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RESEARCH ARTICLE

NEUTRON STAR BINARYAS GRAVITATIONAL QUANTUM SCATTERING SYSTEM

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Abstract

Using ensemble of N gravitons, we evaluate the integrated density of states for gravitons arising from the spiral-in motion of gravitationally bound neutron star (NS) binaries, under the Dirichlet boundary conditions. Consequently, the two massive objects making up of the binary system are immersed in a gravito-bosonic environment. The corrections to the density of states for gravitons give expression to the quantum fluctuations of gravitational fields, i.e., the gravitational Casimir energy. By considering the extended scales of the star components, we find the gravitational Casimir effects are larger than that obtained by treating them as point particles and that the gravitational Casimir energy depends both on the scales of objects and on the separation of the binary, which increases for lager scales of star components in a typical NS binary system, with separation of $10^9 m$, i.e. low-frequency gravitational wave sources.

Introduction:-

The existence of the attraction between two neutral, parallel metallic plates [1] has been confirmed by many experiments with high accuracy [2,3]. The origin of such an attractive force should be attributed to the modification of the spectrum of zero point fluctuations of the electromagnetic field [4,5], when the plates are brought into close distance. Similar phenomena also appear as long-range interactions between macroscopic polarizable systems and give rise nontrivial influence [6-10]. While in large-scale systems subject to the long-range gravitational interactions, the Casimir effect arises in spacetime with nontrivial topology [11,12], which is a geometrical nature and depends on the imposed boundary conditions and the topology. The exact calculations for Casimir energy can be performed in terms of integral or infinite summation under restricted boundary conditions and simple geometries. However, even for the attraction between metallic sphere and a plate, which are much similar to align than two plates, it is still harder to compute. Only the proximity-force approximation [13,14] is applicable for a vanishing separation between a sphere and a plate. While for small separations, several techniques have been employed to calculate the electromagnetic Casimir effects with different boundaries. The semiclassical approach [15,16] to Casimir energies in the framework of the Gutzwiller's trace formula [17] was used to study the contributions just from fluctuations around periodic classical orbits subject to boundary surface. A string-inspired worldline method [18] in the framework of the Feynman path integral was developed and applied to compute the Casimir effect for arbitrary geometries. An optical approximation [19,20] to the Casimir force between arbitrary smooth shapes bases on classical optical paths, which is also accurate and versatile for flat manifolds. In the case of complicated geometries, an evaluation for Casimir problem with any separation resorts to the ensemble of scatters, by utilizing the Krein formula that bridges the special density and the problem of scattering [21-23].

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The problem of modern quantum field theory lies in the way how it responds to topological defects and singularities. By considering that the penetration length of the fluctuating fields inside the objects is much smaller than the separation between them, the Casimir problem falls right in such class of problems. It has been found that the fluctuations of a quantum field cause an attractive force between localized defects [24] in the very same way as Casimir forces between conductors, and that the interaction energy depends on the scattering lengths of the defects and their relative separations. The fluctuations of periodic orbits of the associated classical problems contribute to Casimir energy [15], which inversely relates to the separation of the defects [25] and is not caught in any removal of ultraviolet divergences when involving boundaries. In neutron star (NS) binaries, we treat two star components as 3-dimension spherically topological defects immersed in the flat space-time. Subject to the gravitational interactions, the two objects orbit with each other periodically and move closer and closer in a spiral way, which loses orbital binding energy and releases gravitational waves (GWs). The fluctuating gravitational fields satisfy Dirichlet boundary conditions and disappear at the star surface, which makes the quantum dynamics of the gravitational field fall in the gravitational Casimir problem. The quantum fluctuations of gravitational fields give rise to gravitational Casimir force between defects localized in the binary system. By considering the typically wide separation of $10^9 m$ and thus the cosmological time for coalescence, the radial decay of orbital scale Δr during several periods is much smaller than the binary separation, i.e., $\Delta r << R$. Therefore, the orbital decay Δr during an observable time can be considered as fluctuations of the periodic orbit. Accordingly, we treat the spiral-in motion of wide NS binary as periodically orbital motion with small fluctuations in radial direction and evaluate the gravitational Casimir problem, by relating it to the problem of gravitational scattering.

