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Weibull, κ-Weibull and 3-parameter extended Weibull distributions

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Here we consider a function of κ-statistics, the κ-Weibull distribution. It is compared to the Weibull distribution. We also consider the 3-parameter extended Weibull according to Marshall–Olkin extended distributions. This and the κ-Weibull functions are compared.

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Weibull distribution

The Weibull distribution is a continuous probability distribution. Its probability density function is given by:

$$f(x|\lambda,k)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k} \quad \text{, per} \quad x\geq 0 \qquad (1)$$

If $x<0$, $f(x|\lambda,k)=0$.

In the distribution, $k>0$ is the *shape parameter* and $\lambda>0$ is the *scale parameter*. The Weibull distribution is related to a number of other probability distributions.

If $k=1$, we have the exponential distribution.

The Rayleigh distribution is given by $k=2$. $\lambda=\sqrt{2}\sigma$.

$$f(x|\sigma)=\frac{x}{\sigma^2}e^{-x^2/(2\sigma^2)} \qquad (2)$$

If the quantity x is a "time-to-failure", the Weibull distribution gives a distribution for which the failure rate is proportional to a power of time.

In the formalism of [1], the Weibull probability density function (pdf) is defined as:

$$f(t|B,C,D)=\frac{B}{C}\left(\frac{t-D}{C}\right)^{(B-1)} e^{-\left(\frac{t-D}{C}\right)^{B}} \qquad (3)$$

where $B>0,\ C>0,\ -\infty<D<\infty,\ t>D$.

Symbol t is representing the random variable. The distribution is suitable to analyse time-series, where t is the elapsed time.

Parameter *D* is the threshold, which is therefore representing the minimum value of time.

B is the shape parameter, which controls the overall shape of the probability density function. Its value usually ranges between 0.5 and 8.0 [1]. The Weibull distribution includes other useful distributions [1]. If $B=1$, we have the exponential distribution. For $B=2$, we have the Rayleigh distribution. For $B=2.5$ and $B=3.6$, the Weibull distribution approximates the lognormal distribution and the normal distribution respectively.

The scale parameter *C* changes the scale of the probability density function along the time axis (that is from days to months or from hours to days). It does not change the actual shape of the distribution [1]. Parameter *C* is known as the characteristic life. In [1], it is stressed that "No matter what the shape, 63.2% of the population fails by *t* = *C*+*D* ". It is also told that "Some authors use 1/*C* instead of *C* as the scale parameter ".

Let us put $\alpha=B$, $\gamma=1/C$, $\tau=D$. Eq. (3) becomes:

$$f(t|\alpha,\gamma,\tau)=\alpha\gamma(\gamma\cdot(t-\tau))^{\alpha-1}e^{-(\gamma\cdot(t-\tau))^{\alpha}} \qquad (4)$$

Then, using $\beta=\gamma^{\alpha}$:

$$f(t|\alpha,\gamma,\tau)=\alpha\beta\cdot(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)^{\alpha}} \qquad (5)$$

Then, in the same formalism of Eq.(1):

$$f(x|k,b)=bkx^{k-1}e^{-bx^{k}} \qquad (6)$$

In (6), $x=t-\tau$, $b=\beta$, $k=\alpha$. (6) is the form of the Weibull pdf used for applications in medical statistics and econometrics [2],[3].

κ-Weibull distribution

Let us consider the analogue of Weibull pdf in the κ statistics [4],[5].

The κ-Weibull probability distribution function (pdf) is described by:

$$f_\kappa(x|\alpha,\beta)=\frac{\alpha\,\beta\,x^{\alpha-1}}{\sqrt{1+\kappa^2\beta^2x^{2\alpha}}}\ \exp_\kappa(-\beta\,x^\alpha) \qquad (7)$$

where the κ-exponential is defined in the following manner:

$$\exp_\kappa(u)=\left(\sqrt{1+\kappa^2u^2}+\kappa\,u\right)^{1/\kappa} \qquad (8)$$

Parameters α,β are related to the shape and scale indexes of Weibull distribution, whereas κ is the index of κ-distribution, that is the statistical distribution introduced by G. Kaniadakis, Politecnico di Torino, in [4],[5]. Recently, the use of the distribution has been proposed in epidemiology [6],[7].

In [8], we can find discussed and defined the κ-Weibull. In the formalism of the given reference:

$$f_\kappa=\frac{m}{x_s}\left(\frac{x}{x_s}\right)^{m-1}\ \frac{\exp_\kappa\left(-[x/x_s]^m\right)}{\sqrt{1+\kappa^2(x/x_2)^{2m}}} \qquad (9)$$

In Eq.(9), x is the random variable. In the formalism of [1], with time and threshold, Eq.(9) becomes:

$$f_\kappa(t|B,C,D)=\frac{B}{C}\left(\frac{t-D}{C}\right)^{B-1}\ \frac{\exp_\kappa\{-[(t-D)/C]^B\}}{\sqrt{1+\kappa^2((t-D)/C)^{2B}}} \qquad (10)$$

Let us put $\alpha=B$, $\gamma=1/C$. $\tau=D$, (10) becomes:

$$f_\kappa(t|\alpha,\gamma,\tau)=\alpha\,\gamma\,\gamma^{\alpha-1}(t-\tau)^{\alpha-1}\ \frac{\exp_\kappa\{-\gamma^\alpha(t-\tau)^\alpha\}}{\sqrt{1+\kappa^2\gamma^{2\alpha}(t-\tau)^{2\alpha}}} \qquad (11)$$

Then, using $\beta=\gamma^{\alpha}$:

$$f_k(t|\alpha,\beta,\tau)=\frac{\alpha\beta(t-\tau)^{\alpha-1}}{\sqrt{1+\kappa^2\beta^2(t-\tau)^{2\alpha}}}\ \exp_\kappa(-\beta(t-\tau)^\alpha) \qquad (12)$$

