Extrait of a [second] memoir on double refraction

by Augustin Fresnel

(first published 1822, revised 1825)

with analytical table of contents
by the editors of Fresnel's *Oeuvres complètes*
1866–70

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English translation of A. Fresnel, "Extrait d'un Mémoire sur la double réfraction", *Annales de Chimie et de Physique*, Ser. 2, vol. 28, pp. 263–79 (1825), as reprinted in *Oeuvres complètes d'Augustin Fresnel*, vol. 2 (1868), pp. 465–78, with the corresponding extract from the "Table Analytique" in *Oeuvres complètes...*, vol. 3 (1870), at pp. 659–62.

Translator's abstract

The author's earlier explanation of the double refraction of uniaxial crystals (such as calcite), based on the hypotheses that light consists of transverse waves and that a birefringent medium has different elasticities in different directions, led to the prediction that in a biaxial crystal (such as topaz), there should be no true *ordinary* ray—that is, no ray having a speed independent of direction. This has been confirmed by two experiments, one using interference and the other using refraction.

The optical properties of a uniaxial or biaxial crystal are determined by the lengths of three perpendicular *axes of elasticity*, whose directions are those in which the restoring force is parallel to the displacement, and whose lengths are the principal axes of the *surface of elasticity*, of which each radius vector gives, by its length, the speed of propagation of vibrations parallel thereto. In any plane wavefront, the permitted directions of vibration, into which any other vibration is resolved, are the directions of the longest and shortest radius vectors, these being the directions in which the reaction has a component parallel to the vibration and (at most) another component normal to the wavefront.

If the surface of elasticity has three unequal principal semi-axes, two planes through the center cut the surface in circles. The directions perpendicular to these planes, being the directions of a single wave speed, could be called the *optical axes*. The angle between these axes varies due to dispersion, if the principal semi-axes vary in different proportions.

For an object point so far away that the incident wavefront can be assumed plane, the ordinary and extraordinary images can be located if we know the deviations of the respective wavefronts. For a closer object point, however, we need to know the equation of the *wave surface*, which takes the place of the secondary wavefronts in Huygens' construction. This surface must be tangential to every plane wavefront that travels from the origin in the same time; the equation of the surface that satisfies this condition is given. If two semi-axes of elasticity are equal, the equation of the wave surface can be factored into the equation of a sphere and that of a spheroid; this is the case of uniaxial crystals.

The general wave surface can be constructed from the diametral sections of an ellipsoid having the same principal semi-axes as the surface of elasticity. The directions perpendicular to the two circular sections of this ellipsoid, being different from the corresponding directions for the surface of elasticity, are alternative candidates for the term *optical axes*. The ellipsoid construction leads to the confirmation of Biot's sine-product law (with Biot's ray speeds replaced by their reciprocals), and the approximate confirmation of Biot's dihedral law for the planes of polarization.

The full memoir explains why the refraction of a homogeneous medium is never more than double, and why there cannot be more than two optical axes. However, the theory does not cover the rotation of the plane of polarization in quartz, which seems to imply that the homologous faces of its molecules are not all parallel.

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Translator's preface

On 21 September 1821, Augustin Fresnel wrote a letter to François Arago [7], reporting experimental confirmation of Fresnel's prediction that in the double refraction of topaz (a biaxial crystal), *neither* internal ray was "ordinary" in the sense of having a speed independent of direction. Fresnel accordingly pressed on with what turned out to be his most difficult investigation. He did not get it right on the first attempt, and—with his poor health and competing commitments—did not manage to rectify all the errors and omissions before his untimely death in 1827. Even so, within a decade of his death, leading authorities declared that his study of double refraction "would do honour to the sagacity of Newton" [17, p. 78], and "will. . . be regarded as the finest generalization in physical science which has been made since the discovery of universal gravitation" [16, p. 382]. More recently, Jed Z. Buchwald has called it "Fresnel's greatest and ultimately most influential accomplishment" [4, p. 260].

Modern readers, in assessing this work, need to remember that it was not only the first wave theory of birefringence to account for polarization or for biaxiality, but also the first attempt at a theory of shear waves in anisotropic elastic media. Its shortcomings in the latter capacity were ironically beneficial in the former: when Cauchy, inspired by Fresnel, developed a rigorous theory of elastic waves in anisotropic media, it was not only more complicated, but also harder to reconcile with the requirement of purely transverse waves [5, pp. 228–9], which Fresnel had treated essentially as an independent constraint.

Fresnel submitted what we now call his "First Memoir" on double refraction [12, vol. 2, pp. 261–308] on 19 November 1821. Within a week he realized that his attempted construction of the wave speeds and polarizations from a single ellipsoid did not properly account for the distinction between rays and wave-normals. In a corrected *Extrait* [12, vol. 2, pp. 309–329], read to the Academy of Sciences on 26 November, the ellipsoid was replaced by a 4th-degree ovaloid, later named the *surface of elasticity*. Subsequent documents, or at least the printed versions thereof, refer to the "Memoir" as having been "presented" on 26 November (as in the first "Supplement to the Memoir") or "read" on 26 November (as in the "Second Supplement"), and some corrections to the first manuscript were indeed witnessed on that date [12, vol. 2, p. 261]. The first "Supplement" [12, vol. 2, pp. 343–67], signed on 13 January and submitted on 22 January 1822, formally defined the *surface of elasticity* and gave its equation, derived by assuming three perpendicular "axes of elasticity". This Supplement was accompanied by an *Extrait* [8], which was read on 13 January 1822 according to the editors of the *Oeuvres complètes d'Augustin Fresnel* [12, vol. 2, p. 335 & vol. 3, p. 645], although the *Procès-verbaux* of the Academy [1] do not record any meeting for that date and do not record any such reading at the meetings of 7 and 14 January. A "Second Supplement" [12, vol. 2, pp. 369–442], signed on 31 March and submitted the next day,[1] confirmed the existence of the perpendicular axes, and salvaged the original ellipsoid in connection with the *ray* speeds, with the implication that Biot's sine-product law was exact after all.[2]