The paper is organized as follows: Firstly, we use the Krein formula and relate the gravitational bound NS binaries to gravitational scattering systems by N impenetrable and spherical gravitons. By describing the scattering of N hard cores in the framework of scattering theory [26-30], we define the multi-scattering matrix and evaluate the corrections to the density of states for gravitons in the gravitational bound systems. In section III, we formulate the gravitational Casimir problem in terms of the density of states for the gravitons and calculate the contributions to gravitational Casimir energy for different kinds of NS binary systems with different scattering scales. Finally, we make the conclusion and give a brief summary.

Gravitational scattering in Dirichlet NS binary:-

Let's consider the wide NS binary systems with typical separations of $R \sim 10^9 m$ from surface to surface, consisting of two objects with typical scales of $a \sim 10^4 m$. Accordingly, the shortest center-to-center distance is $L = 2a + R$. The binary systems gravitationally bound via infinite spin-2 gravitons, which are continually emitted due to the gravitation-induced fluctuations of periodic orbits during the inspiralling processes. Consequently, the two objects in a NS binary immersed in the gravito-bosonic environment. The changes in the density of states for gravitons give expression to the quantum fluctuations of gravitational fields, which contributes to gravitational Casimir energy. In order to evaluate the gravitational Casimir energy, we treat the gravitons as impenetrable hard cores, which are nonoverlapping and nontouching, and deal with the N-sphere scattering problem in NS binary systems.

In the case of N spherical scatters, the determinant of scattering matrix S_N factories into the product of the determinants of scattering matrix for single scatter [21-23],

$$\det S_N(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) = \{\prod_{n=1}^{N} \det S_n(\varepsilon, a_n)\} \frac{\det M^+(\varepsilon^*)}{\det M(\varepsilon)}, \quad (1)$$

where a_n denotes the radius of the n th graviton, $\vec{r}_{nn'}$ represents the relative distance between the n th and the n' th gravitons. $M(\varepsilon)$ is a modified Korringa-Kohn-Rostokermultiscattering matrix, which implies that two successive scattering processes have to occur at different gravitons. According to Einstein's general relativity, gravitational fields are massless fields, which obeys the energy dispersion relation $\varepsilon = \hbar\omega = \hbar ck$. Under Dirichlet boundary conditions, the inverse multi-scattering matrix of N nonoverlapping and nontouching spherical gravitons is given by [21]

$$M^{nn'}_{lm,l'm'} = \delta^{nn'}\delta_{ll'}\delta_{mm'} + (1-\delta^{nn'})\sqrt{4\pi}\, i^{2m+l'-l}\sqrt{(2l+1)(2l'+1)}\left(\frac{a_n}{a_{n'}}\right)^2 \frac{j_l(ka_n)}{h^{(1)}_{l'}(ka_{n'})}$$
$$\times \sum_{l''=0}^{\infty}\sum_{m''=-l'}^{l'} D^{l'}_{m',m''}(nn')h^{(1)}_{l''}(kr_{nn'})Y^{m-m''}_{l''}(\hat{\vec{r}}^{(n)}_{nn'})\sqrt{(2l''+1)}\, i^{l''}\begin{pmatrix} l'' & l' & l \\ m-m'' & m'' & -m \end{pmatrix}. \quad (2)$$

Here, $n, n' = 1,2,...,N$ are the labels of N gravitons, $l, l' l'' = 0,1,2,...$ denote the orbital angular momentum quantum numbers, and m, m', m'' represent the pertinent gravitomagnetic quantum numbers. $j_l(x)$ and $h^{(1)}_l(x)$ are spherical Bessel functions and Hankel functions of the first kind, respectively. $Y_{lm}(\hat{\vec{r}}^{(n)}_{nn'})$ denotes spherical harmonic function, where $\hat{\vec{r}}^{(n)}_{nn'}$ is the unit vector measured in the local frame of graviton n. $D^l_{mm'}(nn')$ is the Wigner rotational matrix given in [21], which transforms the local coordinate system from graviton n to that of graviton n'.