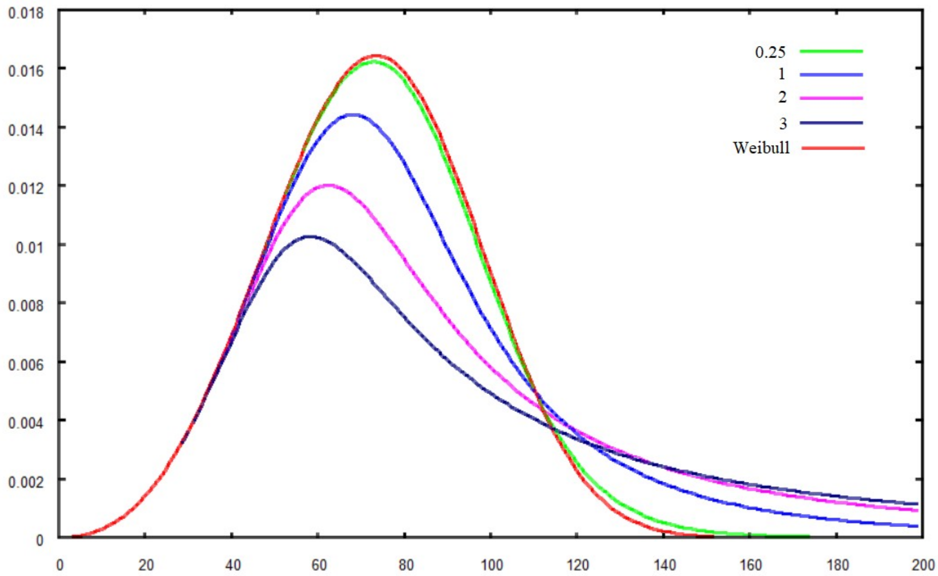


Figure 1 (a) – Comparing Weibull and κ-Weibull. The Weibull pdf is given in red. Parameters are $\alpha=3.5$, $\beta=2.0\times10^{-7}$, and $\tau=0$.

The κ-Weibull curves have different κ values: 0.25, 1, 2 and 3.

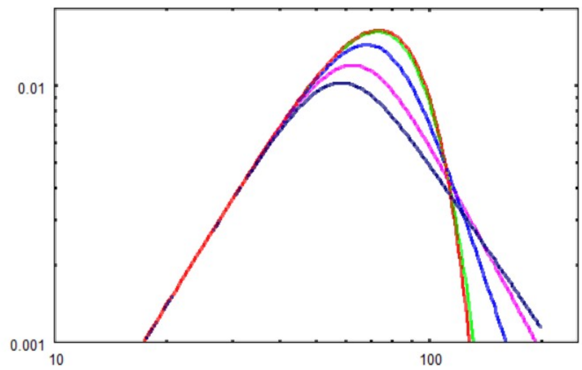


Figure 1 (b) – Comparing Weibull and κ-Weibull in a log-log graph.

Figure 1 shows the comparison of Weibull pdf with that of κ-Weibull. We can see that the value of κ parameter is strongly affecting the tail of the distribution. Increasing the value the tail becomes a "long" tail, that is, a portion of the distribution having many occurrences far from the head of the distribution.

Mixture density

In the case that the distribution is showing two peaks, a mixture of Weibull or κ-Weibull can be considered, in the form:

$$f = f_1 + f_2 = \xi f_{\kappa_1}(t|\alpha_1, \beta_1, \tau_1) + (1-\xi) f_{\kappa_2}(t|\alpha_2, \beta_2, \tau_2) \qquad (13)$$

Parameter ξ, the mixing parameter, is ranging from zero to 1. It is used to generalize the addition of peaks, as proposed for the Weibull distribution [9]. It is also a rough manner to consider the fact that the set of population, involved by pandemic, changed for sure during the considered time period (we will further discuss this point).

In the case that we have three peaks, then (13) becomes:

$$f = f_1 + f_2 + f_3 = \xi_1 f_{\kappa_1}(t|\alpha_1, \beta_1, \tau_1) + \xi_2 f_{\kappa_2}(t|\alpha_2, \beta_2, \tau_2) + \xi_3 f_{\kappa_3}(t|\alpha_3, \beta_3, T\tau_3) \qquad (14)$$

In (14), we must have $\xi_1 + \xi_2 + \xi_3 = 1$.

Being a finite sum, the mixture is known as a finite mixture, and the density is the "mixture density". Usually, "mixture densities" can be used to model a statistical population with subpopulations. Each component is related to a subpopulations, and its weight is proportional to the given subpopulation in the overall population.

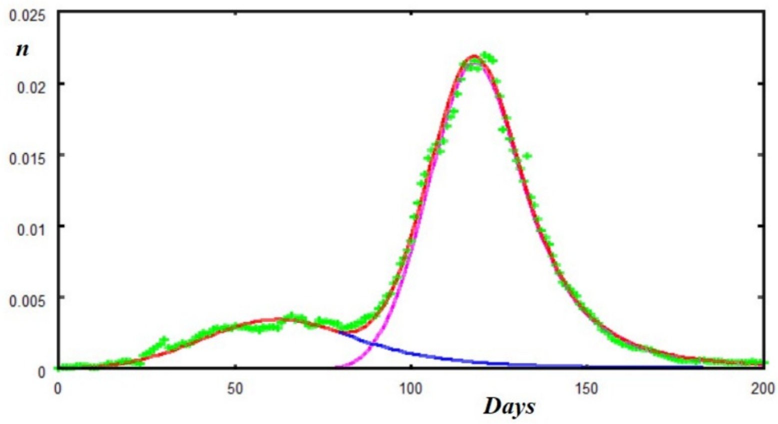


Figure 2 – An example of mixture density (see Ref. [7]).