The patchwork of corrections and additions obviously required a new, consolidated *Extrait*, which duly appeared [9] in the May 1822 issue of the *Bulletin* of the Philomathic Society, under the French title *Extrait d'un Mémoire sur la double réfraction*, and with the printed marginal note

Acad. des Sciences.
26 novembre 1821.

That *Extrait*, in spite of its title, is patently not the one read to the Academy on 26 November. But if the "Memoir" presented or read or excerpted on that date is taken to include the supplementary material, then the new *Extrait* may be fairly described as an updated summary of the composite memoir, compiled immediately after it. This, I submit, is the intended meaning of the date.

An expanded and slightly updated version of that *Extrait*, with the same title, appeared in the March 1825 issue of Arago's *Annales de Chimie et de Physique* (Ser. 2, vol. 28, pp. 263–79). This version, as

[1] In the first footnote to the "Second Memoir" [11], Fresnel refers to this supplement as having been "presented" on 22 April. I have not been able to find any other source for that date.

[2] In the *Oeuvres complètes*, the Second Supplement is followed by a "Note on the agreement of the experiments of MM. Biot and Brewster with the law of speeds given by the ellipsoid" [12, vol. 2, pp. 443–58], signed on 27 May 1822.

reprinted in Fresnel's *Oeuvres complètes* in 1868 [12, vol. 2, pp. 465–78], is the source text of the present translation. The first page of the 1825 printing bears a footnote, presumably by Arago, saying

> (1) We have already given an account of the experimental part of this work in a report made to the Academy of Sciences, and published, in 1822, in volume XX of these *Annales*, page 337.

The second part of footnote (1) in the 1868 reprint [12, vol. 2, p. 465] makes a similar statement, and refers to the preceding reprint of the report [12, vol. 2, pp. 459–64]. The *first* part of that footnote says:

> (1) This *extrait* had already been published in 1822 in the Bulletin de la Société philomatique: only slight changes in its editing have been made here (a).

A note to the note, by H. de Sénarmont, who died six years before publication, adds:

> (a) These changes, purely of form, are unnecessary to point out.

If the comparison is between the 1825 and 1868 printings, the only differences that I have noticed are reformatting a citation, paraphrasing Arago's footnote (already mentioned), and reverting the spelling of "Herschel" to the 1822 version. However, there are bigger differences between the 1822 and 1825 versions, and I do indeed point out some of them in footnotes. I ignore the numerous places where Fresnel refers to himself in the third person in the former version, and the first person in the latter. But I cannot ignore the three new paragraphs appended to the 1825 version, which contain some philosophical reflections found in the later Second Memoir but not in earlier papers. A much smaller addition, concerning the placement of the proof that there are three perpendicular axes of elasticity, matches the Second Memoir rather than the First. Between the two versions of the *Extrait*, Fresnel apparently derived a formula supporting his longstanding opinion (pre-dating the First Memoir) that dispersion could be explained by supposing that the range of action of the elastic forces is not negligible compared with the wavelength. In the 1825 version we find the poignant statement that the explanation is given "in my Memoir": the Second Memoir refers to an attachment containing this explanation—which, however, did not appear in the posthumous printing. A phrase linking the ray form of Biot's dihedral law to the "equation of elasticity" is included in the 1822 version but removed from the 1825 version, with the result that the law is linked to the ellipsoid. This suggests that Fresnel, even by 1825, was not sure that the wave-normal form of the law, derived from the surface of elasticity, is exact—as he showed in the Second Memoir. In summary, the 1822 version does not seem to cover any material unique to the Second Memoir, and the 1825 version covers only some of that material: it offers some philosophical reflections, mentions one organizational feature, and makes a promise that the author apparently did not live to keep, but it does *not* include the main technical advance of the Second Memoir, namely an exact form of the dihedral law.

The subtitle of the 1825 version says "(Read to the Academy of Sciences on 26 November 1821)", which fails to acknowledge even a weak connection with the Second Memoir. Two points, however, can be made in its defense. First, the subtitle essentially reproduces the marginal note from the 1822 version. Second, one may infer from the above that the meaning of the French *extrait* is more rubbery than that of the English cognate "extract": an *extrait* is apparently allowed to include corrections and updates.

Against this background, let us return to the annotations on the first page of the 1868 edition, which conclude with the following note by E. Verdet, who died two years before publication:

> The *Annales de Chimie et de Physique* [meaning the "1825 version"] indicate that the Memoir whose *extrait* we are about to read is that which Fresnel had presented to the Institute on 26 November 1821; but the least attentive comparison of these two pieces suffices to show that the indication is erroneous. This *extrait* contains the definitive summary of Fresnel's last ideas on double refraction, and therefore can only refer to the Memoir printed in volume VII of the *Recueil de l'Académie des sciences*, No. XLVII of this edition [that is, the Second Memoir].

