

An Inconsistency

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Abstract. This paper proves an inconsistency within ZFC by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be deduced.

Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_n denote the natural numbers starting from $n > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation \neg SSGB hold.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g . So, by the definitions we have

$$\text{SSGB} \Leftrightarrow \forall x \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \Leftrightarrow \exists x \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

There are the following two properties of the set S_g .

First, the whole range of \mathbb{N}_3 can be expressed by the triple components of S_g ("covering"), because every integer $x \geq 3$ can be written as some pk with $k = 1$ when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3, k \in \mathbb{N}$. So we have

$$(C) \quad \forall x \in \mathbb{N}_3 \exists (pk, mk, qk) \in S_g \quad x = pk \vee x = mk = 4k.$$

Second, due to the definition of the set S_g , all pairs (p, q) of distinct odd primes are used ("maximality"). So we have

$$(M) \quad \forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g, \text{ where } m = (p + q) / 2.$$

In case of \neg SSGB there is at least one $n \in \mathbb{N}_4$ different from all the numbers m that are defined in S_g . In case of SSGB there is no such n .

Since for n the possibilities $\neg(C)$ and $\neg(M)$ are ruled out, we can proceed with the following argumentation, i.e., $((C) \text{ and } (M))$ implies the contradiction we will derive. The following steps work regardless of the choice of n if there is more than one.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_{g+}(y) \cup S_{g-}(y)$, where

$S_{g+}(y) := \{ (pk', mk', qk') \in S_g \mid \exists k \in \mathbb{N} \quad pk' = yk \vee mk' = yk \vee qk' = yk \}$ and

$S_{g-}(y) := \{ (pk', mk', qk') \in S_g \mid \forall k \in \mathbb{N} \quad pk' \neq yk \wedge mk' \neq yk \wedge qk' \neq yk \}$.

Let $n \in \mathbb{N}_4$ be given by $\neg\text{SSGB}$ as above. Then, we have

$$\neg\text{SSGB} \Rightarrow S_g = S_{g+}(n) \cup S_{g-}(n).$$

More precisely, under the assumption $\neg\text{SSGB}$ with the associated n the set S_g can be written as the union of the following triples.

(i) S_g triples of the form $(pk' = nk, mk', qk')$ with $k' = k$ in case n is prime, due to (C)

(ii) S_g triples of the form $(pk' = nk, mk', qk')$ with $k' \neq k$ in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form $(3k', 4k' = nk, 5k')$ in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form $(pk' = nk, mk', qk')$, $(pk', mk' = nk, qk')$ or $(pk', mk', qk' = nk)$

and

(v) S_g triples of the form $(pk' \neq nk, mk' \neq nk, qk' \neq nk)$, i.e. those S_g triples where none of the nk 's equals a component.

So, $S_{g+}(n)$ is the union of the triples of the above types (i) to (iv) and $S_{g-}(n)$ is the union of the triples of type (v).

Since $S_{g+}(n) \cup S_{g-}(n)$ is independent of n , we have

$$(1) \quad \neg\text{SSGB} \Rightarrow S_g = S_{g+}(y_1) \cup S_{g-}(y_1), \text{ for any } y_1 \in \mathbb{N}_3.$$

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y_2 \in \mathbb{N}_3$ such that

$$(2) \quad \text{SSGB} \Rightarrow S_g = S_{g+}(y_2) \cup S_{g-}(y_2).$$

On the other hand, after defining

$S1 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds} \}$

$S2 := \{ (pk, mk, qk) \in S_g \mid SSGB \text{ holds} \},$

we have

(1') $\neg SSGB \Leftrightarrow S_g = S1$

(2') $SSGB \Leftrightarrow S_g = S2.$

Since in both implications (1) and (2) the premise is assumed to be true, by (1) and (1') we get

(1'') $S1 = S_{g+(y_1)} \cup S_{g-(y_1)}, \text{ for any } y_1 \in \mathbb{N}_3$

and by (2) and (2') we get

(2'') $S2 = S_{g+(y_2)} \cup S_{g-(y_2)}, \text{ for any } y_2 \in \mathbb{N}_3.$

Since $S_{g+(y)} \cup S_{g-(y)} = S_g$ for every $y \in \mathbb{N}_3$, from (1'') and (2'') we obtain $S1 = S2.$

Thereby, (1') and (2') imply that $\neg SSGB \Leftrightarrow SSGB$, which yields the contradiction (SSGB and $\neg SSGB$).

□

Note. The proof is based on the following general principle.

Suppose there is a non-empty set A and a proposition P such that

$\{ a \in A \mid P \text{ holds} \} = \{ a \in A \mid \neg P \text{ holds} \}.$

Then, we have a contradiction for the following reason.

Since A is non-empty, we have

$$\neg P \iff A = A_1 := \{ a \in A \mid \neg P \text{ holds} \}$$

$$P \iff A = A_2 := \{ a \in A \mid P \text{ holds} \}.$$

Since $A_1 = A_2$, we obtain $\neg P \iff P$, which yields $(P \text{ and } \neg P)$.