3-parameter extended Weibull distribution

In [10], we can find an approach, based on Marshall–Olkin extended distributions [11], to the Weibull distribution. In Ref. [10], the 2-parameter Weibull appears as:

$$f(x|\beta,\lambda)=\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}} \quad , \quad x\geq 0 \qquad (15)$$

The 3-parameters extended distribution is given as:

$$f(x|\alpha,\beta,\lambda)=\frac{\alpha\beta\lambda(\lambda x)^{\beta-1}\, e^{-(\lambda x)^{\beta}}}{\left[1-\tilde{\alpha}\, e^{-(\lambda x)^{\beta}}\right]^{2}} \qquad (16)$$

In (16), $x>0$, $\alpha,\beta,\lambda>0$, $\tilde{\alpha}=1-\alpha$.

Let us compare to κ-Weibull. Here we rewrite Eq.(7) in the same formalism as (16):

$$f_{\kappa}(x|\beta,\lambda)=\frac{\beta\lambda(\lambda x)^{\beta-1}}{\sqrt{1+\kappa^{2}\lambda^{2\beta}x^{2\beta}}}\ \exp_{\kappa}(-\lambda^{\beta}x^{\beta}) \qquad (17).$$

Let us compare (16) and (17). In the following figure, $\xi=\lambda x$.

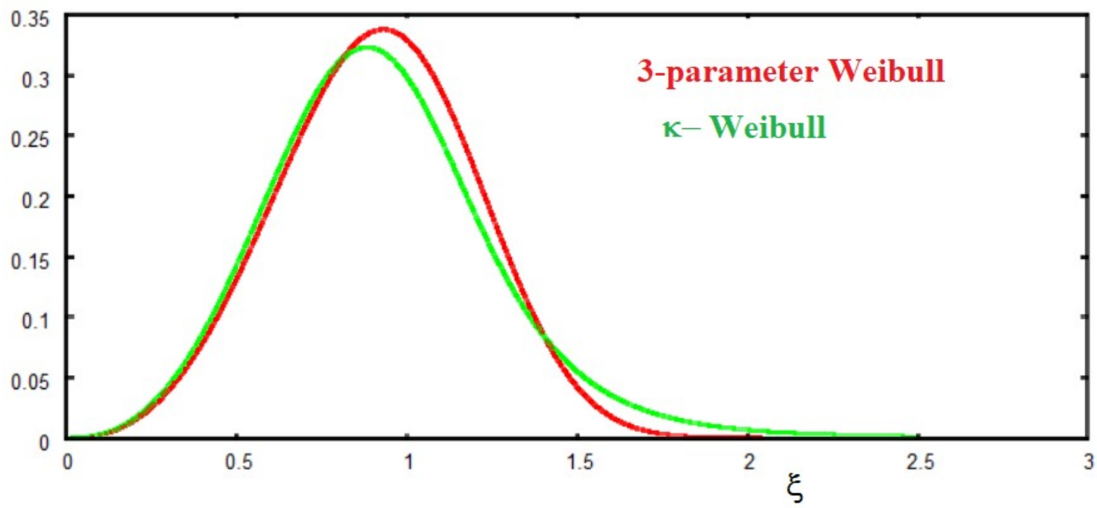


Figure 3: Comparing functions (16) and (17). Parameters used for the calculation: $\beta=3.5$, $\lambda=0.25$, $\kappa=0.5$, $\alpha=1.1$.

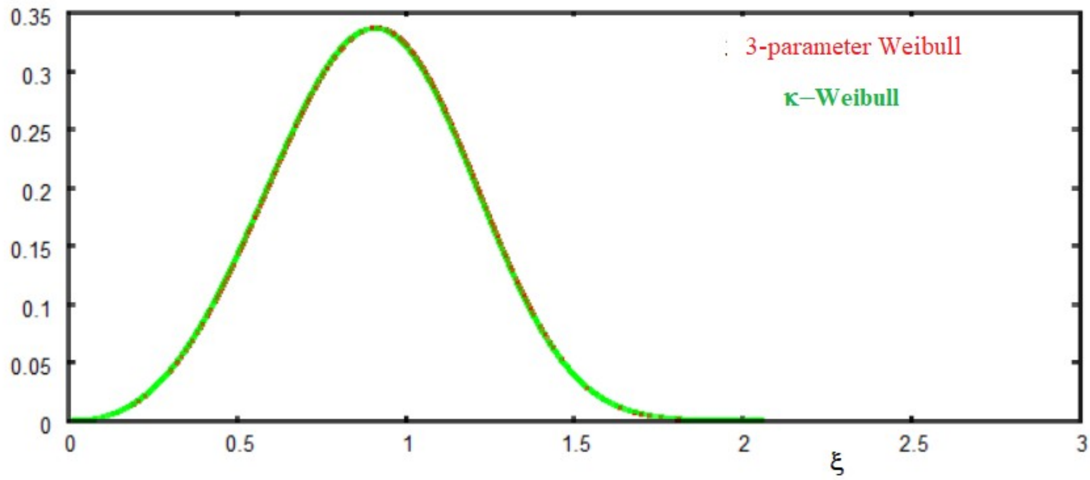


Fig. 4: In the case we change parameter κ in $\kappa = 0.05$, with the same other parameters ($\beta = 3.5$, $\lambda = 0.25$, $\alpha = 1.01$), the curves are indistinguishable.

We can note again the role of parameter κ in determining the tail of the function.