The "least attentive comparison" would indeed show that the present *Extrait* is not the one read on 26 November 1821, and that the memoir reported in the present *Extrait* surpasses what had been submitted by that date. But, as I hope I have shown, it takes considerably more attentiveness to establish how the present *Extrait* points beyond the First Memoir *with its Supplements*: the 1822 version does not seem to point beyond it at all, and the 1825 version still has not caught up with the Second Memoir.

Accordingly, whereas the title of the 1822 and 1825 versions may be translated "*Extrait* of a memoir on double refraction", and whereas the 1868 reprint, following Verdet's opinion, changes the words "a memoir" to "the second memoir", I have compromised by changing them to "a [second] memoir", where the square brackets acknowledge the editorial addition. And in the current version of this translation, I have belatedly decided not to translate the word *extrait*.

The French text bears only one footnote by Fresnel himself, marked in the translation by its original number (1) in addition to its sequential number. Other footnotes are mine; some are signed "*Translator*" for emphasis. Items in *square brackets*, including citations such as "[9, p.64]", are also mine. Unusually, the editors of the *Oeuvres complètes* have not added section numbers. Whereas all cited authors have the title "M." in the French text, I have given them their usual English titles.

— *Translator*.

Analytical Table of Contents

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It had been assumed until now that in all crystals which divide light into two beams, one of these beams always follows the laws of ordinary refraction. The experiments of Huygens, Wollaston, and Malus having demonstrated this principle for calcareous spar [calcite], it had been extended by analogy to all other substances endowed with double refraction. The mechanical considerations by means of which I managed to explain it for crystals of one axis, which I have set forth in these *Annales* (vol. XVII, p. 179 and following),[3] made me sense that the same principle was no longer applicable to crystals with two axes,[4] and that in the latter, neither of the two beams should follow the laws of ordinary refraction, or, in other words, that the rays called *ordinary* should themselves suffer variations in speed analogous to those of the extraordinary rays; this is also what experiment has confirmed.

The wave theory indicated these variations to me in no vague way: it gave me the means of calculating their extent according to the elements of the double refraction of the crystal, namely its degree of energy and the angle of the two axes. I had done this calculation in advance for limpid topaz, according to the data drawn from the observations of Mr. Biot; and the experiment agreed satisfactorily with the calculation, or at least the small difference that I observed can be attributed to some inaccuracy in the sections of the crystal or the direction of the rays, and perhaps also to some slight difference in optical properties between my topaz and that of Mr. Biot.

To measure the variations in speed of the ordinary rays, I have successively employed the method of interference furnished by diffraction, and the procedure that Mr. Biot followed in his researches on double refraction. In order to compare more easily, by both methods, the progress of the rays that crossed the two plates or the two prisms taken from the same crystal, I had employed together the two plates glued edge-to-edge, like the two prisms, so that in each apparatus the faces of the two contiguous parts were exactly on the same plane, as had been verified by reflection, and by means of the colored rings generated on the surface of the two crystals due to contact with a slightly convex glass. Each apparatus was then lightly pressed between two flat glasses coated with a thin layer of turpentine [French: *térébenthine*], which completed the polish and served at the same time to compensate almost exactly for the small defect of continuity of the two contiguous surfaces. The flat glasses glued to the topaz prisms were themselves prismatic, and each of them presented, in the opposite sense to the angle of the crystal, an angle equal to half of that, so as to achromatize it. In the apparatus consisting of two topaz plates, these glasses were plates with parallel faces.[5]

To obtain the greatest difference in refraction between the ordinary beams, both must be perpendicular to the line that bisects the acute angle of the two axes, one of the beams being parallel and the other perpendicular to the plane of the axes. It should be noted that in the same circumstance, the extraordinary rays on the contrary maintain a constant speed, in accordance with the theory. Thus when the beam of light, remaining perpendicular to the intermediate axis [*l'axe moyen*],[6] turns around this axis, the speed of the extraordinary rays remains constant, and that of the ordinary rays suffers the greatest variations to which it is susceptible; and conversely, when the beam of light rotates around the line that bisects the obtuse angle of the two axes, while remaining perpendicular to this line, the ordinary rays maintain the same speed, and the extraordinary refraction passes from the *maximum* to the *minimum*.

The theoretical ideas that led me to this discovery rest on the hypothesis that the vibrations of light[7] are executed only in directions parallel to the wavefront[8]. In the note already cited, where I presented this hypothesis with some development, I showed that in order to conceive of the absence of longitudinal vibrations, it was enough to suppose that the aether had a sufficiently high resistance to compression. According to this supposition on the nature of the luminous vibrations, polarized light is

[3] The cited paper is now available in English: see [6], §§ 10–14.

[4] These *axes* are the directions in which light travels with a speed independent of polarization (and depend on whether that single speed is taken as the ray speed or the wave-normal speed).

[5] The 1822 version of this paragraph [9, p. 64] lacks the last sentence and does not specify the cause of the colored rings.

[6] Here the context implies that "the intermediate axis" (*l'axe moyen*) means the bisector of the acute angle of the aforesaid two axes. But compare footnote 23 below.

[7] In modern terms, Fresnel's "vibrations" are those of the electric displacement vector **D**.

