FRACTAL DIMENSION

Bekmirzayev D.A
t.f.d; DSc, Institute of Mechanics and Seismic Resistance of Structures named after M.T.Urazbaev of the Academy of Sciences of the Republic of Uzbekistan
Uralova I.A
Magistr of Computer Engineering Faculty, Tashkent University of Information Technologies named after Muhammad al-Khwarizmi, Tashkent, Uzbekistan

Abstract: Fractal geometry provides a powerful tool for scale-free spatial analysis of cities, but the fractal dimension calculation results always depend on methods and scopes of the study area. This phenomenon has been puzzling many researchers. This paper is devoted to discussing the problem of uncertainty of fractal dimension estimation and the potential solutions to it. Using regular fractals as archetypes, we can reveal the causes and effects of the diversity of fractal dimension estimation results by analogy.

Keywords: Fractal, fractal geometry, Koch coastline, fractal objects, Hausdorff-Besicovitch dimension.

Introduction

A fractal dimension is an index for characterizing fractal patterns or sets by quantifying their complexity as a ratio of the change in detail to the change in scale. Several types of fractal dimension can be measured theoretically and empirically. Fractal dimensions are used to characterize a broad spectrum of objects ranging from the abstract to practical phenomena, including turbulence, river networks, urban growth, human physiology, medicine, and market trends. The essential idea of fractional or fractal dimensions has a long history in mathematics that can be traced back to the 1600s, but the terms fractal and fractal dimension were coined by mathematician Benoit Mandelbrot in 1975.

Fractal dimensions were first applied as an index characterizing complicated geometric forms for which the details seemed more important than the gross picture. For sets describing ordinary geometric shapes, the theoretical fractal dimension equals the set's familiar Euclidean or topological dimension. Thus, it is 0 for sets describing points (0-dimensional sets); 1 for sets describing lines (1-dimensional sets having length only); 2 for sets describing surfaces (2-dimensional sets having length and width); and 3 for sets describing volumes (3-dimensional sets having length, width, and height). But this changes for fractal sets. If the theoretical fractal dimension of a set exceeds its topological dimension, the set is considered to have fractal geometry.

In mathematics, more specifically in fractal geometry, a fractal dimension is a ratio providing a statistical index of complexity comparing how detail in a pattern (strictly speaking, a fractal pattern) changes with the scale at which it is measured. It has also been characterized as a measure of the space-filling capacity of a pattern that tells how a fractal scales differently from the space it is embedded in; a fractal dimension does not have to be an integer.

The essential idea of "fractured" dimensions has a long history in mathematics, but the term itself was brought to the fore by Benoit Mandelbrot based on his 1967 paper on self-similarity in which he discussed fractional dimensions. Mandelbrot cited previous work by Lewis Fry Richardson describing the counter-intuitive notion that a coastline's measured length changes with the length of the measuring stick used. In terms of that notion, the fractal dimension of a coastline quantifies how the number of scaled measuring sticks required to measure the coastline changes with the scale applied to the stick.

Take the Koch coastline and examine it through a badly focused lens. It appears to have a certain length. Let's call it 1 unit. Sharpen the focus a bit so that you can resolve details that are ⅓ as big as those seen with the first approximation. The curve is now four times longer or 4 units. Double the resolution by the same factor. Using a focus that reveals details 1/9 the first focus gives us a coastline 16 times longer and so on. Such an activity hints at the existence of a quantifiable characteristic.

To be a bit more precise, every space that feels "real" has associated with it a sense of distance between any two points. On a line segment like the Koch coastline, we arbitrarily chose the length of one side of the first iterate as a unit length. On the Euclidean coordinate plane the distance between any two points is given by the Pythagorean theorem

$$s^2 = x^2 + y^2$$

In relativity, the "distance" between any two events in space-time is given by the proper time

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

Such distance establishing relationships are called metrics and a space that has a metric associated with it is called a metric space. One of the more famous, non-euclidean metrics is the Manhattan metric (or taxicab metric). How far is the corner of 33rd and 1st from 69th and 5th? Answer: 36 blocks and 4 avenues or 40 units. (We have to bend reality a bit and assume that city blocks in Manhattan are square and not rectangular.) Metrics are also used to create neighborhoods in a space. Pick a point in a metric space. This point plus all others lying less than or equal to a certain distance away comprise a region of the space called a closed disk. The term disk is used because such regions are disk-shaped in the coordinate plane with the usual metric but any shape is possible. In euclidean three-space disks would be balls while in a two-space with a Manhattan metric they would be squares.

