An Inconsistency

Ralf Wüsthofen

Abstract. This paper proves an inconsistency within ZFC by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be deduced.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold*.

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g. So, by the definitions we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \;\; \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \;\; \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

There are the following two properties of the set S_g.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \;\; \exists (pk, mk, qk) \in S_g \quad x = pk \lor x = mk = 4k.$

Second, due to the definition of the set S_g, all pairs (p, q) of distinct odd primes are used ("*maximality*"). So we have

(M) $\forall p, q \in \mathbb{P}_3, p < q \;\; \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g$, where $m = (p + q) / 2$.

In case of ¬SSGB there is at least one $n \in \mathbb{N}_4$ different from all the numbers m that are defined in S_g. In case of SSGB there is no such n.

Since for an n given by $\neg$SSGB the possibilities $\neg$(C) and $\neg$(M) are ruled out, we can proceed with the proof, that is, ((C) and (M)) implies the contradiction we will derive. The following steps work regardless of the choice of n if there is more than one.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, where

$S_g+(y) := \{ (pk', mk', qk') \in S_g \mid \exists\, k \in \mathbb{N}\ \ pk' = yk \ \vee\ mk' = yk \ \vee\ qk' = yk \}$ and

$S_g-(y) := \{ (pk', mk', qk') \in S_g \mid \forall\, k \in \mathbb{N}\ \ pk' \neq yk \ \wedge\ mk' \neq yk \ \wedge\ qk' \neq yk \}$.

Then, we have

$\neg$SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption $\neg$SSGB the set S_g can be written as the union of the following triples.

(i) S_g triples of the form (pk' = nk, mk', qk') with k' = k in case n is prime, due to (C)

(ii) S_g triples of the form (pk' = nk, mk', qk') with k' ≠ k in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k', 4k' = nk, 5k') in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk' = nk, mk', qk'), (pk', mk' = nk, qk') or (pk', mk', qk' = nk)

and

(v) S_g triples of the form (pk' ≠ nk, mk' ≠ nk, qk' ≠ nk), i.e. those S_g triples where none of the nk’s equals a component.

So, $S_g+(n)$ is the union of the triples of the above types (i) to (iv) and $S_g-(n)$ is the union of the triples of type (v).

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we have

(1) $\neg$SSGB => $S_g = S_g+(y_1) \cup S_g-(y_1)$, for any $y_1 \in \mathbb{N}_3$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose any $y_2 \in \mathbb{N}_3$ such that

(2) SSGB => $S_g = S_g+(y_2) \cup S_g-(y_2)$.

We take note of the correct interpretation of (1) and (2): We have assumed $\neg$SSGB and SSGB, respectively, and then we have proved in each case that $S_g = S_g+(y_i) \cup S_g-(y_i)$ follows from the assumption. So, we have proofs of the implications (1) and (2) where (1) and (2) are neither the tautology "False => Q" nor the tautology "Q => A = A".

On the other hand, after defining $S := \{ (pk, mk, qk) \in S_g \mid \text{SSGB holds} \}$ we have

(3) SSGB <=> $S_g = S$.

Using (3) in (1) and (2), we get

(1') $S_g \neq S \Rightarrow S_g = S_g+(y_1) \cup S_g-(y_1)$, for any $y_1 \in \mathbb{N}_3$

(2') $S_g = S \Rightarrow S_g = S_g+(y_2) \cup S_g-(y_2)$, for any $y_2 \in \mathbb{N}_3$.

Since (1') and (2') are neither the tautology "False => Q" nor the tautology "Q => A = A", $S_g = S_g+(y_1) \cup S_g-(y_1)$ follows from $S_g \neq S$ and $S_g = S_g+(y_2) \cup S_g-(y_2)$ follows from $S_g = S$.

Thus, we conclude that $S_g+(y_1) \cup S_g-(y_1) \neq S$ and that $S_g+(y_2) \cup S_g-(y_2) = S$.

Since in (1) and (2) the variables y_1 and y_2 are arbitrary, we have $S_g+(y_1) \cup S_g-(y_1) = S_g+(y_2) \cup S_g-(y_2)$. So, we get $S_g+(y_2) \cup S_g-(y_2) = S$ and $S_g+(y_2) \cup S_g-(y_2) \neq S$.

Because of $S_g+(y) \cup S_g-(y) = S_g$ for every $y \in \mathbb{N}_3$, we obtain the contradiction ($S_g = S$ and $S_g \neq S$), which is the same as (SSGB and $\neg$SSGB).

□

Note. The proof is based on the following general principle.

Suppose there is a non-empty set A and a proposition P such that there are sets A1, A2 and proofs of the implications

(1) $\neg$P => A = A1

(2) P => A = A2,

where (1) and (2) are neither the tautology "False => Q" nor the tautology "Q => True", i.e. "Q => A = A".

Suppose further there is a proof that A1 = A2.

Then, we have a contradiction for the following reason.

After defining $A3 := \{ a \in A \mid P \text{ holds} \}$ we have

(3) $P \iff A = A3$.

Using (3) in (1) and (2), we get

(1') $A \neq A3 \Rightarrow A = A1$

(2') $A = A3 \Rightarrow A = A2$.

Since (1') and (2') are neither the tautology "False => Q" nor the tautology "Q => A = A", $A = A1$ follows from $A \neq A3$ and $A = A2$ follows from $A = A3$.

Thus, we conclude that $A1 \neq A3$ and that $A2 = A3$.

Since $A1 = A2$ and so $A = A1 = A2$, we obtain ($A = A3$ and $A \neq A3$), which is the same as (P and $\neg P$).