[8] Literally "surface of the waves"—mixing singular and plural.

that in which the transverse oscillations are executed constantly in the same direction, and ordinary light is the union and the rapid succession of an infinitude of wavetrains[9] polarized in all directions. The act of polarization consists not in creating these transverse vibrations, but in decomposing them into two constant perpendicular directions, and in separating the two wavetrains thus produced, either by their difference in speed alone, as in crystallized plates, or also by a difference in the inclination of the waves and the rays, as in crystals cut as prisms or in thick plates of carbonate of lime [calcite]; for wherever there is a difference in speed between the rays, refraction can make them diverge. Finally, according to the same theory, the plane of polarization is the plane perpendicular to which the transverse vibrations are executed.[10]

That being said, I consider a medium endowed with double refraction as presenting different elasticities in the various directions; and here by *elasticity* I mean the greater or lesser force with which the displacement of one slice of the vibrating medium induces the displacement of the next slice. I always suppose that these slices neither approach nor move away from each other, but only slide in their respective planes, and by a very small distance relative to that which separates two consecutive molecules of the aether.

When light passes through a transparent body, do the molecules belonging to this body participate in the luminous vibrations, or are the vibrations propagated only by the aether contained in the body? This is a question that is not yet resolved. But even if the aether were the only vehicle of light waves, the hypothesis just stated would be admissible because a particular arrangement of the molecules of the body can modify the elasticity of the aether—that is, modify the mutual dependence of its consecutive layers, so that its elasticity does not have the same energy in all directions. Thus, without trying to discover whether the whole refractive medium or only a portion of this medium participates in the luminous vibrations, we shall consider only the vibrating part, whatever it may be; and the mutual dependence of its molecules is what we shall call the *elasticity of the medium*.

When we displace a molecule in an elastic medium, the resultant of the forces which tend to restore it to its first position is not generally parallel to the direction in which it has been displaced: such parallelism requires that the resultants of the forces which push this molecule from the right and from the left, in each azimuth, have the same intensity. The directions for which this condition is fulfilled—that is, along which the molecule is pushed back in the very direction of its displacement—are what I call the *axes of elasticity* of the medium, and what I consider to be the *true axes* of the crystal.[11]

In the Memoir whose *extrait* I present here, I first demonstrate[12] that *when any system of material points is in equilibrium, there are always, for each of them, three perpendicular axes of elasticity*. To suppose that these axes are parallel in all the extent of the medium, and that the small displacements of the molecules do not meet the same resistance in these three perpendicular directions, is then enough to represent all the optical properties of the substances that we call *crystals of one axis* or *of two axes*.

If we take, on each of the three perpendicular axes of elasticity and on radius vectors led in all directions, lengths proportional to the square roots of the elasticities brought into play by small displacements parallel to each of these directions, we will thus form a surface which will represent the law of elasticities of the medium, and which, for this reason, we shall call the *surface of elasticity*.[13] The length of each of its radius vectors will immediately give the speed of propagation of the vibrations parallel thereto,[14] because this speed will also be proportional to the square root of the elasticity. In this construction, the square of the radius vector is assumed to be not the entire resultant of the forces that

[9] Literally "systems of waves".

[10] This plane contains the wave-normal direction but not necessarily the ray direction, whereas the emissionists had defined their "plane of polarization" as containing the ray direction. In isotropic media, the two definitions merge.

[11] That is, the principal axes or coordinate axes, not to be confused with the previously mentioned optical axes.

[12] The opening phrase and the word "first" (*d'abord*) do not appear in the 1822 version [9, p. 66]. These additions tend to identify the subject memoir as the Second Memoir [11], which indeed presents this demonstration early [pp. 264–73, 281–4; Hobson's translation], whereas the First Memoir relegates it to the Second Supplement [12, vol. 2, pp. 371–8].

[13] The 1822 version [9] omits the clause introducing the term, but goes on to use the term.

[14] The radius vector is in the direction of vibration—not the direction of wave propagation, which is normal thereto.

push back the molecule displaced in its direction, but only the component parallel to the radius vector: this resultant can always be decomposed into two forces, one parallel and the other perpendicular to the radius vector.[15] When the molecule is obliged to follow the radius vector—that is, when the plane of the wave is perpendicular to the other component—that component has no influence on the speed of propagation,[16] since it cannot contribute to the displacement of the layers of the medium parallel to the wavefront; we then only have to consider the force directed along the radius vector. Now it is always to this case that I reduce all the questions of propagation of waves in the crystal.

Notice here that if the plane of the wave is not normal to the component perpendicular to the radius vector, then, between one slice and another, this component tends to alter the direction of the vibration, to which we consequently cannot apply the ordinary laws of wave propagation. But its progress becomes easy to follow if the vibration is decomposed into two other perpendicular motions directed along the largest and smallest radius vectors included in the plane of the wave; for these the second component is normal to this plane (as calculation proves), and therefore can no longer influence the direction of the vibratory motion, which is then executed and propagated as in media of uniform elasticity—except that the two wavetrains thus produced, as they develop different accelerating forces, do not propagate both at the same speed, and the interval which separates their corresponding points becomes the more sensible as they have traversed a greater thickness of the crystal.[17] It is these two wavetrains which give rise to the phenomena of coloring of crystallized plates, or to the bifurcation of the rays when the crystal is cut as a prism,[18] because their difference in speed necessarily entails a difference in refraction. When we know the law of the propagation speeds of each wavetrain, we can always determine the change in inclination that they suffer at their entrance into the prism and at their exit, and thus calculate the relative inclinations of the incident and emerging beams: this is why we shall here concern ourselves only with the search for the law of speeds.

First it should be remarked that knowing the three axes of the surface of elasticity is enough to determine the lengths of all its radius vectors, whatever the nature of the reciprocal action of the molecules of the vibrating medium—at least if one considers only very small displacements of these molecules, as we have done until now. If we represent by a, b, and c the three semi-axes of the surface, by X, Y, and Z the angles that a general radius vector makes with these axes, and by v the length of this radius vector, the equation of the surface of elasticity is

$$v^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z\,.$$

The calculation that leads to this result is based on the easily demonstrated principle that *any small displacement of a molecule, in any direction, produces a repulsive force which is rigorously equivalent, in magnitude and direction, to the resultant of the three repulsive forces that would be produced separately by, and parallel to the directions of, three perpendicular displacements equal to the respective statical*

[15] The clause after the colon does not appear in the 1822 version.

