HIGHER ORDER STATISTICS FOR LASER-EXTINCTION MEASUREMENTS

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ABSTRACT

Recent laser technology provides accurate measures of the dynamics of fluids and embedded particles. For instance, the laser-extinction measurements (LEM) use a laser beam passing across the fluid and measure the residual laser light intensity at the fluid output. Some particle and fluid properties are estimated from these measurements such as concentration or velocity. However, the flow is submitted to fluctuations. The received intensity is more appropriately modeled by a random process. This paper first models the particle flow by a queueing process. The measured intensity statistics are derived according to this model. A particular case is the scintillation index which is independent of the emitted intensity. The proposed model allows to define higher order scintillation indices with the same property. Alternative estimates of the fluid and particle properties can be deduced from these quantities.

Index Terms: laser measurements, queueing analysis, higher order statistics.

1. INTRODUCTION

Laser techniques such as the laser-induced incandescence (LII) [10], the laser-induced scattering (LIS) [10], the laser doppler anemometry (LDA) [1] or the laser-extinction measurements (LEM) [13] are currently used to study the dynamics of fluid and embedded particles. This paper deals more specifically with the LEM technique. An application is the observation of emitted particles for *in situ* monitoring of combustion effluents by measurement of the soot volume fraction [3]. Such measurements are motivated by the need to reduce pollutant emissions for environmental purposes all the more that legislated emission limits are imposed by environmental protection agencies worldwide [13].

The LEM system is composed of a laser beam crossing the flow and embedded particles (Fig.1). When the laser beam crosses the fluid, the light is partially absorbed by the particles. Residual light intensity measurements in the emission direction provide estimates of the particle concentration, size and/or velocity. An optical device thus measures either the received beam intensity power for opacity monitors or its temporal variation for more recent scintillation monitors [3]. The measurements of the first and second order moments of the intensity lead to the so-called scintillation index. However, it has been shown that the autocorrelation function and the power spectrum of the light intensity provide additional information on the observed medium [8], [9]. This paper generalizes these results to higher order statistics of the received intensity. This generalization provides alternative estimates for the fluid and particle properties. Moreover, the particular behavior of the higher order scintillation indices should allow to check the model validity through practical measurements.

Section 2 models the physical phenomenon as an appropriate queueing process. Section 3 proposes measurements based on higher order statistics of the received intensity. Section 4 studies the particular cases of a fluid with constant velocity and a rectangular laser profile, of small and/or identical particles and of a circular cylindrical laser beam. The proofs (provided in appendix) are based on the properties of Poisson processes and of the multinomial distribution.

2. PHYSICAL PHENOMENON MODEL

Figure 1 displays the measurement apparatus in the particular case of a rectangular laser beam profile developed in section 4. However, the model presented in this section is independent of the profile. Indeed, optics allows to build a large variety of profiles (see for instance http://www.ophiropt.com). The intersection between the laser beam and the particle flow is a piece of cylinder $\mathscr{V}$ with constant cross-section $\mathscr{S}$.

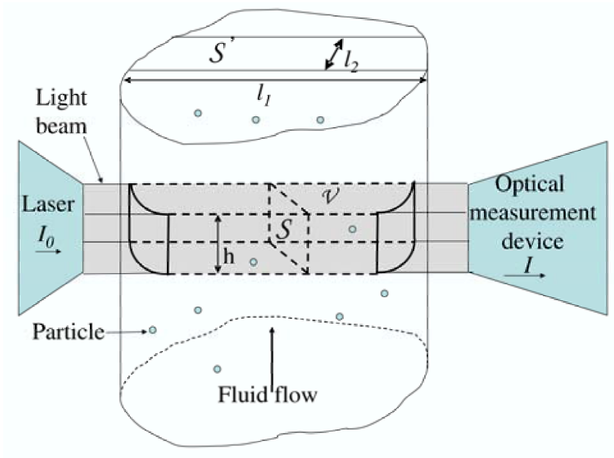

Figure 1: Laser extinction measurement system

According to Mie theory, a given particle scatters and absorbs a small amount of light. From [3] and others, the received intensity I can be expressed by:

$$I = I_0 \prod_{k \in J_t} A_k \ \text{ with } \ A_k = 1 - \frac{\mathscr{E}_k}{\mathscr{S}} \tag{1}$$

where I_0 denotes the incident laser beam intensity.