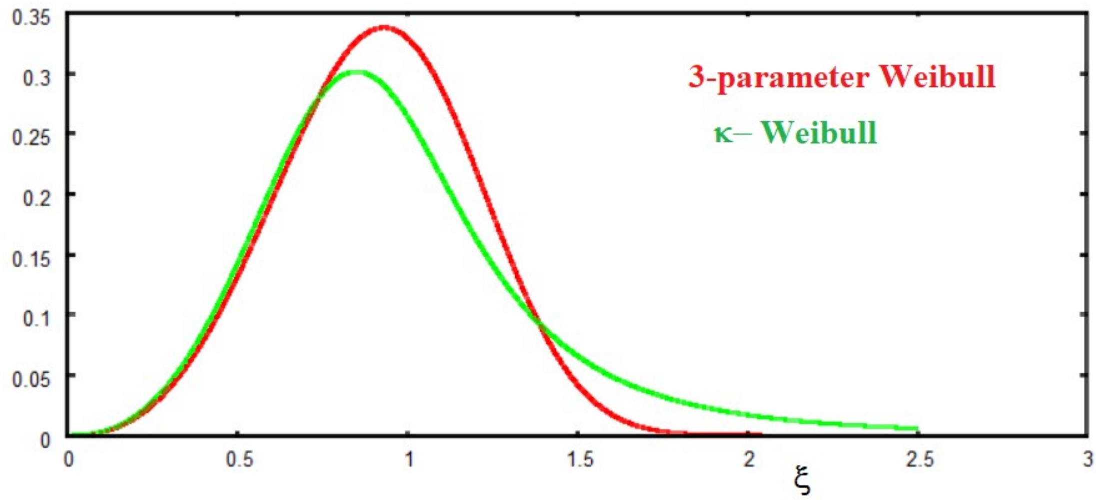


Fig. 5 : In the plot given above, parameters are $\beta = 3.5$, $\lambda = 0.25$, $\kappa = 0.9$, $\alpha = 1.1$.

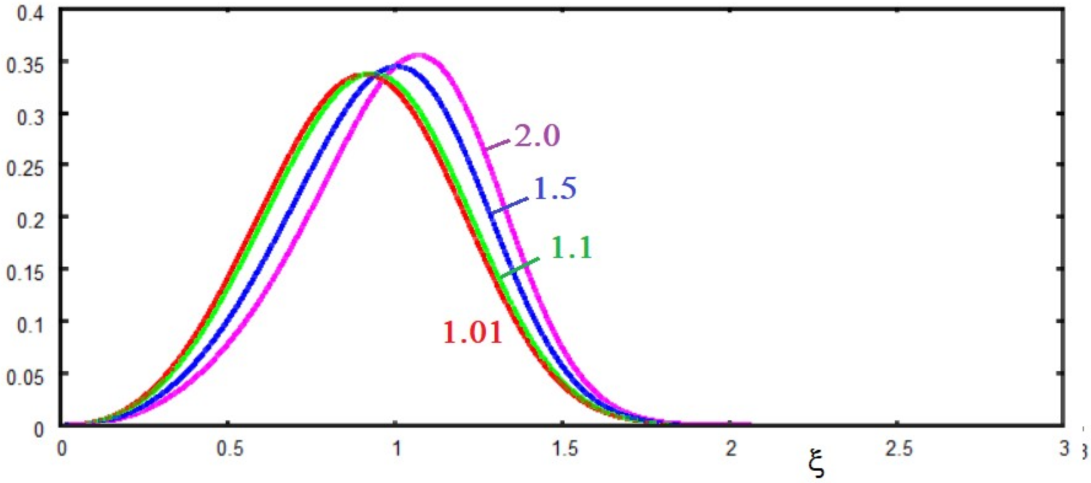


Fig. 6 : In the plot given above, the 3-parameter Weibull is given for parameters $\beta=3.5$, $\lambda=0.25$. Values of α . are 1.01 (red), 1.1 (green=, 1.5 (blue) and 2.0 (violet).

Cumulative κ-Weibull

In the following Figure, and in the formalism of (17), the cumulative function of κ-Weibull, for three different values of parameter κ.

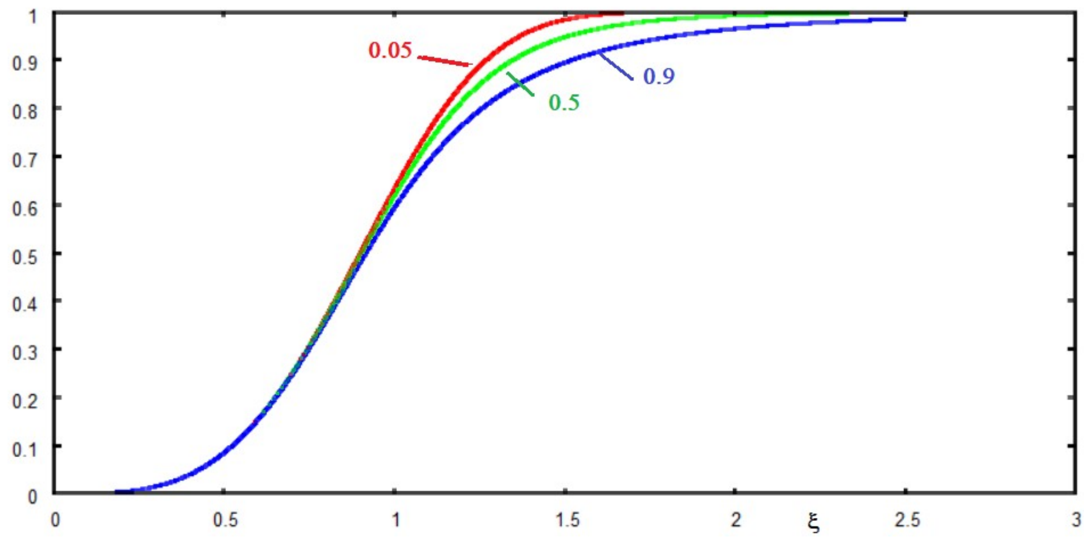


Fig. 7 : Cumulative function of κ-Weibull for parameters $\beta=3.5$, $\lambda=0.25$. Numbers in the image are referring to the values of parameter κ.