[16] Or on the wavelength, as the 1822 version adds.

[17] The key claim is that for a given orientation of the plane wavefront, the directions of vibration that can propagate unchanged are those for which the wave-normal speed is an extremum: let us call these the *principal directions*. Fresnel has justified this claim by showing, in the First Supplement to the First Memoir [12, vol. 2, pp. 351–3] and in the Second Memoir [11, pp. 286–8], that if the vibration is in a principal direction, the developed forces have no tendency to change the direction of vibration within the wavefront. The same claim can be supported more generally by noting that the directions of vibration for which the wave-normal speed is an extremum are also those for which the propagation time is stationary with respect to variations of the direction of vibration: hence, if a vibration could be resolved into components with a continuum of directions, components with directions near a principal direction would propagate with better synchronism, and therefore interfere more constructively, than components with other directions. —*Translator*.

[18] The 1822 version has instead "... bifurcation of the rays when they are inclined to the surface of the crystal".

components of the first displacement.[19] Here we assume that there is only one displaced molecule, and that all the others have remained in their original positions.[20]

It is then easy to demonstrate that the elasticities brought into play by the complex displacements of molecules in infinite plane waves follow the same law as the elasticities brought into play by the displacement of a single molecule, independently of any hypothesis on the nature of the molecular forces, at least as long as the plane of the wave is not varied, but only the direction of the vibrations. To prove that the equation representing the law of elasticities developed in the case of the displacement of a single molecule is also applicable to the elasticities brought into play in light waves, whatever the direction of their fronts, it suffices to establish furthermore that the developed elasticity depends only on the direction of the small molecular displacements, and remains constant notwithstanding the variations of the plane of the wave, as long as the direction of displacement does not change.[21] This is what I have verified on topaz by several very careful experiments, in which I have compared the speeds of propagation of rays traversing the crystal in different directions, but whose vibratory movements or planes of polarization took the same direction: I have always found that then the speeds are the same—that is, they depend not on the direction of the ray, but only on that of its plane of polarization. Is this second theorem as general as the first and independent of any hypothesis on the law of the reciprocal action of the molecules of the vibrating medium? This is what I have not yet fathomed; I have found only that one might make sense of it by very simple and very admissible hypotheses.[22]

With the aid of the above equation, we simultaneously determine the propagation speeds of the ordinary and extraordinary waves and the direction of their planes of polarization: it suffices to calculate the curve of intersection of the surface of elasticity with a diametral plane parallel to the wave[front]: the largest and the smallest radius vectors included in the cutting plane will give by their directions those of the ordinary and extraordinary vibrations (and hence those of their planes of polarization, which are perpendicular to these vibrations), while their lengths will represent the propagation speeds of the ordinary and extraordinary waves, measured perpendicular to the cutting plane.

There are always two diametral planes which cut this surface in a circle; they pass through the intermediate axis [*l'axe moyen*] and are equally inclined to each of the other two axes.[23] The waves parallel to these planes can have only one propagation speed because, all the radius vectors contained in these planes being equal, the parallel vibrations will always develop the same accelerating forces, in whatever direction they are executed. Moreover, as the components perpendicular to the radius vectors are all perpendicular to the cutting plane for the special case of circular sections, the refractive medium will no longer be able to deflect the oscillatory movements of the parallel waves, nor consequently to change their plane of polarization. So, if we cut the crystal parallel to the plane of one of the circular sections and introduce perpendicularly incident rays polarized in any azimuth, then, as the incident wave[front] is parallel to the face of entry, it will still be parallel in the interior of the crystal, and will

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[19] The parallelism of the forces with the components of the first displacement implies that these components are in the directions of the axes of elasticity.

[20] This sentence and the paragraph break do not appear in the 1822 version.

[21] The "variations of the plane of the wave" are implicitly restricted by the requirement that the given direction of displacement (vibration) is parallel to the wavefront.

[22] The 1822 version of this paragraph [9, p. 67, after the italics] is generally more concise. The question at the end does not appear in the 1822 version, or in the corresponding passage in the Second Memoir [11, p. 329]. In the context, neither the direction of vibration for a given orientation of the wavefront, nor the orientation of the wavefront for a given direction of vibration, can be chosen arbitrary; the permitted combinations are those for which the propagation speed (given by the length of the radius vector of the surface of elasticity) is an extremum for the same orientation of the wavefront (as in the next paragraph). Hence, in the second theorem, and in footnote 21 above, not all wavefront orientations parallel to the given vibration are permitted, unless the vibration is in the direction of an axis of elasticity—in which case the length of the radius vector is an unconstrained extremum, and therefore an extremum in any plane containing that axis. —*Translator*.

[23] That is, they contain the intermediate-length principal axis (aligned with the y axis), and are symmetrically inclined to the longest axis of the surface of elasticity (the x axis) and likewise to the shortest axis (the z axis); *cf.* [11] at pp. 288–90.

therefore suffer neither double refraction nor deviation of its plane of polarization.[24] Thus the two directions perpendicular to the circular sections present all the characteristics of what we call "the axes of the crystal". I have proposed to give them the name *optical axes* [*axes optiques*], to distinguish them from the axes of elasticity. Experiment confirms the relation which this construction establishes between the angle of the two optical axes and the other elements of the double refraction of the crystal.