How many disks does it take to cover the Koch coastline? Well, it depends on their size of course. 1 disk with diameter 1 is sufficient to cover the whole thing, 4 disks with diameter ⅓, 16 disks with diameter 1/9, 64 disks with diameter 1/27, and so on. In general, it takes 4^n disks of radius $(⅓)^n$ to cover the Koch coastline. If we apply this procedure to any entity in any metric space we can define a quantity that is the equivalent of a dimension. The Hausdorff-Besicovitch dimension of an object in a metric space is given by the formula

$$D = \lim_{h \to 0} \frac{\log N(h)}{\log (1/h)}$$

where N(h) is the number of disks of size h needed to cover the object. Thus the Koch coastline has a Hausdorff-Besicovitch dimension which is the limit of the sequence $\frac{\log 1}{\log 1}, \frac{\log 4}{\log 3}, \frac{\log 16}{\log 9}, \frac{\log 64}{\log 27}, \ldots$

$$\frac{\log 4^n}{\log 3^n} = \frac{n \log 4}{n \log 3} = \frac{\log 4}{\log 3} = 1.261859507 \ldots$$

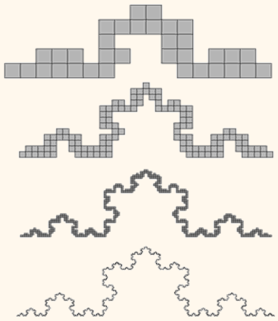
Is this really a dimension? Apply the procedure to the unit line segment. It takes 1 disk of diameter 1, 2 disks of diameter ½, 4 disks of diameter ¼, and so on to cover the unit line segment. In the limit we find a dimension of

$$\frac{\log 2^n}{\log 2^n} = \frac{n \log 2}{n \log 2} = \frac{\log 2}{\log 2} = 1$$

This agrees with the topological dimension of the space.

The problem now is, how do we interpret a result like 1.261859507…? This does not agree with the topological dimension of 1 but neither is it 2. The Koch coastline is somewhere between a line and a plane. Its dimension is not a whole number but a fraction. It is a fractal. Actually fractals can have whole number dimensions so this is a bit of a misnomer. A better definition is that a fractal is any entity whose Hausdorff-Besicovitch dimension strictly exceeds its topological dimension ($D > D_T$). Thus, the Peano space-filling curve is also a fractal as we would expect it to be. Even though its Hausdorff-Besicovitch dimension is a whole number ($D = 2$) its topological dimension ($D_T = 1$) is strictly less than this. The monster has been tamed.

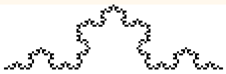
It should be possible to use analytic methods like those described above on all sorts of fractal objects. Whether this is convenient or simple is another matter. Fractals produced by simple iterative scaling procedures like the Koch coastline are very easy to handle analytically. Julia and Mandelbrot sets, fractals produced by the iterated mapping of continuous complex functions, are another matter. There's no obvious fractal structure to the quadratic mapping, no hint that a "monster" curve lurks inside, and no simple way to extract an exact fractal dimension. If there are analytic techniques for calculating the fractal dimension of an arbitrary Julia set they are well hidden. A narrow and quick search of the popular literature reveals nothing on the ease or impossibility of this task. There are, however, experimental techniques.



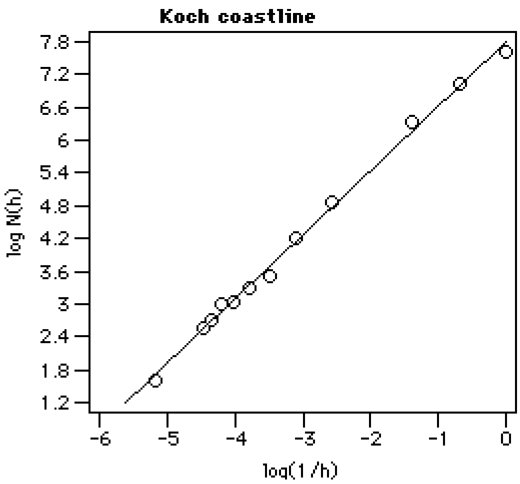
Surrounding the Koch coastline with boxes is a way to determine its dimension

Take any plane geometric object of finite extent (fractal or otherwise) and cover it with a single closed disk. Any type of disk will do, so to make life easy we will use a square; the disk of the Manhattan metric in the plane. Record its dimension and call it "h". Repeat the procedure with a smaller box. Record its dimension and the number of boxes "N(h)" required to cover the object. Repeat with ever smaller boxes until you have reached the limit of your resolving power as shown in the figure to the right. Plot the results on a graph with "log N(h)" on the vertical axis and "log (1/h)" on the horizontal axis. The slope of the best fit line of the data will be an approximation of the Hausdorff-Besicovitch dimension of the object. The following are the results of a few sample experiments using this box-counting method. I think with a bit of refinement, the deviations could all be brought below 5%.

