

# CLASSICAL AND $\mathbb{C}$ -MOTIVIC ADAMS CHARTS

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**ABSTRACT.** This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the  $\mathbb{C}$ -motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 110-stem.

This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the  $\mathbb{C}$ -motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 110-stem.

The charts are intended to be viewed electronically. The authors can supply versions that are suitable for printing.

Justifications for these computations appear in [3], [4], [5], [7], [9], [10], [12], [17], [18], and [19]. Older references that justify many of the classical computations include [1], [2], [6], [13], [14], [15], and [16].

This document supersedes [8], which included Adams charts for the cofiber of  $\tau$ . The charts associated to the cofiber of  $\tau$  now appear in the separate manuscript [11].

## 1. THE $E_2$ -PAGE OF THE CLASSICAL ADAMS SPECTRAL SEQUENCE

This chart shows the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 110-stem.

- (1) Gray dots indicate copies of  $\mathbb{F}_2$ .
- (2) Vertical lines indicate  $h_0$  multiplications.
- (3) Lines of slope 1 indicate  $h_1$  multiplications.
- (4) Lines of slope  $1/3$  indicate  $h_2$  multiplications.
- (5) Light blue lines of slope  $-2$  indicate Adams  $d_2$  differentials.
- (6) Red lines of slope  $-3$  indicate Adams  $d_3$  differentials.
- (7) Green lines of slope  $-4$  indicate Adams  $d_4$  differentials.
- (8) Blue lines of slope  $-5$  indicate Adams  $d_5$  differentials.
- (9) Orange lines of slope less than  $-5$  indicate higher Adams differentials.
- (10) Dashed lines indicate possible Adams differentials. Beyond the 90-stem, possible Adams differentials are not indicated.

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## 2. THE $E_3$ -PAGE OF THE CLASSICAL ADAMS SPECTRAL SEQUENCE

This chart shows the  $E_3$ -page of the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 110-stem. See Section 1 for instructions on interpreting the chart. Beyond the 90-stem, possible Adams differentials are not indicated.

## 3. THE $E_\infty$ -PAGE OF THE CLASSICAL ADAMS SPECTRAL SEQUENCE

This chart indicates the  $E_\infty$ -page of the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem. Beyond the 90-stem, possible Adams differentials are not indicated. Beyond the 70-stem, not all hidden extensions have been resolved; see [10] for more details.

See Section 1 for instructions on interpreting the chart. In addition:

- (1) Red lines indicate hidden 2 extensions.
- (2) Blue lines indicate hidden  $\eta$  extensions.
- (3) Green lines indicate hidden  $\nu$  extensions.

## 4. THE $E_2$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the cohomology of the Steenrod algebra, i.e., the  $\mathbb{C}$ -motivic Adams  $E_2$ -page, as well as the Adams  $d_2$  differentials, through the 110-stem. For legibility, the chart is divided into two pages with different scales.

- (1) Gray dots indicate copies of  $\mathbb{M}_2$ .
- (2) Red dots indicate copies of  $\mathbb{M}_2/\tau$ .
- (3) Blue dots indicate copies of  $\mathbb{M}_2/\tau^2$ .
- (4) Green dots indicate copies of  $\mathbb{M}_2/\tau^3$ .
- (5) Purple dots indicate copies of  $\mathbb{M}_2/\tau^k$  for some  $k \geq 4$ .
- (6) Vertical lines indicate  $h_0$  multiplications. For legibility, the colors of these lines matches the colors of their targets.
- (7) Lines of slope 1 indicate  $h_1$  multiplications. For legibility, the colors of these lines matches the colors of their targets.
- (8) Lines of slope  $1/3$  indicate  $h_2$  multiplications. For legibility, the colors of these lines matches the colors of their targets.
- (9) Red arrows indicate infinite towers of  $h_1$  multiplications, all of which are annihilated by  $\tau$ .
- (10) Magenta lines indicate that an extension hits  $\tau$  times a generator. For example,  $h_0 \cdot h_0 h_2 = \tau h_1^3$  in the 3-stem.
- (11) Orange lines indicate that an extension hits  $\tau^k$  times a generator, for some  $k \geq 2$ . For example,  $h_0 \cdot h_0^3 x = \tau^2 h_0 e_0 g$  in the 37-stem.
- (12) Blue lines of slope  $-2$  indicate Adams  $d_2$  differentials.
- (13) Magenta lines of slope  $-2$  indicate that an Adams  $d_2$  differential hits  $\tau$  times a generator. For example,  $d_2(h_0 c_2) = \tau h_1^2 e_1$  in the 40-stem.
- (14) Orange lines of slope  $-2$  indicate that an Adams  $d_2$  differential hits  $\tau^k$  times a generator for some  $k \geq 2$ . For example,  $d_2(h_0 y) = \tau^2 h_0 e_0 g$  in the 37-stem.