Reliability Function (Weibull)

In the formalism of [1], we have seen before that the Weibull pdf is:

$$f(t|B,C,D)=\frac{B}{C}\left(\frac{t-D}{C}\right)^{(B-1)}e^{-\left(\frac{t-D}{C}\right)^{B}}$$

where $B>0,\ C>0,\ -\infty<D<\infty,\ t>D$.

The reliability (or survivorship) function, $R(t)$, is giving the probability of surviving beyond the time t .

For the Weibull pdf, we have:

$$R(t)=e^{-\left(\frac{t-D}{C}\right)^{B}} \qquad (19)$$

The reliability function is one minus the cumulative distribution function.

That is:

$$R(t)=1-F(t) \qquad (20)$$

where $F(t|B,C,D)=\int_{-\infty}^{t} f(t'|B,C,D)\,dt'$.

Reliability Function (κ-Weibull)

Defining the function as in (20) and using the formalism of Ee. (17), we have that the reliability of κ-Weibull is depending on κ parameter as in the following figure.

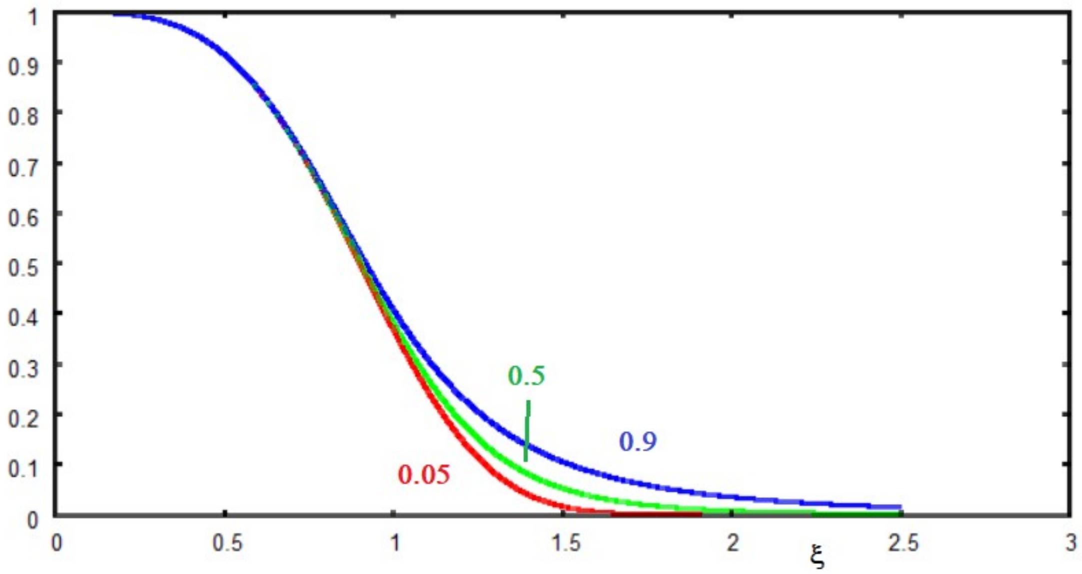


Fig. 8: Reliability function of κ-Weibull for parameters $\beta = 3.5$, $\lambda = 0.25$. Numbers in the image are referring to the values of parameter κ.

Hazards in reliability analysis

The hazard function is a conditional failure rate. In fact, it is conditional because an organism or device had actually survived until time t . In this manner, the function at year 10 only applies to organisms or devices (items) who were actually alive in year 10. It doesn't count those who died or failed in previous periods.

The hazard function are frequently associated with products and applications.

Usually, the hazard functions are featured as increasing, constant and decreasing function. An increasing hazard function indicates that items are more likely to fail with time. For instance, mechanical items subjected to stress or fatigue have an increased risk of failure over the lifetime of the product. A decreasing hazard function indicates failures that are more likely to occur early in the life of an item. An example is errors in a computer program; they are more likely near the release of a new software program, decreasing as time passes with improved releases. A constant hazard function indicates failures that are equally likely to occur at any time in the item's life.

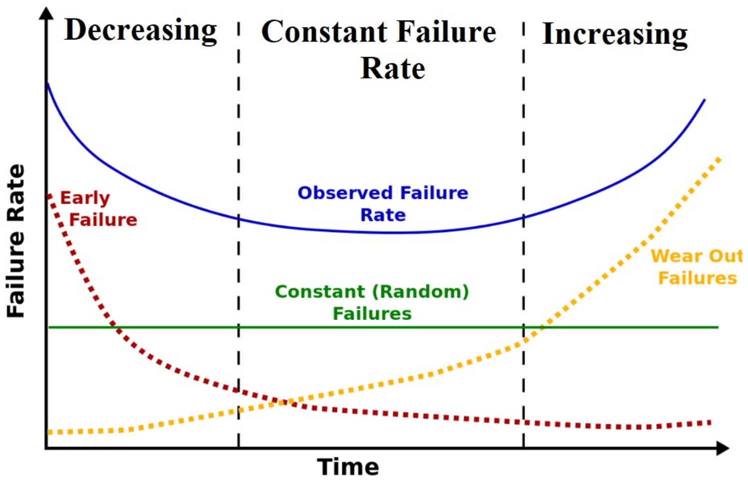
Products exist having failure rates that follow a "bathtub" curve. The name of the curve is derived from the cross-sectional shape of a bathtub: steep sides and a flat bottom. These items have hazard rate which is high initially and low in the centre. Then the hazard is high again at the end of item's life. For this reason, a bathtub curve is widely used in reliability engineering and in the models of deterioration (Wikipedia)

In particular, a bathtub hazard function which comprises three parts:

1) The first part is a decreasing failure rate, known as early failures.

2) The second part is a constant failure rate, known as random failures.