The total density of states for gravitons produced by an NS binary can be expressed as a function of the energy ε,

$$g(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) = g_I(\varepsilon) + g_D(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) + \sum_{n=1}^{N} g_W(\varepsilon, a_n). \quad (3)$$

$g_I(\varepsilon)$ represents the density of states for gravitons mediating the Newtonian gravitational interactions at the initial state of the system. $\sum_{n=1}^{N} g_W(\varepsilon, a_n)$ denotes the corrections to the density of states arising from N gravitons infinitely apart from each other when the GWs propagate outside the whole space that indicates the total phase shift of the gravitational scattering problem in the binary, and $g_W(\varepsilon, a_n)$ is identified as a Weyl-type formula with the phase shift of the n th graviton. The term of $g_D(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\})$ reflects changes in geometry of the binary arising from the orbital fluctuations and thus the dynamically gravitational Casimir effects, which is the only part showing an energy dependence and subsequently have influence on the frequencies of released GWs. The changes in density of states for N gravitons due to the presence of two star components in the binary can be expressed as N-body scattering matrix connected by the Krein formula [31],

$$\begin{aligned}\delta g(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) &= g(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) - g_I(\varepsilon) = g_D(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) + \sum_{n=1}^{N} g_W(\varepsilon, a_n) \\ &= \frac{1}{2\pi i}\frac{d}{d\varepsilon}\ln\det S_N(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\}) \\ &= \frac{1}{2\pi i}\frac{d}{d\varepsilon}\ln\prod_{n=1}^{N}\det S_n(\varepsilon, a_n) + \frac{1}{2\pi i}\frac{d}{d\varepsilon}\ln\frac{\det M^+(\varepsilon^*)}{\det M(\varepsilon)}.\end{aligned} \quad (4)$$

The geometry-dependent part of the density of states $g_D(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\})$ can be extracted from the inverse multi-scattering matrix of Eq. (2), which depends on the relative arrangement of the scatterers and is finite. By considering that the transmission of gravitational interactions depends on infinite gravitons, the total number of gravitons

$$N = \int_0^{\varepsilon} g(\varepsilon')d\varepsilon' \to \infty, \quad (5)$$

which implies that the density of states $\sum_{n=1}^{N} g_W(\varepsilon, a_n)$ are infinite. In addition, the term $\sum_{n=1}^{N} g_W(\varepsilon, a_n)$ is proportional to the volume of the entire space, whose changes are divergent, and thus a divergence of $\delta g(\varepsilon, \{a_n\}, \{\vec{r}_{nn'}\})$. However, the changes in density of states becomes a finite one in well-defined systems, e.g. the Dirichlet NS binary systems.

Gravitational Casimir problem in NS binary:-

In inspiraling NS binaries, the transmission of gravitational interactions between two objects can be described by Green's function, and the propagator for gravitons is written as $G_0(\vec{r}_1, \vec{r}_2, k) = -\frac{e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}}{2\pi\hbar^2|\vec{r}_1-\vec{r}_2|}$, where $|\vec{r}_1-\vec{r}_2|$ describes the relative location of two objects and depends on the multi-scattering paths of gravitons. The changes of relative location $|\vec{r}_1-\vec{r}_2|$ in the binary modify the strength of gravitational interactions and subsequently contribute to the corrections to the density of states for gravitons, which is responsible for the quantum fluctuations of gravitational fields. What we are interested in is the gravitational Casimir energy arising from the orbital decay in radial direction subject to gravitational interactions.

According to the constraints from LIGO, the mass of gravitons during the merger phase of black hole binary is $m_g \le 1.2\times 10^{-22}\, eV/c^2$ [32]. Our investigations for gravitational Higgs mechanism in inspiraling double NS binaries [33] and NS-white dwarf [34] systems, with wide separation of $10^9 m$, demonstrate that the masses of gravitons vary with the intrinsic properties of the binary sources, and that a relatively wider system provides more massive gravitons. However, the order of masses from wide inspiraling NS binaries with typical separation of $10^9 m$ is about $10^{-23} eV/c^2$. In light of these results, the wavelength of GWs should be $\lambda_g \sim 10^{16} m$, which is much larger than both the binary separation and the radius of two star components making up of the system, i.e. $\lambda_g >> R >> a$. Therefore, we are allowed to simplify the inverse scattering matrix (2) in the large-distance limit [35],

