1). But, note, due to the linearity of (3.5), we have $G(x(t)) \equiv 0$ in (3.1). Moreover, $\delta A \equiv 0$ in (3.1), since we do not assume here that the Laplace operator is affected by a perturbation.

For this linear heat flow model with boundary output feedback control input, the benchmark collection contains two different data sets of the linear control system of the form (3.1). The first data set corresponds to a case study for copper, while the second data set comes from a case study for platinum. Table 3.1 lists the parameters that we have used for generating the corresponding data sets for these two linear control system examples. In our test environment, (HF2D3) (copper)

TABLE 3.1
Parameter: linear heat flow model

Material	n_x	n_u	n_y	ρ	C	λ	α_3	α_4	v_3^a	v_4^a	c_1	d_1
copper	4489	2	4	8.94	0.0914	0.05	0.1	0.2	1600	900	100	10
platinum	2025	2	4	21.4519	0.1297	0.7264	0.51	0.32	2500	1500	1	100

and (HF2D4) (platinum) refers to the large scale approximation of the linear heat flow model, while the corresponding POD approximations can be found in (HF2D12) (copper) and (HF2D13) (platinum), respectively.

3.2.2. Perturbed linear heat flow with boundary control . In this example, we change the linear model defined in Paragraph 3.2.1 by adding a perturbation operator to the linear heat equation in (3.5). Particularly, on the rectangular domain Ω we consider now the following perturbed linear heat flow model:

$$\begin{array}{rcll} v_t(\xi,\eta;t) & = & \kappa(\Delta+\delta)v(\xi,\eta;t), & \text{in } \Omega, t>0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & 0, & \text{on } \Gamma_j, j=1,2,t>0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & \alpha_j(v(\xi,\eta;t)-v_j^a+u_j(t)), & \text{on } \Gamma_j, j=3,4,t>0, \\ v(\xi,\eta,0) & = & v_0(\xi,\eta), & \text{in } \Omega, \end{array} \tag{3.6}$$

where Δ denotes the Laplace operator and δ is a perturbation operator. The other quantities are defined as in Paragraph 3.2.1.

Similar as above, a discretization scheme yields a large scale linear output feedback control system of the form (3.1). Again, due to the linearity of (3.6), we have $G(x(t)) \equiv 0$ in (3.1), but now, $\delta A \neq 0$ in (3.1), since we assume $\delta \neq 0$.

For the perturbed linear heat flow model with boundary output feedback control input, the benchmark collection contains two different data sets of the linear control system of the form (3.1) too. The first data set corresponds to a case study for copper, while the second data set comes from a case study for platinum. Table 3.2 lists the parameters that we have used for generating the corresponding data sets for these two linear control system examples. For the perturbation operator

TABLE 3.2
Parameter: perturbed linear heat flow model

Material	n_x	n_u	n_y	ρ	C	λ	α_3	α_4	v_3^a	v_4^a	c_1	d_1	s_δ
copper	4489	2	4	8.94	0.0914	0.05	0.1	0.2	1600	900	10	100	0.3825
platinum	2025	2	4	21.4519	0.1297	0.7264	0.51	0.32	2500	1500	10	50	1.725

δ we used a scaled identity operator, i. e. $\delta := s_\delta \mathcal{I}$, where $\mathcal{I}$ denotes the identity mapping and s_δ is a given positive scalar. For generating the examples in the test set collection, s_δ can be found in Table 3.2. For both instances, the perturbation operator shifts the original stable system to an unstable linear control system. In example, there is at least one eigenvalue of $(A+\delta A)$ with positive real part. Note, in our case, the discrete perturbation operator is given by $\delta A = s_\delta I_{n_x}$. Moreover, in the test set, (HF2D5) (copper) and (HF2D6) (platinum) denotes the large scale approximation of the perturbed linear model, while the POD approximations of these two instances are contained in (HF2D14) (copper) and (HF2D15) (platinum), respectively.

3.2.3. Perturbed nonlinear heat flow with boundary control . Finally, we extend the perturbed linear heat flow model stated in Paragraph 3.2.2 to a nonlinear model. Especially, in (3.8) we add the nonlinear radiation term

$$\varepsilon_j \; \sigma \; \left(v(\xi,\eta;t)^4 - \left(v_j^a\right)^4 \right), \quad j = 3,4, \tag{3.7}$$

to the boundary terms on $\Gamma_j, j = 3,4$, where

ε_j :	radiation coefficient on $\Gamma_j, j = 3,4$,	$\sigma = 5.6697 \cdot 10^{-8}$:	Stefan–Boltzmann constant
v_j^a :	ambient temperature on $\Gamma_j, j = 3,4$,	v :	temperature in $^\circ K$ on Ω.