For a given particle with index k, $\mathscr{E}_k$ denotes the so-called extinction cross-section. Uncertainties on the particle shape and orientation result in a random model for $\mathscr{E}_k$. According to [4], in some practical cases the particle sizes follow a logarithmic normal or a Rosin-Rammler distribution. The factor A_k such that $0 < A_k < 1$ models the laser beam attenuation induced by this particle. Note that this model assumes a constant beam intensity in the emission surface. This hypothesis is currently made in the literature. However, if the intensity is not constant, which is the case for Gaussian Schell-model beams [16], the extinction cross-section $\mathscr{E}_k$ distribution depend also on the place where the particle enters the laser beam.

The LII and LIS problems admit similar models. In the case of the LII, the laser heats the particles up to incandescence: laser-heated soot particles originate near-blackbody emissions [14]. Then measurement of the light produced by combustion is performed. Indeed, for a sufficient laser pulse, LII emission in the visible wavelength range is approximately proportional to the local soot concentration [14], [17]. The LIS rather measures the light scattered by the encountered particles [2]. In both cases, the received intensity I' is the sum of all the particle contributions:

$$I' = I'_0 \sum_{k \in J_t} A'_k \tag{2}$$

Note that this expression appears as the logarithm of (1). Though the logarithm of the spectral density of I is not the spectral density of the logarithm of I (the same holds for the autocorrelation functions), the method proposed in this paper can be applied to the LII and LIS problems. The LDA is based upon a different principle [1]: the laser beam is split into two beams which create an interference region inside the fluid. When crossing this region, a given particle modulates the light at a given frequency. This frequency is a function of the particle celerity i.e. a parameter to be measured. The measurement accuracy requires that at most one particle is in the interference region at a time. Consequently, the LDA cannot be treated with the same mathematical models than LII, LIS and LEM.

Let $\mathbf{J}_t$ denote the set of illuminated particle indices at time t. If a particle of index k enters the beam at time t_k and goes out at the time t'_k then

$$\mathbf{J}_t = \{k; t_k < t, t'_k > t\} \tag{3}$$

The particle flow in the illuminated volume $\mathscr{V}$ can be described by a queueing process with service times $L_k = t'_k - t_k$ corresponding to the lightening or beam crossing duration [5], [8], [9]. L_k depends on $\mathscr{S}$ and on the possibly time-varying particle velocity $\overrightarrow{v}$.

For a reasonably weak particle concentration ρ, the arrival time sequence $\mathbf{t} = \{t_k; k \in Z\}$ can be modeled by a Homogeneous Poisson Process (HPP) with parameter λ [15]. The Poisson parameter λ is a function of ρ.

Now, let $|F|$ denote the cardinal number of a given set F. Then, $N(t,\tau)$ defined by

$$N(t,\tau) = |\{k; t \le t_k < t+\tau\}| \tag{4}$$

denotes the number of particles entering the beam in the interval $[t, t+\tau[$. $N(t,\tau)$ is a Poisson random variable with parameter $\lambda\tau$. For non-overlapping intervals $[t, t+\tau[$ and $[t', t'+\tau'[$, the random variables $N(t,\tau)$ and $N(t',\tau')$ are assumed independent. Moreover, the probability distribution of $N(t,\tau)$ is assumed independent of t and thus stationary. Finally, $N(t,\tau)$ verifies:

$$\lim_{\tau \to 0} \frac{\Pr[N(t,\tau) \ge 2]}{\tau} = 0 \tag{5}$$

Moreover, if the L_k's are independent and identically distributed, the process is a $M/G/\infty$ queue. According to queueing notations, M is for an HPP for particle arrival times, G denotes the unspecified cumulative distribution of the independent service durations:

$$G(x) = \Pr[L_k < x] \tag{6}$$

G depends on the particle velocity $\overrightarrow{v}$ and on $\mathscr{S}$ geometric properties. Finally, ∞ is for an infinite number of servers. An unlimited number of servers i.e. "the service is instantaneous and equivalent for each customer" means that the particle is not stopped when entering the volume $\mathscr{V}$.