$$M(\varepsilon)_{nn'} \approx \delta_{nn'} - (1-\delta_{nn'}) f_n(\varepsilon)\frac{e^{ikr_{nn'}}}{r_{nn'}}, \quad (6)$$

where $f_n(\varepsilon)$ is the scattering amplitude of gravitons that transfer the Newtonian gravitational interactions. The determinant of a typical NS binary system in the total domain factorizes into two subdeterminants,

$$\begin{aligned}\det M(k)^d &= (1+\frac{a_1}{R+a_1+a_2}e^{ik(R+a_1)})(1-\frac{a_2}{R+a_1+a_2}e^{ik(R+a_2)}) \\ &= 1-\frac{a_1 a_2}{R+a_1+a_2}e^{ik(2R+a_1+a_2)},\end{aligned} \quad (7)$$

where a_1 and a_2 are the radii of two objects in the binary system, respectively. For a system consisting of two approximately identical stars, e.g. double NS (DNS) binaries, the determinant of inverse scattering matrix becomes

$$\begin{aligned}\det M(k)^i &= (1+\frac{a}{R+2a}e^{ik(R+a)})(1-\frac{a}{R+2a}e^{ik(R+a)}) \\ &= 1-\frac{a^2}{(R+2a)^2}e^{2ik(R+a)},\end{aligned} \quad (8)$$

As consequence, the gravitational Casimir energy can be computed by

$$\mathrm{E}_{Cas} = \int_0^\infty \frac{1}{2}\delta g(\varepsilon) d\varepsilon, \quad (9)$$

which rest with the integrated corrections to density of states for gravitons.
For approximately identical objects comprising systems, the gravitational Casimir energy is

$$\mathrm{E}_{Cas}^i = \frac{\hbar c}{2\pi}\int_0^\infty dk \ln[1-\frac{a^2}{(R+2a)^2}e^{2ik(R+a)}], \quad (10)$$

while for the binaries formed by different components, it is calculated as

$$\mathrm{E}_{Cas}^{d} = \frac{\hbar c}{2\pi}\int_0^\infty dk \ln[1 - \frac{a_1 a_2}{(R+a_1+a_2)^2} e^{ik(2R+a_1+a_2)}]. \quad (11)$$

The part of corrections to the density of states arising from the radial decay during the spiral-in orbital motion depend on the relative location $\sim \frac{1}{|\vec{r}_1 - \vec{r}_2|}$ of two objects making up of the binary system, which is determined by the inverse multiple scattering matrix. For binaries consisting of two approximately identical stars, e.g. DNS systems, the corresponding integrated density of states can be computed as

$$\begin{aligned} N_D^i(\varepsilon) &= \int_0^\varepsilon d\varepsilon' \delta g(\varepsilon', |\vec{r}_1 - \vec{r}_2|) \\ &= 2 \times \frac{a^2}{\pi (R+2a)^2} \sin[2k(R+a)] + \mathrm{O}(ka^3, \frac{a^4}{(R+2a)^4}), \end{aligned} \quad (12)$$

where the factor "2" in the second line denotes the spin factor of graviton. Consequently, the gravitational Casimir energy coming from the orbital decay of the binary system consisting of two identical Dirichlet NS components is obtained as

$$\begin{aligned} \mathrm{E}_{Cas}^{i} &= -\frac{1}{2}\int_0^\infty d\varepsilon N_D^i(\varepsilon) = \frac{\hbar c}{2\pi}\int_0^\infty dk \ln[1 - \frac{a^2}{(R+2a)^2} e^{2ik(R+a)}] \\ &= -\frac{\hbar c}{2\pi} \frac{a^2}{(R+a)(R+2a)^2} \\ &= -\frac{\hbar c \pi^2}{1440} \frac{4\pi a^2}{R^3} \frac{90}{\pi^4} \frac{2}{(1+\frac{a}{R})(1+\frac{2a}{R})^2}. \end{aligned} \quad (13)$$

By considering the typical separation of $R \sim 10^9 m$ and star radius of $a \sim 10^4 m$, the leading-order gravitational Casimir energy arising from the radial decay during the spiral-in orbital motion is

$$\mathrm{E}_{LC}^{i,D} = -2 \times \frac{\hbar c \pi^2}{1440} \frac{4\pi a^2}{R^3} \frac{90}{\pi^4}. \quad (14)$$