It is known that rays of different colors, or, in other words, waves of different lengths, do not propagate with equal speeds in the same medium,[25] and that their speed of propagation is smaller if they are shorter: this phenomenon can be explained by supposing that the spheres of activity of the forces tending to restore the molecules of the medium to their equilibrium positions extend to distances that are significant compared with the wavelengths of the light, of which the longest are less than a thousandth of a millimeter: then we find, as I show in my Memoir,[26] that the shortest waves must propagate a little more slowly than the others. Consequently the three semi-axes a, b, c, which represent the speeds of propagation of vibrations parallel thereto, must vary a little for waves of different lengths, according to theory as well as to experiment. Now it is quite possible that this variation does not occur in the same ratio for the three axes, in which case the angle between the two circular sections, and hence between the two optical axes, would no longer be the same for rays of different colors—as Dr. Brewster and Mr. Herschel have observed in several crystals.[27]

If, as we have supposed so far, the object point on which we observe the effects of double refraction is sufficiently distant that the incident wave can be assumed plane, it is still so after its refraction in the crystal; and in order to determine the divergence of the ordinary and extraordinary rays, which can be appreciable only if the crystal is prismatic, it suffices to know the changes of inclination of the two wavetrains at their entry into the prism and at their exit. We can calculate each angle of refraction with the aid of the equation of elasticities, or rather speeds, according to the general principle that the sines of the angles of the incident and refracted waves with the surface of the refractive medium are to each other as the propagation speeds of these waves inside and outside the medium;[28] it will be in a direction perpendicular to the emergent wave[front] that we will see the image of the object point.

But when this point is close enough and the double refraction strong enough, it becomes necessary to know the law of curvature of the light waves in the interior of the crystal—that is, the equation of their surface—in order to calculate the directions in which we will see the two images of the object point through the crystal. It follows from the principle of the composition of small movements that any plane tangent to the wave surface[29] (assumed entirely in the same medium) must be distant from its center by a quantity equal to the space traversed at the same instant by an infinite plane wave, from this point at the origin of the movement, and parallel to the element of the curved wave[front] located in the tangent plane.[30] Now these spaces traversed by infinite plane waves, measured perpendicular to their fronts, are proportional, for all directions, to the largest and the smallest radius vectors of the diametral sections of the surface of elasticity parallel to these plane waves. If the equation of the cutting plane is $z = mx + ny$,

[24] Indeed it does not suffer *double* refraction. Fresnel was not in a position to see that it suffers *internal conical refraction*: the single wave-normal direction inside the crystal corresponds to a cone of ray directions, one of which coincides with the wave-normal; and a half-turn of the polarization of the incident ray causes a full turn of the refracted ray about the cone.

[25] The 1822 version adds "although its density and its elasticity remain the same".

[26] This clause first appears in the 1825 version. Two footnotes in the posthumous Second Memoir [11, pp. 277–8, 331] refer to an attached note containing this demonstration. No such note appeared in print. But Fresnel's manuscripts contain calculations indicating that, by 1824, he was in possession of a theoretical formula describing dispersion [12, vol. 1, p. xcvi; vol. 2, p. 411]. The basic idea that dispersion might be explained by the distance of action of the forces was floated in the First Memoir [12, vol. 2, p. 289*n*] and even earlier [6, § 18*n*].

[27] In the Second Memoir [11, p. 331], "several crystals" becomes "the greater number of bi-axal crystals". The relevant works of J. Herschel are footnoted in the 1868 reprint of that Memoir [12, vol. 2, p. 595].

[28] This is not quite Snell's law, because the ratio of speeds is not said to be independent of the angles (as it would be in isotropic media) and because the ray and wave-normal directions do not generally coincide inside the crystal—although they coincide in the air outside the crystal, as the next clause assumes.

[29] French: *surface de l'onde*.

[30] That is, the "wave surface" is one of the secondary wavefronts that are tangential to the plane wave according to Huygens' construction [15, p. 64].

the largest and the smallest radius vectors of the section are given by the relation[31]

$$(a^2 - v^2)(c^2 - v^2)n^2 + (b^2 - v^2)(c^2 - v^2)m^2 + (a^2 - v^2)(b^2 - v^2) = 0\,,$$

where v represents both the largest and the smallest radius vector. Thus the curved wave surface is touched by each plane parallel to the cutting plane and distant from the origin by a quantity equal to the value of v drawn from the above equation. Now this condition is satisfied by the following equation, which is consequently that of the wave surface:

$$\left(a^2x^2 + b^2y^2 + c^2z^2\right)\left(x^2 + y^2 + z^2\right) - a^2(b^2{+}c^2)x^2 - b^2(a^2{+}c^2)y^2 - c^2(a^2{+}b^2)z^2 + a^2b^2c^2 = 0\,.$$

If in the construction which Huygens gave for determining the direction of the rays refracted by Iceland spar [calcite], and which is applicable to any wavefront shape,[32] we replace the sphere and the ellipsoid of revolution by the two-sheeted surface represented by this last equation, and use it in the same way, we will have two tangent planes, whose points of contact when joined [by lines] to the center of the wave will give the direction of the *ordinary ray* and of the *extraordinary ray*.