The use of color is well-illustrated by the element  $h_2 g^2$  in the 43-stem. The dot is green, indicating that  $\tau^3 h_2 g^2$  is zero. The outgoing blue lines indicate that  $h_0 \cdot h_2 g^2$  and  $h_2 \cdot h_2 g^2$  are annihilated by  $\tau^2$ . The incoming magenta line indicates

that  $h_2 \cdot \tau g^2$  equals  $\tau h_2 g^2$ , and the incoming orange line indicates that  $h_1 \cdot Ph_1^3 h_5$  equals  $\tau^2 h_2 g^2$ .

#### 5. THE $E_3$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the Adams  $d_3$  differentials on the  $E_3$ -page of the motivic Adams spectral sequence through the 110-stem. The chart is complete through the 95-stem. Beyond the 95-stem, possible differentials are not indicated.

See Section 4 for instructions on interpreting the chart. In addition:

- (1) Blue lines of slope  $-3$  indicate Adams  $d_3$  differentials.
- (2) Magenta lines of slope  $-3$  indicate that an Adams  $d_3$  differential hits  $\tau$  times a generator.
- (3) Orange lines of slope  $-3$  indicate that an Adams  $d_3$  differential hits  $\tau^k$  times a generator for some  $k \geq 2$ .

#### 6. THE $E_4$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the Adams  $d_4$  differentials on the  $E_4$ -page of the motivic Adams spectral sequence through the 110-stem. The chart is complete through the 95-stem. Beyond the 95-stem, possible differentials are not indicated.

See Section 4 for instructions on interpreting the chart. In addition:

- (1) Blue lines of slope  $-4$  indicate Adams  $d_4$  differentials.
- (2) Magenta lines of negative slope indicate that an Adams differential hits  $\tau$  times a generator.
- (3) Orange lines of negative slope indicate that an Adams differential hits  $\tau^k$  times a generator for some  $k \geq 2$ .

#### 7. THE $E_5$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the Adams  $d_5$  differentials on the  $E_5$ -page of the motivic Adams spectral sequence. The chart is complete through the 95-stem.

See Section 4 for instructions on interpreting the chart. In addition:

- (1) Blue lines of slope  $-5$  indicate Adams  $d_5$  differentials.
- (2) Magenta lines of negative slope indicate that an Adams differential hits  $\tau$  times a generator.
- (3) Orange lines of negative slope indicate that an Adams differential hits  $\tau^k$  times a generator for some  $k \geq 2$ .

#### 8. THE $E_6$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the higher Adams differentials on the  $E_6$ -page of the motivic Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem.

See Section 4 for instructions on interpreting the chart. In addition:

- (1) Blue lines of negative slope indicate Adams  $d_r$  differentials for some  $r \geq 6$ .
- (2) Magenta lines of negative slope indicate that an Adams differential hits  $\tau$  times a generator.
- (3) Orange lines of negative slope indicate that an Adams differential hits  $\tau^k$  times a generator for some  $k \geq 2$ .
- (4) Dashed lines indicate possible Adams differentials.

9. THE  $E_\infty$ -PAGE OF THE MOTIVIC ADAMS SPECTRAL SEQUENCE

This chart indicates the  $E_\infty$ -page of the motivic Adams spectral sequence. The chart is complete through the 90-stem, with partial results through the 95-stem.

For clarity, hidden extensions by 2,  $\eta$ , and  $\nu$  are not shown on this chart.

See Section 4 for instructions on interpreting the chart. In addition:

- (1) Green vertical lines indicate hidden  $\tau$  extensions.
- (2) Dashed lines of negative slope indicate unknown Adams differentials. Beyond the 90-stem, unknown differentials are not shown.

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