As far as contributions from the propagation of GWs all over the global space, the changes of the density of states arise from the gravitons infinitely apart from each other, which are obtained by the expansion of relativistic periodic orbits under Dirichlet boundary conditions and can be expressed as the separation of the binary. We finally derive the result,

$$N_W^i(\varepsilon) \approx 2 \times \frac{a^2}{4\pi R(R+2a)} \sin(2kR). \quad (15)$$

As consequence, the contributions to the gravitational Casimir energy arising from the global propagation of GWs in binaries consisting of two identical components is

$$\begin{aligned} \mathrm{E}_{Cas}^{i,W} &= -2 \times \frac{a^3}{16\pi R^2 (R+2a)} j_1(2kR) \\ &= -2 \times \frac{\hbar c \pi^2}{1440} \frac{4\pi a^2}{R^3} \frac{90}{\pi^4} \frac{j_1(2kR)}{1+\frac{2a}{R}}. \end{aligned} \quad (16)$$

where $j_1(2kR)$ is the first order spherical Bessel function.

If the binary system consists of two different components, e.g., NS-white dwarf (WD) binary systems, the integrated density of states for gravitons coming from the orbital decay is calculated as,

$$\begin{aligned} N_D^d(\varepsilon) &\approx -\frac{1}{\pi}\,\mathrm{Im}\ln[1-\frac{a_1a_2}{4\pi(R+a_1+a_2)^2}e^{2ik(R+a_1+a_2)}] \\ &\approx 2\times\frac{a_1a_2}{\pi(R+a_1+a_2)^2}\sin[2k(R+a_1+a_2)], \end{aligned} \tag{17}$$

where a_1 and a_2 represent the radius of NS and WD, respectively. That arising from the propagations of GWs all over the space is then given by

$$N_W^d(\varepsilon) \approx 2\times\frac{a_1a_2}{4\pi R(R+a_1+a_2)}\sin(2kR). \tag{18}$$

Therefore, the corresponding gravitational Casimir energy can be evaluated as

$$\mathrm{E}_{Cas}^{d,D} = -\frac{\hbar c}{2\pi}\frac{a_1a_2}{(R+a_1+a_2)^3}, \tag{19}$$

and

$$\mathrm{E}_{Cas}^{d,D} = -\frac{\hbar c}{8\pi}\frac{a_1a_2}{R^2(R+a_1+a_2)}j_1(2kR), \tag{20}$$

respectively.

Conclusion:-

In this work, we present an evaluation of the gravitational Casimir energy for wide inspiraling NS binaries, with separation of $R \sim 10^9 m$, by taking the extended scales of star components into account. We treat the spiral-in orbital motion of NS binaries as periodic orbits with radial fluctuations, by comparing the radial shrink during an observable time with typical separation of $R \sim 10^9 m$ that coalesces on the cosmological timescale. According to the hypothesis of quantum field theory, the gravitational interactions between two massive objects are transferred by infinite gravitons, which are continual released due to the spiral-in motion and the increasing gravitational interactions. As a result, the two objects making up of the binary are immersed in a gravito-bosonic environment. The problem for quantum fluctuations of gravitational field in gravitationally bound NS binaries is therefore related to the scattering of N gravitons, which are considered as non-overlapping, nontouching, and impenetrable hard cores. The changes of density of states for gravitons give expression to the fluctuations of gravitational fields and thus contribute to gravitational Casimir energy. We calculate the integrated corrections to the density of states under Dirichlet boundary conditions, by using the Krein type formula. It is found that the contributions to corrections to the density of states and subsequently to the gravitational Casimir energy include both orbital decay in radial direction and the propagations of GWs all over the space, both of which give rise to quantum effects with the same order. However, by considering the extended scales of objects, the gravitational Casimir effects are larger than that obtained by treating them as point particles. The gravitational Casimir energy depends both on the scales of objects and on the separation of the binary, which increases for lager scales of star component in an NS binary system, with typical separation of $R \sim 10^9 m$, i.e., low-frequency gravitational wave sources.

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