If two of the [semi-]axes of elasticity, say b and c, are equal, this equation can be put in the form

$$\left(x^2 + y^2 + z^2 - b^2\right)\left(a^2x^2 + b^2(y^2 + z^2) - a^2b^2\right) = 0\,,$$

which is the product of the equation of a sphere with that of an ellipsoid of revolution. Then the two circular sections of the surface of elasticity merge with the yz plane, and the two optical axes with the x axis; this is the case of crystals *of one axis*, such as calcareous spar. But if the three axes are unequal, the general equation is no longer decomposable into rational factors of the second degree.

The luminous wave surface in crystals for which a, b, and c are unequal can be generated by a very simple construction, which establishes an immediate relation between the length and the direction of its radius vectors. If we conceive an ellipsoid having the same semi-axes a, b, and c, and if, having cut it by any diametral plane, we erect on this plane, from the center of the ellipsoid, a perpendicular equal to the smallest or to the largest radius vector of the section, the end of this perpendicular will belong to the wave surface, or, in other words, the length of this perpendicular will be that of the corresponding radius vector of the wave surface, and will thus give the speed of the *rays of light* that propagate in this direction; for these radius vectors must indeed present, according to the wave theory, all the optical characteristics that we attach to the words *ray of light*.[33] This is a principle which we could not demonstrate without entering into somewhat lengthy details, but which it was necessary to state here to facilitate translation of the consequences of the wave theory into the better known language of the emission system.

If we divide the unit by the squares of the two semi-axes of a diametral section of the ellipsoid, the difference between these quotients[34] is proportional to the product of the sines of the angles that the perpendicular to this section makes with the two normals of the planes of the circular sections—that is, with the two optical axes of the crystal.[35] This consequence of the wave theory, translated into the language of emission, in which the ratios of the speeds attributed to the rays are inverse, is

[31] The 1822 version adds “deduced from the equation of elasticity”.

[32] Literally “. . . any form of wave”.

[33] The 1822 version has “. . . to the word *ray* in the emission system.”

[34] That is, the difference between the inverse squares of the ray speeds.

[35, (1)] [Note by Fresnel:] The planes of the circular sections of the ellipsoid and of the surface of elasticity do not strictly coincide, and consequently the normals to these planes differ somewhat, but by an angle which is very small for all the crystals of two axes known until now: one can equally well give the name *optical axis* [*axe optique*] to one or the other of these normals. [The normals to the circular sections of the ellipsoid, being the directions of a single ray velocity, are now called the **biradial** axes or the *ray* axes, whereas the normals to the circular sections of the surface of elasticity, being the directions of a single wave-normal velocity, are now called the **binormal** axes or the *optic* axes. In the Second Supplement [12, vol. 2, pp. 395–6] and in the Second Memoir [11, pp. 312–13, 317], as a concession to the emissionists, Fresnel gave the name *axes optiques* to the biradials, whereas nowadays we give the name *optic axes* to the binormals (as Fresnel proposed to do in the First Supplement to the First Memoir [12, vol. 2, pp. 354–5]; *cf.* Buchwald [4], pp. 285–6). —*Translator*.]

precisely the law of the difference of the squares of the speeds which Dr. Brewster had deduced from his experiments [3], and which had since been confirmed by those of Mr. Biot [2, pp. 13–14], to whom we owe the simple form of the product of the two sines.[36]

The rule given by Mr. Biot [2, pp. 15–16] to determine the direction of the planes of polarization of the ordinary and extraordinary rays also agrees with the construction just stated;[37] or at least the slight difference which one will notice by reflecting on it does not seem susceptible to being captured by observation.[38] Thus the exactitude of this construction is found to be established both by the previous experiments of Dr. Brewster and Mr. Biot, and by the new observations that I have made to verify it.

[The following paragraphs were added in the 1825 version.]

The theory of double refraction whose main results we have expounded in this *extrait*, and the simple construction that we deduce from it, exhibit the remarkable property that all the unknowns are determined simultaneously by the solution of the problem: we find at once the speeds of the ordinary and

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[36] Hence this law has become known as *Biot's sine-product law* or *Biot's sine law*—although, as Fresnel here implies, the correct statement of it involves the difference of the *inverse* squares of the ray speeds, not the difference of the squares. Fresnel's proof dates from the First Memoir (see Grattan-Guinness [13], pp. 886–7, and Buchwald [4], pp. 269–72, citing Fresnel [12], vol. 2, pp. 293–8). The law was thought to be an approximation in the first *Extrait* (Buchwald [4], p. 279; *cf.* [12], vol. 2, pp. 322–3); but it was reinstated as "rigorous" in the Second Supplement [12, vol. 2, pp. 395–6], and duly reappears in the Second Memoir [11, pp. 312–17]. In the supporting diagram in the original edition of the Second Memoir, and in the English translation [11, p. 313], 'y' should be 'z'; the corresponding diagram in the 1868 French edition [12, vol. 2, p. 573] is correct.