Thus, we get the following perturbed nonlinear model:

$$\begin{array}{rcll} v_t(\xi,\eta;t) & = & \kappa(\Delta+\delta)v(\xi,\eta;t), & \text{in } \Omega, t>0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & 0, & \text{on } \Gamma_j, j=1,2, t>0, \\ -\lambda\frac{\partial v}{\partial n}(\xi,\eta;t) & = & \alpha_j(v(\xi,\eta;t) - v_j^a + u_j(t)) + \varepsilon_j\;\sigma\;\left(v(\xi,\eta;t)^4 - \left(v_j^a\right)^4\right), & \text{on } \Gamma_j, j=3,4, t>0, \\ v(\xi,\eta,0) & = & v_0(\xi,\eta), & \text{in } \Omega, \end{array} \tag{3.8}$$

where the variables and parameters are denoted as in the previous two Paragraphs 3.2.1 and 3.2.2, respectively. Then, a discretization of the infinite dimensional nonlinear model (3.8) yields a large finite dimensional nonlinear control system of the form (3.1). Due to the nonlinearity of (3.8), in (3.1) the nonlinear function

$$G : \boldsymbol{R}^{n_x} \to \boldsymbol{R}^{n_x}, \quad G(x(t)) := Nx(t)^4, \quad N \in \boldsymbol{R}^{n_x \times n_x}$$

models the approximation of the nonlinear boundary terms $\varepsilon_j \sigma v(\cdot;t)^4$ on Γ_j, $j = 3,4$. Moreover, $\delta A \neq 0$, where $\delta A = s_\delta I_{n_x}$ and $s_\delta > 0$ given.

For the perturbed nonlinear heat flow model the benchmark collection also contains two different data sets of the nonlinear control system (3.1). One data set corresponds to a case study for copper, while the other data set comes from a case study for platinum. Table 3.3 lists the parameters that we have used for generating the corresponding data set of the nonlinear control system example. Moreover, the values for ρ, C and λ are defined as in Table 3.2. Note, due to the

TABLE 3.3
Parameter: perturbed nonlinear heat flow model

Material	n_x	n_u	n_y	ε_3	ε_4	α_3	α_4	v_3^a	v_4^a	c_1	d_1	s_δ
copper	4489	2	4	$2.3 \cdot 10^{-4}$	$2.3 \cdot 10^{-4}$	0.1	0.2	1600	900	10	100	0.2775
platinum	2025	2	4	$4.0 \cdot 10^{-6}$	$4.0 \cdot 10^{-6}$	0.51	0.32	2500	1500	50	50	0.7575

choice of s_δ (see Table 3.3), the perturbation operator also shifts the original stable system to an unstable control system. In example, there is at least one eigenvalue of $(A + \delta A)$ with positive real part.

In the test set (HF2D7) (copper) and (HF2D8) (platinum) refers to the large dimensional approximation of this model, while (HF2D16) and (HF2D18) denotes the corresponding POD reduced order approximation of the large control system, respectively.

3.3. Distributed control of a perturbed linear heat flow model . In this subsection we consider a two dimensional model of the linear heat equation with n_u distributed control input functions in the domain $\Omega = [0,a] \times [0,b]$, where $a = 1, b = 1$. Moreover, we assume that the heating process is affected by a perturbation operator acting on Ω. Then the two dimensional perturbed linear model is defined by:

$$\begin{array}{rcll} v_t(\xi,\eta;t) & = & \kappa(\Delta+\delta)v(\xi,\eta;t) + \sum_{i=1}^{n_u} u_i(t)b_i, & \text{in } \Omega, t>0, \\ v(\xi,\eta;t) & = & 0, & \text{on } \partial\Omega, t>0, \\ v(\xi,\eta,0) & = & v_0(\xi,\eta), & \text{in } \Omega, \end{array} \tag{3.9}$$

where Δ denotes the Laplace operator, δ is a perturbation operator, $\partial\Omega$ denotes the boundary of Ω, $\kappa > 0$ is the diffusion coefficient, $b_i, i = 1, \ldots, n_u$ are given shape functions for the control inputs $u_1, \ldots, u_{n_u}$ and $v_0(\cdot)$ is the initial temperature distribution in Ω at $t = 0$. After a spatial finite difference discretization (with uniform mesh size $h = 0.0167$) of (3.9) we end up with a linear control system of the form