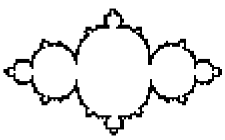
log (1/h)	log N(h)
0	7.60837
−0.69315	7.04054
−1.38629	6.32972
−2.56495	4.85981
−3.09104	4.21951
−3.49651	3.52636
−3.78419	3.29584
−4.00733	3.04452
−4.18965	2.99573
−4.34381	2.70805
−4.47734	2.56495
−5.17615	1.60944


Koch Coastline

dimension (experimental) = 1.18
dimension (analytical) = 1.26
deviation = 6%

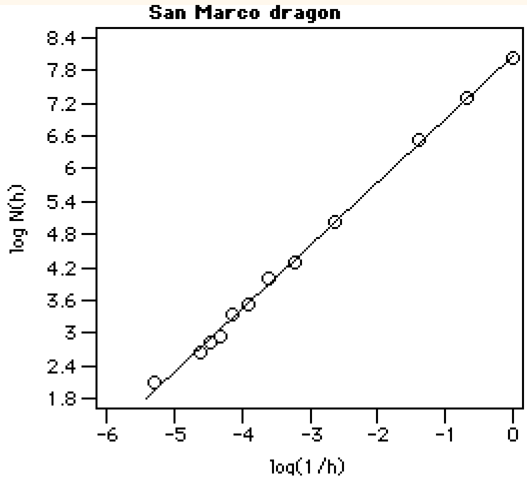


log (1/h)	log N(h)
0	8.02355
−0.69315	7.29438
−1.38629	6.52209
−2.63906	5.03044
−3.21888	4.29046
−3.61092	4.00733


San Marco Dragon

dimension (experimental) = 1.16
dimension (analytical) = ??
deviation = ??

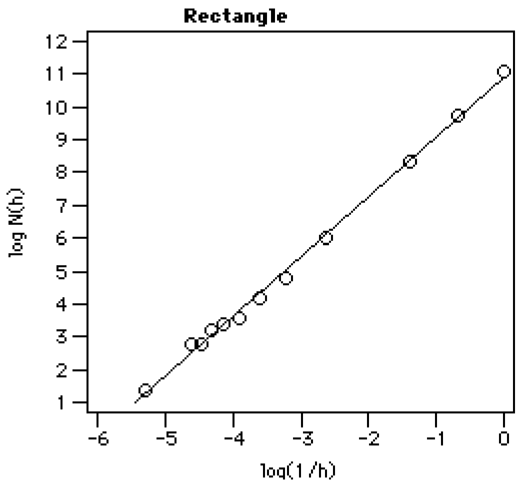
−3.91202	3.52636
−4.12713	3.33220
−4.31749	2.94444
−4.46591	2.83321
−4.60517	2.63906
−5.29832	2.07944

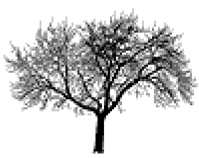



Rectangle

log (1/h)	log N(h)
0	11.0904
−0.69315	9.71962
−1.38629	8.31777
−2.63906	5.99146
−3.21888	4.79579
−3.61092	4.15888
−3.91202	3.58352
−4.12713	3.40120
−4.31749	3.21888
−4.46591	2.77259
−4.60517	2.77259
−5.29832	1.38629

dimension (experimental) = 1.82
dimension (analytical) = 2.00
deviation = 9%

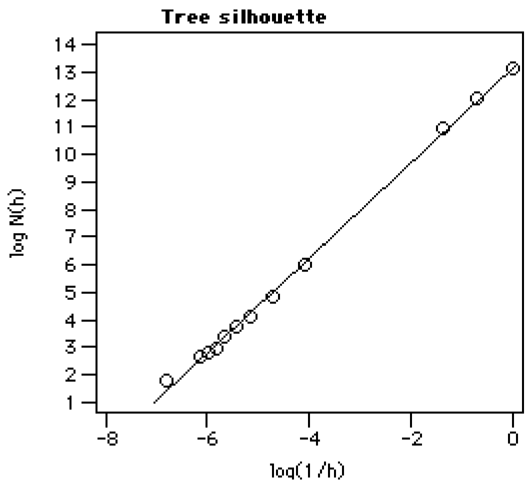



Tree Silhouette

log (1/h)	log N(h)

0	13.1055
−0.69315	12.0209
−1.38629	10.9253
−4.06044	5.99894
−4.71850	4.82831
−5.12396	4.14313
−5.41165	3.73767
−5.63479	3.36730
−5.81711	2.94444
−5.97126	2.83321
−6.10479	2.63906
−6.79794	1.79176

dimension (experimental) = 1.73
dimension (analytical) = ??
deviation = ??



Many real-world phenomena exhibit limited or statistical fractal properties and fractal dimensions that have been estimated from sampled data using computer based fractal analysis techniques. Practically, measurements of fractal dimension are affected by various methodological issues, and are sensitive to numerical or experimental noise and limitations in the amount of data. Nonetheless, the field is rapidly growing as estimated fractal dimensions for statistically self-similar phenomena may have many practical applications in various fields including astronomy, acoustics, geology and earth sciences, diagnostic imaging, ecology, electrochemical processes, image analysis, biology and medicine, neuroscience, network analysis, physiology, physics, and Riemann zeta zeros. Fractal dimension estimates have also been shown to correlate with Lempel-Ziv complexity in real-world data sets from psychoacoustics and neuroscience.

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