

An Antinomy

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Notations. Let \mathbb{N} denote the natural numbers starting from 1, let \mathbb{N}_n denote the natural numbers starting from $n > 1$ and let \mathbb{P}_3 denote the prime numbers starting from 3.

SSGB: *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation \neg SSGB hold.*

Proof. $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB $\Leftrightarrow \forall x \in \mathbb{N}_4 \exists (pk, mk, qk) \in S_g \quad x = m$.

\neg SSGB $\Leftrightarrow \exists n \in \mathbb{N}_4 \forall (pk, mk, qk) \in S_g \quad n \neq m$.

(C): $\forall k \in \mathbb{N} \exists (pk', mk', qk') \in S_g \quad nk = pk' \vee nk = mk' = 4k'$.

(M): $\nexists p, q \in \mathbb{P}_3, p < q \quad n = (p + q) / 2$.

$\forall y \in \mathbb{N}_3 \quad S_g = S_{g+}(y) \cup S_{g-}(y)$, where

$S_{g+}(y) := \{ (pk', mk', qk') \in S_g \mid \exists k \in \mathbb{N} \quad pk' = yk \vee mk' = yk \vee qk' = yk \}$,

$S_{g-}(y) := \{ (pk', mk', qk') \in S_g \mid \forall k \in \mathbb{N} \quad pk' \neq yk \wedge mk' \neq yk \wedge qk' \neq yk \}$.

$S_{g+} := S_{g+}(n), S_{g-} := S_{g-}(n)$.

\neg SSGB $\Rightarrow ((S_g = S_{g+} \cup S_{g-}) \text{ or } \neg(C) \text{ or } \neg(M))$.

\neg SSGB $\Rightarrow S_g = S_{g+} \cup S_{g-}$.

(1): $\forall y \in \mathbb{N}_3 \quad \neg$ SSGB $\Rightarrow S_g = S_{g+}(y) \cup S_{g-}(y)$.

(2): $\forall y \in \mathbb{N}_3 \quad$ SSGB $\Rightarrow S_g = S_{g+}(y) \cup S_{g-}(y)$.

(3): $\forall y \in \mathbb{N}_3 \quad ((\text{SSGB} \Rightarrow S_g = S_{g+}(y) \cup S_{g-}(y)) \text{ and } (\neg \text{SSGB} \Rightarrow S_g = S_{g+}(y) \cup S_{g-}(y)))$.

(1) \Rightarrow (4): $\forall y \in \mathbb{N}_3 \quad (\forall S \quad (\neg \text{SSGB} \Rightarrow S_g = S) \Leftrightarrow (\neg \text{SSGB} \Rightarrow S_{g+}(y) \cup S_{g-}(y) = S))$.

(3), (4) $\Rightarrow (\forall S \quad (\neg \text{SSGB} \Rightarrow S_g = S) \Leftrightarrow (S_g = S)) \Leftrightarrow \neg \text{SSGB}$.

(2) \Rightarrow (5): $\forall y \in \mathbb{N}_3 \quad (\forall S \quad (\text{SSGB} \Rightarrow S_g = S) \Leftrightarrow (\text{SSGB} \Rightarrow S_{g+}(y) \cup S_{g-}(y) = S))$.

(3), (5) $\Rightarrow (\forall S \quad (\text{SSGB} \Rightarrow S_g = S) \Leftrightarrow (S_g = S)) \Leftrightarrow \text{SSGB}$. □