$$\begin{array}{rcl} \dot{x}(t) & = & (A + \delta A)x(t) + B_1 w(t) + Bu(t), \quad x(0) = x_0, \\ z(t) & = & C_1 x(t) + D_{12} u(t), \\ y(t) & = & Cx(t), \\ u(t) & = & Fy(t), \end{array} \tag{3.10}$$

where, here, $n_x = 3481$ and the variables and constant data matrices are defined as in (3.1), respectively. We choose $n_u = 2$ and $b_i = \chi_{\Omega_i^u}$, $i = 1, 2$, where $\chi_{\Omega_i^u}$ denotes the characteristic function on the control input domain $\Omega_i^u \subset \Omega$ of u_i and

$$\Omega_1^u = [0.1, 0.4] \times [0.1, 0.4], \quad \Omega_2^u = [0.5, 0.7] \times [0.5, 0.7].$$

Moreover, we set $n_y = 2$ and measure the state on the observation domains $\Omega_i^y \subset \Omega, i = 1, 2$ of $y(t) = (y_1(t), y_2(t))^T$, where

$$\Omega_1^y = [0.1, 0.2] \times [0.1, 0.2], \quad \Omega_2^y = [0.4, 0.6] \times [0.4, 0.6].$$

Hence, the data matrices $B \in \boldsymbol{R}^{n_x \times n_u}$, $C \in \boldsymbol{R}^{n_y \times n_x}$ only contain zeros and ones with ones at grid points within the control input and observation domains, respectively.

Table 3.4 lists the parameters that we have used for generating the corresponding data set for the linear control system of the form (3.1). Note, due to the choice of s_δ (see Table 3.4), the

TABLE 3.4
Parameter: perturbed distributed control heat flow model

n_x	n_u	n_y	κ	c_1	d_1	s_δ
3481	2	2	0.01	1	1	0.47813

perturbation operator $\delta A := s_\delta I_{n_x}$ shifts the original linear stable system to an unstable control system. In example, the real part of the largest eigenvalue of $(A + \delta A)$ is positive. Figure 3.12

FIG. 3.12. *Distributed control input at $t = 1.5, 3, 10, 20$.*

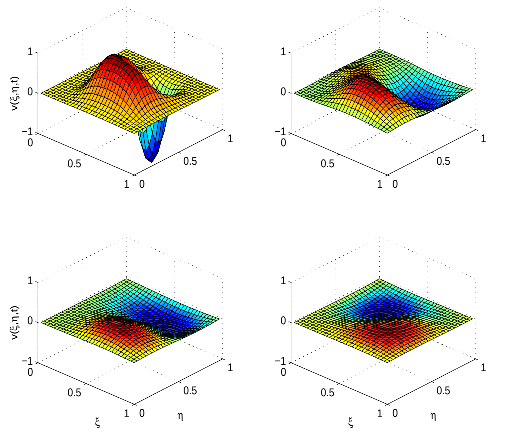
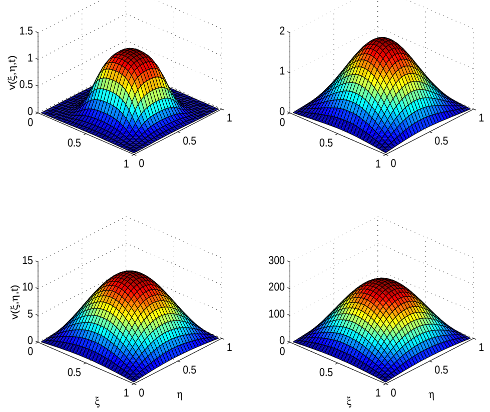


FIG. 3.13. *No distributed control at $t = 1.5, 3, 10, 20$.*



shows the temperature distribution of the linear perturbed heat flow example (3.9) if we use the output feedback control law computed by IPCTR in combination with POD. This figure illustrates very well that the simple SOF controller stabilizes the unstable control system pretty good. On the other hand, due to the perturbation in the heat equation, Figure 3.13 visualizes the instability of this heat flow process. In the uncontrolled case, we observe a rapid increase of the temperature.