According to this model, the number of illuminated particles at time t i.e. $|\mathbf{J}_t|$ is Poisson distributed independently of the distribution G. Note that the independence with respect to G holds only when $\mathbf{t} = \{t_k; k \in \mathbb{Z}\}$ is a HPP.

A Poisson distribution for $|\mathbf{J}_t|$ justifies the results obtained in [2], [3] and allows to derive the higher order moments of the received intensity leading to extensions of the scintillation index.

3. HIGHER ORDER SCINTILLATION INDICES

Particle emission is classically monitored with opacity systems: the particle concentration is directly measured through the light beam intensity power attenuation on the fluid path. However, the major drawback of opacity monitors is their sensitivity to dust accumulation on the lens. The opacity system reliability can only be achieved at the cost of a high level of maintenance. The scintillation index has been recently introduced because of its insensitivity to the optical receiver contamination [3], [4]. Scintillation monitors rather measure the temporal variation of the light beam intensity power. Measurements during an extended period without maintenance is thus possible. A linear relationship between scintillation and particle concentration has been verified through experiments.

The scintillation index is defined as:

$$\Delta = \frac{var[I(t)]}{E^2[I(t)]} = \frac{E[I^2(t)]}{E^2[I(t)]} - 1 \tag{7}$$

where $E[..]$ denote the mathematical expectation and var denotes the variance. A similar index called coefficient of variation is used in acoustics [12]. Δ is independent of I_0 which prevents sensitivity to dust accumulation and to slow variations of the laser intensity. In practice, scintillation monitors provide the empirical variance of the intensity measurements normalized by their squared empirical mean measured over a given time period.

Let define the generalized scintillation index by:

$$\Delta_{pq}(\tau) = \frac{E[I^p(t) I^q(t+\tau)]}{E[I^p(t)]E[I^q(t)]} - 1, \forall p,q \in \mathbb{N}^2 \tag{8}$$

This index is also independent of the laser beam intensity I_0. The classical scintillation index is a particular case obtained by $\Delta = \Delta_{11}(0)$. Note that, whatever the integers p and q:

$$\lim_{\tau\to\infty} E[I^p(t)I^q(t+\tau)] = E[I^p(t)]E[I^q(t)] \quad (9)$$

Let $m_n = E\left[A_k^n\right]$. The appendix proves the following formula:

$$\Delta_{pq}(\tau) = \exp[\lambda\mu_{pq}\int_\tau^\infty (1-G(u))\,du] - 1 \quad (10)$$

with:

$$\mu_{pq} = m_{p+q} - m_p - m_q + 1 > 0. \quad (11)$$

Indeed, since $0 < A_k < 1$:

$$A_k^{p+q} - A_k^q - A_k^p + 1 = (1-A_k^p)(1-A_k^q) > 0 \quad (12)$$

A remarkable property of $\Delta_{pq}(\tau)$ is that its second order derivative is positive for all $\tau > 0$. $\Delta_{pq}(\tau)$ is a positive convex function. As such, $\Delta_{pq}(\tau)$ belongs to the class of Polya-type characteristic functions [11] whatever the integers p and q. As a result, its Fourier transform (related to higher order spectra) is positive whatever the integers p and q. Moreover, the shape of the higher order scintillation indices only depends on $G(u)$ i.e. on the particle velocity and on the laser beam geometry. Consequently, for a given system, the generalized scintillation indices measured for different integers p and q, $\Delta_{pq}(\tau)$, only differ by the factor μ_{pq} and have the same shapes and thus same Fourier transform shapes. These shapes as functions of the delay τ provide useful information on the parameters of interest. Moreover, plotting these curves for different p and q values should allow to validate the queueing process model through experimental measurements. The generalized scintillation index is function of λ and hence of the laser profile provided by constructors.