3) The third part is an increasing failure rate, known as wear-out failures.



Courtesy: Wikipedia, McSush

Many electronic consumer product life cycles strongly exhibit the bathtub curve [12]. In reliability engineering, the cumulative distribution function corresponding to a bathtub curve may be analysed using a Weibull chart [12].

Hazard Function (Weibull)

The hazard function represents the instantaneous failure rate. The rate is given by the function:

$$h(t)=\frac{f(t)}{R(t)}=\frac{B}{C}\left(\frac{t-D}{C}\right)^{B-1} \qquad (21)$$

In the following figure, it is shown the behaviour of function $(\lambda x)^{(\beta-1)}$, for three different values of β. As discussed in [13], it is helpful to visualise the differences between values of β in the hazard function by using a "bathtub" diagram shown that given in the Figure 2 of [13]. Hazard function for $\beta<1$ we have a "likely to fall at the start". When $\beta \sim 1$. "failure rate is fairly constant". When $\beta>1$, the "failure rate increases as the time goes by".

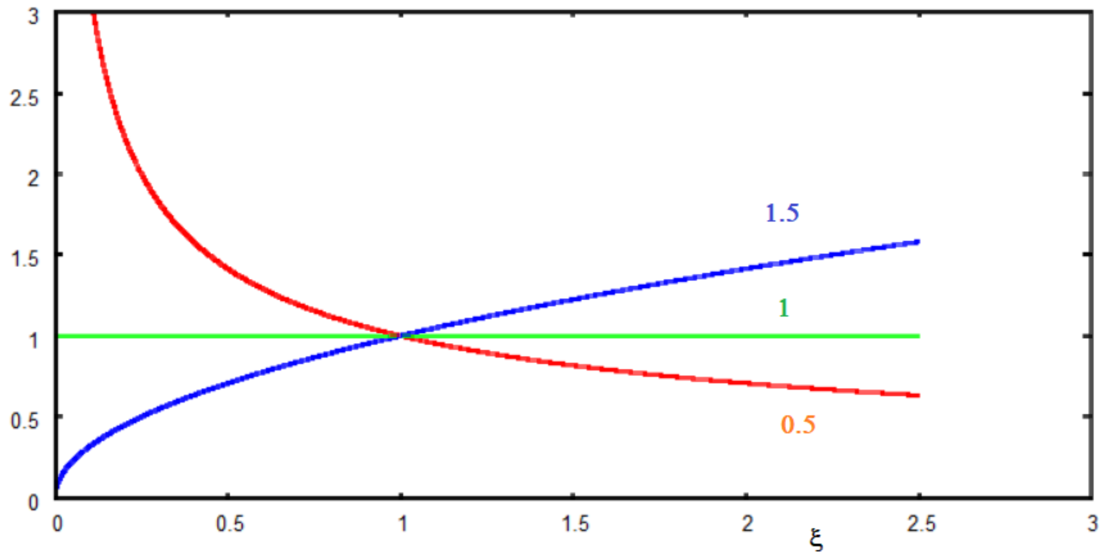


Fig. 9: Function $(\lambda x)^{(\beta-1)}$ for parameter $\lambda=0.25$. Numbers in the image are referring to the values of parameter β.

Hazard Function (κ-Weibull)

The hazard function of the κ-Weibull, given by: $h_\kappa(x|\lambda,\beta)=\dfrac{f_\kappa(x|\lambda,\beta)}{R_\kappa}$

In the following figure, the hazard function is given for three different values of parameter κ. The formalism is the same of the Figures 3 to 7.

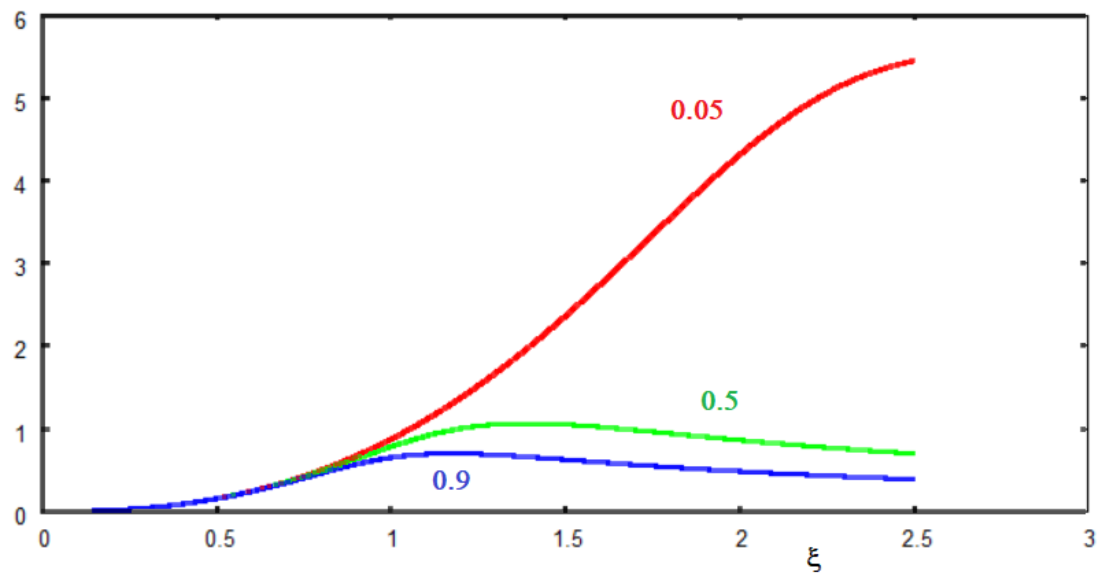


Fig. 10: Hazard function of κ-Weibull for parameters $\beta=3.5$, $\lambda=0.25$. Numbers in the image are referring to the values of parameter κ.