In our test set collection (HF2D9) refers to the large scale perturbed approximation of this example. On the other hand, the corresponding POD model is stated in (HF2D18).

4. Short description of all benchmark examples in *COMPleib* 1.0. In this section we present a short overview of all 124 test examples which are currently implemented in release 1.0 of the test suite *COMPleib* . For more detailed information about the source and the application (if any) of the *COMPleib* 1.0 benchmark problems, we refer to [21]. At the current stage, *COMPleib* is divided into problem sets and the problem sets are grouped into problem classes.

Table 4.1 provides a list of the first problem class. In this class are those examples which are static output feedback (SOF) stabilizable (i. e. see Section 1).

Table 4.1: *Static output feedback control examples*

Example	n_x	n_u	n_y	Structure of A	Example	n_x	n_u	n_y	Structure of A
(AC1)	5	3	3	dense	(WEC3)	10	3	4	dense
(AC2)	5	3	3	dense	(HF1)	130	1	2	sparse
(AC3)	5	2	4	dense	(BDT1)	11	3	3	sparse
(AC4)	4	1	2	dense	(BDT2)	82	4	4	sparse
(AC5)	4	2	2	dense	(MFP)	4	3	2	dense
(AC6)	7	2	4	dense	(UWV)	8	2	2	dense
(AC7)	9	1	2	dense	(IH)	21	11	10	sparse
(AC8)	9	1	5	dense	(CSE1)	20	2	10	sparse
(AC9)	10	4	5	dense	(CSE2)	60	2	30	sparse
(AC10)	55	2	2	sparse	(EB1)	10	1	1	sparse
(AC11)	5	2	4	dense	(EB2)	10	1	1	sparse
(AC12)	4	3	4	dense	(EB3)	10	1	1	sparse
(AC13)	28	3	4	sparse	(EB4)	20	1	1	sparse
(AC14)	40	3	4	sparse	(EB5)	40	1	1	sparse
(AC15)	4	2	3	dense	(EB6)	160	1	1	sparse
(AC16)	4	2	4	dense	(PAS)	5	1	3	dense
(AC17)	4	1	2	dense	(TF1)	7	2	4	dense
(AC18)	10	2	2	dense	(TF2)	7	2	3	dense
(HE1)	4	2	1	dense	(TF3)	7	2	3	dense
(HE2)	4	2	2	dense	(PSM)	7	2	3	dense
(HE3)	8	4	6	dense	(TL)	256	2	2	dense
(HE4)	8	4	6	dense	(CDP)	120	2	2	sparse
(HE5)	8	4	2	dense	(NN1)	3	1	2	dense
(HE6)	20	4	6	dense	(NN2)	2	1	1	dense
(HE7)	20	4	6	dense	(NN3)	4	1	1	dense
(JE1)	30	3	5	partly sparse	(NN4)	4	2	3	dense
(JE2)	21	3	3	dense	(NN5)	7	1	2	dense
(JE3)	24	3	6	dense	(NN6)	9	1	4	dense
(REA1)	4	2	3	dense	(NN7)	9	1	4	dense
(REA2)	4	2	2	dense	(NN8)	3	2	2	dense
(REA3)	12	1	3	dense	(NN9)	5	3	2	dense
(REA4)	8	1	1	dense	(NN10)	8	3	3	dense
(DIS1)	8	4	4	dense	(NN11)	16	3	5	dense
(DIS2)	3	2	2	dense	(NN12)	6	2	2	dense
(DIS3)	6	4	4	dense	(NN13)	6	2	2	dense
(DIS4)	6	4	6	dense	(NN14)	6	2	2	dense
(DIS5)	4	2	2	dense	(NN15)	3	2	2	dense
(TG1)	10	2	2	dense	(NN16)	8	4	4	dense
(AGS)	12	2	2	sparse	(NN17)	3	2	1	dense
(WEC1)	10	3	4	dense	(NN18)	1006	1	1	sparse
(WEC2)	10	3	4	dense					

Note, we subdivide the example class of *COMPleib* listed in Table 4.1 into the following problem sets:

- Aircraft models *(AC)*
- Helicopter models *(HE)*
- Jet engine models *(JE)*
- Reactor models *(REA)*
- Decentralized interconnected systems *(DIS)*
- Euler Bernoulli beams *(EB)*
- Academic test problems *(NN)*.

Moreover, some further examples from different applications in this class are, i. e. , wind energy conversion models *(WEC)*, binary distillation towers *(BDT)*, terrain following models *(TF)*, and a compact disk player *(CDP)*.