Note that the main drawback of higher order statistics is the usually higher mean square error of their empirical estimates. Hence, high p and q values will lead to poor estimates. Consequently, since the increasing order scintillation indices should provide the same information, relatively small p and q values should be preferred for parameter estimation. The following section considers particular cases of interest leading to simplified expressions. In such cases, estimates of the system parameters are proposed. In general, there exists a closed form expression of $\Delta_{pq}(\tau)$ leading to estimates of v and ρ when the Poissonian property and model (1) hold i.e. for small ρ values.

4. PARTICULAR CASES

4.1 Constant fluid velocity and rectangular laser profile

The formula (10) holds even when the particle velocity is random or variable during the lightening duration. This happens when the fluid is turbulent or when edge effects cannot be neglected. However, formula (10) leads to a simplified expression when the fluid velocity is constant. When the celerity $\overrightarrow{v}$ of the particles can be assumed constant and is perpendicular to the laser beam profile, the Poisson parameter λ can be expressed as

$$\lambda = \rho v \left|\mathscr{S}'\right| \quad (13)$$

where $v = |\overrightarrow{v}|$ and $\mathscr{S}'$ is the beam projection on a plane perpendicular to $\overrightarrow{v}$ (Fig.1).

Consider a rectangular beam as displayed in Fig. 1 such that $\mathscr{S}' = l_1 l_2$. The lightening duration $L_k = h/v$ of particle k is constant, so that, using (10) and taking into account that the arrivals are uniformly distributed on $\mathscr{S}'$:

$$\Delta_{pq}(\tau) = \begin{cases} \exp[\rho\mu_{pq} l_1 l_2 (h/v - \tau)] - 1 & \tau < h/v \\ 0 & \tau > h/v \end{cases} \quad (14)$$

Equation (14) provides estimates of the celerity v and of the product $\rho\mu_{pq}$. μ_{pq} depends on the particles and on the laser wavelength through the $\mathscr{E}_k$ in (1). μ_{pq} has to be determined by an appropriate device calibration before measurements. Indeed it depends only on the particle size which is part of the required a priori knowledge on the physical phenomenon. The concentration ρ can then be measured from the shape of the higher order scintillation index as a function of τ. In particular, the slope of the higher order scintillation index logarithm is proportional to ρ.

There exist various laser beam profiles, but which do not generally lead to a simplified expression of (10). However, whatever the profile, $\Delta_{pq}(\tau)$ exhibits an angular point for $\tau = \tau_0 > 0$. τ_0 corresponds to the maximum service time and varies like $1/v$. This provides an estimate of the fluid and particle velocity.

Figure 2 displays the higher order scintillation index provided in Eq. (14) for different values of p and q and a given concentration $\rho = 10$. The distribution of the laser beam attenuation induced by a given particle A_k is determined from the particle size distribution. This distribution is log-normal in many applications (see [10] for instance). However, the proposed measurement method only requires the first and second order statistics of the laser beam attenuation. The simulations have been performed with mean $E[A_k] = 0.1$ and standard deviation $\sigma_{A_k} = 0.01$. The parameter μ_{pq} is estimated through 1000 runs. The area $\mathscr{S}' = l_1 l_2$ is normalized. The curves, displayed in logarithmic scale, show the approximately linear variation of the higher order scintillation index up to the point $\tau = h/v = 0.8$. This point position provides an estimate of the particle velocity. The line slopes are proportional to the particle concentration ρ.