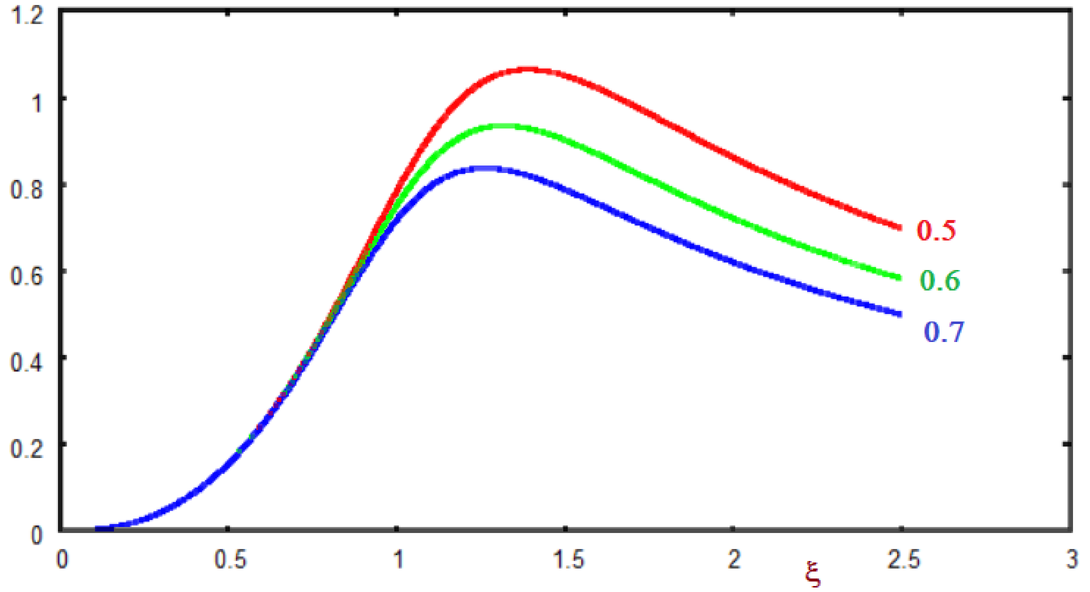


Fig. 11: Hazard function of κ-Weibull for parameters $\beta=3.5$, $\lambda=0.25$. Numbers in the image are referring to the values of parameter κ.

In [6], this function is proposed as the following function:

$$f_\kappa(t|\alpha,\beta)=\frac{\alpha\,\beta\,t^{\alpha-1}}{\sqrt{1+\kappa^2\beta^2t^{2\alpha}}}\ \exp_\kappa(-\beta\,t^\alpha)$$

Hazard function in [6] is proposed as:

$$h_\kappa(t|\alpha,\beta)=\frac{\alpha\,\beta\,t^{\alpha-1}}{\sqrt{1+\kappa^2\beta^2t^{2\alpha}}} \qquad (22)$$

Eq. (22) in the formalism of (17) becomes:

$$h_\kappa(t|\beta,\lambda)=\frac{\beta\,\lambda(\lambda\,x)^{\beta-1}}{\sqrt{1+\kappa^2(\lambda\,x)^{2\alpha}}} \qquad (23)$$

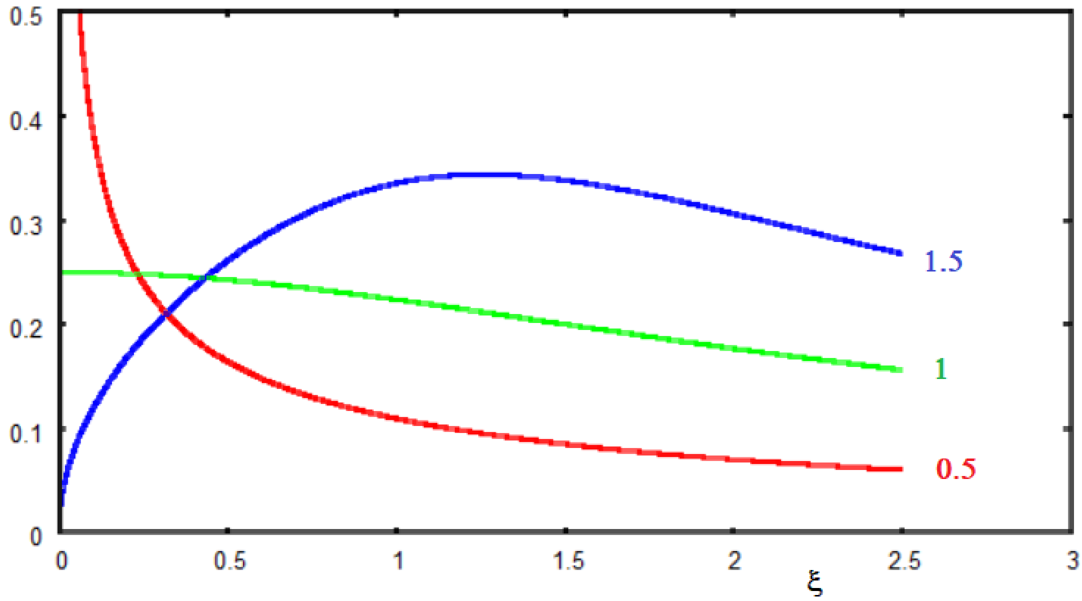
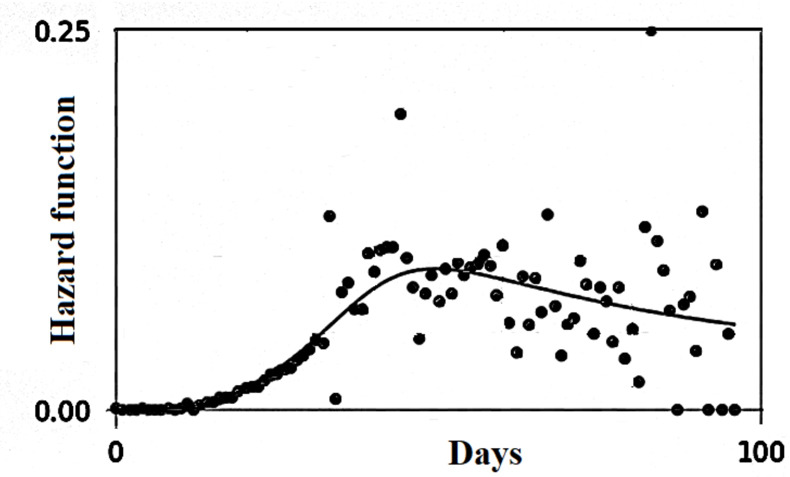