The second class of *COMPl$_e$ib* examples are the 2D heat flow models as intensely discussed in Section 3. As listed in Table 4.2, the first nine examples represent the approximation of the discretized 2D heat flow models, while the other nine are the corresponding highly reduced order approximations of the large dimensional systems gained by the proper orthogonal decomposition (POD) approach as discussed in [25].

Table 4.2
2D heat flow models [Section 3]

Large model (sparse)					POD model (dense)					Property of	
Example	n_x	n_u	n_y	δ	Example	n_x	n_u	n_y	δ	A	model
(HF2D1)	3796	2	3	0.3825	(HF2D10)	5	2	3	0.3825	unstable	nonlinear
(HF2D2)	3796	2	3	0.5325	(HF2D11)	5	2	3	0.5325	unstable	nonlinear
(HF2D3)	4489	2	4	0	(HF2D12)	5	2	4	0	stable	linear
(HF2D4)	2025	2	4	0	(HF2D13)	5	2	4	0	stable	linear
(HF2D5)	4489	2	4	0.3825	(HF2D14)	5	2	4	0.3825	unstable	linear
(HF2D6)	2025	2	4	1.725	(HF2D15)	5	2	4	1.725	unstable	linear
(HF2D7)	4489	2	4	0.2775	(HF2D16)	5	2	4	0.2775	unstable	nonlinear
(HF2D8)	2025	2	4	0.7575	(HF2D17)	5	2	4	0.7575	unstable	nonlinear
(HF2D9)	3481	2	2	0.47813	(HF2D18)	5	2	2	0.47813	unstable	linear

The examples listed in Table 4.3 represent so–called second order models which can be rewritten into first order ODEs (i. e. see [21] and the references therein). Note, in this case, the system matrices have a special structure. This is the reason why we have collected those problems in an extra class. But, note, all currently *COMPl$_e$ib* examples in this problem class are also SOF stabilizable which, in general, is not always true for second order models.

Table 4.3
Second order models

Example	n_x	n_u	n_y	Structure of A	Example	n_x	n_u	n_y	Structure of A
(CM1)	20	1	2	partly sparse	(DLR1)	10	2	2	dense
(CM2)	60	1	2	partly sparse	(DLR2)	40	2	2	sparse
(CM3)	120	1	2	partly sparse	(DLR3)	40	2	2	sparse
(CM4)	240	1	2	partly sparse	(ISS1)	270	3	3	sparse
(CM5)	480	1	2	partly sparse	(ISS2)	270	3	3	sparse
(CM6)	960	1	2	partly sparse	(CBM)	348	1	1	partly sparse
(TMD)	6	2	4	dense	(LAH)	48	1	1	partly sparse
(FS)	5	1	3	dense					

Briefly this class is divided into the following problem sets:

- six so–called cable mass models with very low damping *(CM)*
- three models of a space structure developed by the "Deutsche Forschungsanstalt für Luft- und Raumfahrt" *(DLR)*
- two instances of a component of the International Space Station *(ISS)*
- some other second order models, i. e. a tuned mass damper *(TMD)* example, a clamped beam model *(CBM)*, a flexible satellite *(FS)* example, and, finally, a model of the Los Angeles (university) hospital *(LAH)*

Table 4.4
Reduced order control problems

Example	n_x	n_u	n_y	n_c	Structure of A	Example	n_x	n_u	n_y	n_c	Structure of A
(ROC1)	9	2	2	1	dense	(ROC6)	5	3	3	2	dense
(ROC2)	10	2	3	1	dense	(ROC7)	5	2	3	1	dense
(ROC3)	11	4	4	2	dense	(ROC8)	9	4	4	3	dense
(ROC4)	9	2	2	1	dense	(ROC9)	6	3	3	2	dense
(ROC5)	7	3	5	1	dense	(ROC10)	6	2	4	1	dense

The last class of test examples in *COMPl$_e$ib* 1.0 are the reduced order control (ROC) problems (i. e. see Section 1). These instances are not SOF stabilizable, but they are at least stabilizable by a reduced order output feedback control law of order n_c. Table 4.4 gives an overview of the currently implemented ROC problems. In this table, n_c denotes the smallest possible order of the reduced output feedback controller which can be used for stabilizing the control system. For more details on ROC problems, we refer the interested reader back to Section 1 or to [25] and the references therein.

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