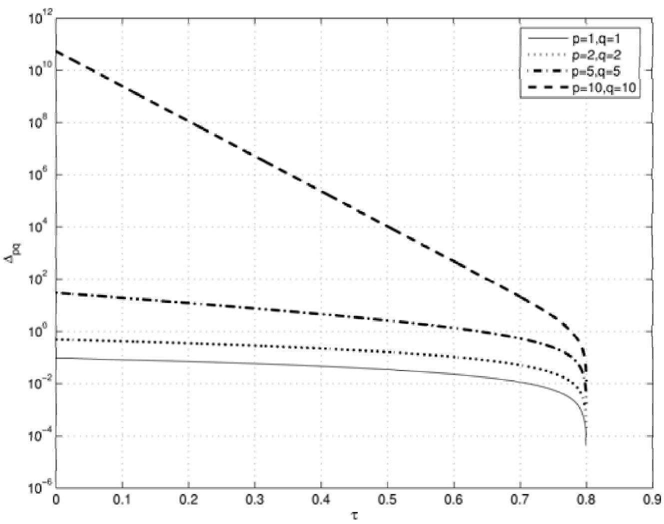

Figure 2: Higher order scintillation indices

4.2 Small particles

Formulas (10) and (14) depend on the embedded particle geometrical properties through parameter μ_{pq}. When particles are small with respect to the beam section ($A_k \approx 1$ for all k), the following approximation holds :

$$\mu_{pq} = E[(1-A_k^p)(1-A_k^q)] \approx pqE[(1-A_k)^2] \qquad (15)$$

Furthermore, if the particle are identical, the attenuation A_k can be identified to a constant a. Then $m_n = a^n$ and parameter μ_{pq} can be approximated as follows:

$$\mu_{pq} = (1-a^p)(1-a^q) \approx pq(1-a)^2 \qquad (16)$$

4.3 Circular cylindrical laser profile

As shown previously, a rectangular laser profile perpendicular to the constant particle velocity, leads to a constant lightening duration. However, for a constant velocity but a non rectangular laser profile, the lightening duration is a random variable. The case of a circular cylindrical beam is the most ubiquitous. In this case, the lightening duration is determined by the place where the particle enters the beam. This duration is maximum when the particle trajectory cuts the cylinder axis and is minimum at a distance equal to the cylinder radius r. Assume that the particle with index k crosses the beam at the distance x_k of the cylinder axis ($-r < x_k < r$), the lightening duration is:

$$L_k = \frac{2r}{v}\sqrt{1-\left(\frac{x_k}{r}\right)^2}. \qquad (17)$$

Now suppose that the particle density ρ is uniform. The lightening duration probability density function is given by:

$$g(u) = \sqrt{1-\left(\frac{vu}{2r}\right)^2},\ |u| < \frac{2r}{v}. \qquad (18)$$

The lightening duration distribution can be obtained whatever the laser beam profile but the derivations are generally not straightforward.

5. CONCLUSION

In this paper, particles crossing a laser beam have been modeled as customers in a queueing process with an infinite number of stations and Poissonian system inputs. The service time is the time spent by the particle in the beam. This time depends on the fluid velocity and on the beam profile. If the entering times are Poissonian, the queueing process is $M/G/\infty$, and associated statistics are well-known.
This paper relates these statistics to the physical phenomenon and proposes extinction measurements based on the higher order statistics $E\left[I^p(t)I^q(t+\tau)\right]$ and the generalized scintillation indices $\Delta_{pq}(\tau)$ which are related to these moments. We have shown that the $\Delta_{pq}(\tau)$ belong to the same class and are positive definite functions with positive Fourier transforms. Moreover, simplified expressions have been provided in some cases of interest. The properties of the generalized scintillation index may allow to check the model validity from practical measurements. This paper is theoretical as an extension of papers [2] and [3]. It provides a general framework that can be applied to many different measurement problems. Indeed, a similar queueing model has been used for LII and LDA [8] and could be extended to more complex systems. Higher order statistics could be studied in those cases as well.

6. APPENDIX

1) This appendix demonstrates the relation (10). Some useful properties are recalled in the second part of the appendix. Recall that $\mathbf{J}_t$ is the set of illuminated particles at time t. Let define the following sets:

$$\mathbf{B} = \mathbf{J}_t \cap \mathbf{J}_{t+\tau} = \left\{k; t_k < t, t'_k \geq t+\tau\right\} \qquad (19)$$

$$\mathbf{C} = \mathbf{J}_t \cap \overline{\mathbf{J}}_{t+\tau} = \left\{k; t_k < t, t \leq t'_k < t+\tau\right\} \qquad (20)$$

$$\mathbf{D} = \overline{\mathbf{J}}_t \cap \mathbf{J}_{t+\tau} = \left\{k; t \leq t_k < t+\tau, t'_k \geq t+\tau\right\} \qquad (21)$$