In his derivation of Biot's sine law, Fresnel comes excruciatingly close to discovering *external conical refraction*. He shows that if the equation of the ellipsoid is $fx^2+gy^2+hz^2=1$, with $f<g<h$, while n and m are the angles between the normal to the general diametral section (the ray direction) and the normals to the circular sections (the ray axes), then the major and minor semi-axes r of the diametral section (the ray speeds) are given by

$$\frac{1}{r^2} = A(n,m) \pm B(n)\sin m\,, \tag{1}$$

where

$$A(n,m) = \tfrac{1}{2}(f+h) - \tfrac{1}{2}(f-h)\cos n\cos m \quad ; \qquad B(n) = \tfrac{1}{2}(f-h)\sin n \tag{2}$$

(my abbreviations for Fresnel's expressions, implying that B is negative). To show that this represents a double cone for small m, to first order, we do not need Hamilton's rotation of the coordinates [14, pp. 132–3]. From (1),

$$r = \Big(A(n,m) \pm B(n)\sin m\Big)^{-1/2}. \tag{3}$$

If m is small and/or the birefringence is weak, the second term in the parentheses is much smaller than the first, so that we can use the first-order Taylor/binomial expansion

$$r \approx \Big\{A^{-1/2}(n,m)\Big\} \mp \Big\{\tfrac{1}{2}A^{-3/2}(n,m)\,B(n)\Big\}\sin m\,. \tag{4}$$

If m is small, A and B, and hence the expressions in braces, are smoothly-varying non-zero functions, and the first term represents a smooth surface, so that the double surface (4) departs from that smooth surface by a radial distance proportional to $\pm\sin m$, where m is the angular departure from the ray axis as seen from the origin. As $m \to 0$, this double surface approaches a double cone. Thus a single ray direction corresponds to a cone of wave-normal directions, which, on refraction out of the crystal, becomes a cone of wave-normal directions in an isotropic medium, hence a cone of ray directions. However, all Fresnel does with his expression for the roots $1/r^2$ is to take the difference between the roots, obtaining Biot's sine law. We should also notice, in retrospect, that Fresnel's formula for the roots—which dates back to the First Memoir—is a form of the equation for the wave surface, which can be recognized in other forms in the course of the derivation. —*Translator*.

[37] The 1822 version has ". . . the construction deduced from the equation of elasticity" [9, p. 70].

[38] Biot's dihedral law, as Buchwald calls it [4, pp. 418–19], says that for a given direction of propagation, the permitted planes of polarization bisect the dihedral angles between two other planes, each of which contains the direction of propagation and one of the optical axes. As Fresnel shows in his Second Memoir [11, pp. 320–22], this law is exact if the propagation direction is taken as the wave-normal direction, and the optical axes as the directions of a single wave-normal velocity (the binormals); but it is only approximate if the propagation direction is taken as the ray direction and the optical axes as the directions of a single ray velocity (the biradials), because the plane of polarization, defined as normal to the vibration, does not generally contain the ray. Fresnel had given the latter version, as an approximation, in the *Extrait* of 26 November 1821 [12, vol. 2, pp. 322–3]. However, both versions of the law become exact if each plane of polarization is replaced by the *plane of vibration*, which contains both propagation directions and the direction of vibration (see [18], searching for "dihedral law for rays" and "dihedral law for wave-normals").

extraordinary rays and their planes of polarization. Physicists who have often reflected on the laws of nature will sense that this simplicity and these intimate relations between the various parts of the phenomenon lend great plausibility to the theory that underlies them.[39]

Long before having conceived it, and by mere meditation on the facts, I had felt that one could not discover the true explanation of double refraction without simultaneously explaining the phenomenon of polarization which always accompanies it: so it was after finding which mode of vibration constituted the polarization of light that I promptly glimpsed the mechanical causes of double refraction [6, §§ 10–14]. It seemed to me even more evident that the speeds of the ordinary and extraordinary beams had to be, in some way, the two roots of the same equation; and not for one moment have I been able to accept the explanation whereby there would be two different media—for example[40] the refractive body, and the aether contained therein—of which one would transmit extraordinary waves and the other ordinary waves;[41] indeed if these two media could transmit light waves separately, one does not see why the two propagation speeds would be rigorously equal in most refractive bodies,[42] and why prisms of glass, water, alcohol, etc., would not also divide the light into two distinct beams.

In the Memoir of which I have just given an *extrait* (which I would have liked to be able to develop further), I also explain by the same theory why the refraction of a homogeneous medium never divides the light into three or four beams, but only into two,[43] and why there cannot be more than two optical axes in the crystals,[44] at least as long as the three axes of elasticity at each of the points of the refractive medium are parallel in all its extent, as must happen when the homologous lines or faces of its molecules are parallel. It would seem at first glance that this parallelism must be the constant result of regular crystallization. However, some perfectly crystallized bodies, such as quartz[45], present optical phenomena which cannot be reconciled with complete parallelism of the molecular lines, and which would seem to indicate a progressive and regular deviation of these lines in the passage from one slice of the medium to the next slice; indeed we can imagine, in addition to the case of parallelism, a multitude of other molecular arrangements which would preserve in the body all the characteristics of homogeneity and of a regular organization.[46] So far, however, I have calculated the laws of refraction only for the particular case where the axes of elasticity have the same direction at each point of the vibrating medium.

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[40] In the Second Memoir [11, p. 330], Fresnel deletes "for example", leaving only one version of the hypothesis, which is indeed the one refuted by his next argument.

[41] Huygens supposed the existence of "two different kinds of matter which serve for the two species of refraction" [15, p. 94], but then could not explain to his own satisfaction why one ray (ordinary or extraordinary) emerging from one crystal could move either kind or both kinds of matter in a second crystal, according to the relative orientations of the crystals [15, pp. 92–4].

[42] That is, in optically isotropic media.

[43] Second Supplement to the First Memoir [12, vol. 2, pp. 379–83], and Second Memoir [11, pp. 299–300]; *cf.* [13, p. 892].

[44] First Memoir [12, vol. 2, pp. 287–9], First Supplement [12, vol. 2, pp. 353–5], and Second Memoir [11, pp. 288–91]; *cf.* [13, pp. 886, 890].

[45] Literally "crystal of rock". Fresnel's memoir on the circular birefringence of quartz is now available in English [10].

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