Fig. 12: Hazard function of κ-Weibull for parameters $\kappa=0.5$, $\lambda=0.25$. Numbers in the image are referring to the values of parameter β .

In the Figure 12, it is given the hazard function of κ-Weibull. Parameter κ has a value $\kappa=0.5$. As in the previous figures, $\lambda=0.25$. Numbers in the image are referring to the values of parameter β . We can see that, due to the role of κ parameter in the tail of the distribution, the behaviour is different from that given in the Figure 8 for the Weibull distribution. Then, also the “bathtub” diagram of [12],[13] is modified for the κ-Weibull.

In the following Figure, a real case is proposed as that in the Figure 2 of Ref. [6].



In that figure, among other functions, the theoretical continuous curve and empirical dots are plotted for the κ-Weibull hazard function versus time (see [6] for parameters). The data are those concerning Covid-19 in China.

The hazard function is unimodal with a large tail, as in the Figure 11.

Let us consider Ref. [14]. “A Weibull distribution allows a monotonic (either continuously increasing or decreasing hazard) and a log-logistic distribution allows either a monotonic or a unimodal hazard function.”.

Fig. 11 shows unimodal behaviours for κ-Weibull. For this reason, a further comparison to log-logistic function is interesting.

Applications

A plot model for Weibull

The cumulative distribution function is:

$$F(t)=1-e^{-\left(\frac{t-D}{C}\right)^B}$$

Let us assume $D=0$:

$$\ln(1-F(t))=-(t/C)^B \quad \text{then:} \quad \ln(-\ln(1-F(t)))=-B\ln C+B\ln(t)$$

Let us introduce: $y=\ln(-\ln(1-F(t)))$, $x=\ln t$, we have:

$$y=-B\ln C+Bx$$

In this manner, the plot is that of a straight line.

Weibull distribution for pseudorandom numbers

Routine RNWIB can be used to generate pseudorandom numbers, starting from a Weibull distribution, with shape parameter A and unit scale parameter, so that:

$$f(x)=A x^{A-1} e^{-x^A} \quad , \quad x\geq 0 \qquad (18)$$

https://help.imsl.com/fortran/6.0/stat/default.htm?turl=rnwib.htm

References

[1] NCSS Statistical Software NCSS.com, Chapter 550. Distribution (Weibull) Fitting. https://ncss-wpengine.netdna-ssl.com/wp-content/themes/ncss/pdf/Procedures/NCSS/Distribution-Weibull-Fitting.pdf

[2] Collett, D. (2015). Modelling survival data in medical research (3rd ed.). Boca Raton: Chapman and Hall / CRC. ISBN 978-1439856789.

[3] Cameron, A. C., & Trivedi, P. K. (2005). Microeconometrics: methods and applications. p. 584. ISBN 978-0-521-84805-3.

[4] Kaniadakis, G. (2002). Statistical mechanics in the context of special relativity. Physical review E, 66(5), 056125.

[5] Kaniadakis, G. (2001). Non-linear kinetics underlying generalized statistics. Physica A: Statistical mechanics and its applications, 296(3-4), 405-425.

[6] Kaniadakis, G., Baldi, M. M., Deisboeck, T. S., Grisolia, G., Hristopulos, D. T., Scarfone, A. M., Sparavigna, A. C., Wada, T. and Lucia, U., 2020. The κ-statistics approach to epidemiology. Scientific Reports, 10(1), pp.1-14. https://www.nature.com/articles/s41598-020-76673-3

[7] Sparavigna, A. C. (2021). A decomposition of waves in time series of data related to Covid-19, applied to study the role of Alpha variant in the spread of infection. Zenodo. https://doi.org/10.5281/zenodo.5745008

[8] Hristopulos, D. T., Petrakis, M. P., and Kaniadakis, G. (2015). Weakest-link scaling and extreme events in finite-sized systems. Entropy, 17(3), 1103-1122.

[9] Razali, A. M., & Salih, A. A. (2009). Combining two Weibull distributions using a mixing parameter. European Journal of Scientific Research, 31(2), 296-305.

[10] Caroni, C. (2010). Testing for the Marshall–Olkin extended form of the Weibull distribution. Statistical Papers, 51(2), 325-336.

[11] Marshall, A. W., and Olkin, I. (1997) A new method of adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika 84, 641–652.

[12] Lienig, J., and Bruemmer, H. (2017). Fundamentals of Electronic Systems Design. Springer International Publishing. p. 54. doi:10.1007/978-3-319-55840-0. ISBN 978-3-319-55839-4.

[13] Kızılersü, A., Kreer, M. and Thomas, A.W. (2018), The Weibull distribution. Significance, 15, 10-11. https://doi.org/10.1111/j.1740-9713.2018.01123.x

[14] Christiana Kartsonaki, C. (2016). Survival analysis, Diagnostic Histopathology, Volume 22, Issue 7, Pages 263-270, ISSN 1756-2317, DOI

https://doi.org/10.1016/j.mpdhp.2016.06.005.