Let $K_{p,q}(t,\tau)$ denote $E\left[I^p(t)I^q(t+\tau)\right]$. With the previous definitions:

$$K_{p,q}(t,\tau) = E\left[\prod_{k\in\mathbf{B}} A_k^{p+q} \prod_{j\in\mathbf{C}} A_j^p \prod_{m\in\mathbf{D}} A_m^q\right]. \qquad (22)$$

A basic property of the HPP is the independence of $N(0,t)$ and $N(t,\tau)$. This implies the independence of $|\mathbf{B}|$ and $|\mathbf{C}|$ with respect to $|\mathbf{D}|$.
Using conditional expectations, (22) leads to

$$K_{p,q}(t,\tau) = E\left[m_{p+q}^{|\mathbf{B}|} m_p^{|\mathbf{C}|}\right] E\left[m_q^{|\mathbf{D}|}\right]$$

where $m_n = E[A^n]$. Conditionally to the event "$N(0,t) = n$", the random variables t_k in the interval $(0,t)$ are independent and uniformly distributed (when the indices k take values in $\{1,...,n\}$ with probability $\frac{1}{n}$). Consequently, the two-dimensional random variable $(|\mathbf{B}|,|\mathbf{C}|\,|N(0,t)=n)$ follows a trinomial distribution with respective parameters [6]:

$$b_t = \frac{1}{t}\int_0^t \Pr\left[t'_k - t_k > t+\tau-u\right] du = \frac{1}{t}\int_0^t [1-G(u+\tau)]\,du \qquad (23)$$

and

$$c_t = \frac{1}{t}\int_0^t \Pr\left[t-u < t'_k - t_k > t+\tau-u\right] du = \frac{1}{t}\int_0^t [G(u+\tau)-G(u)]\,du \qquad (24)$$

Also, $(|\mathbf{D}|\,|N(t,\tau)=n)$ is binomial with parameter:

$$d = \frac{1}{\tau}\int_t^{t+\tau} \Pr\left[t'_k - t_k > t+\tau-u\right] du = \frac{1}{\tau}\int_0^\tau [1-G(u)]\,du. \qquad (25)$$

The multinomial generating function yields [7],[15]:

$$K_{p,q}(t,\tau) = I_0^{p+q} E\left[(m_{p+q}b_t + m_p c_t + 1 - b_t - c_t)^{N(0,t)}\right] \times E\left[(m_q d + 1 - d)^{N(t,\tau)}\right] \qquad (26)$$

Since $N(0,t)$ and $N(t,\tau)$ are Poisson distributed with respective parameters λt and $\lambda\tau$, equation (26) leads to:

$$K_{p,q}(t,\tau) = I_0^{p+q}\exp[\lambda t(m_{p+q}b_t + m_p c_t - b_t - c_t) + \lambda\tau d(m_q - 1)] \quad (27)$$

The stationary limit is obtained when $t \to \infty$. Then

$$\begin{aligned} K_{p,q}(\tau) &= \lim_{t\to\infty} K_{p,q}(t,\tau) \\ &= I_0^{p+q}\exp\left[\alpha + \lambda\mu_{pq}\int_{\tau}^{\infty}[1-G(u)]du\right] \quad (28) \end{aligned}$$

with:

$$\alpha = \lambda E\left[t'_k - t_k\right](m_p + m_q - 2) \quad (29)$$

2) The previous proofs use properties of the Poisson process and multinomial probability distribution [6], [15]. The HPP definition is recalled in section 2 [5]. The main hypothesis is the independence of the random variables $N(t,\tau)$ and $N(t',\tau')$ when the set $(t,\tau)\cap(t',\tau')$ is empty. The properties of the particle queueing process are closely dependent of this hypothesis validity. The hypothesis of stationarity as well as property (5) may be unnecessary. Note that the stationarity hypothesis can be questioned for periodic excitation as gas emission in diesel engines [18]. However, the cycle is generally short with respect to the measurement duration. Consequently, the non-stationarity can be neglected.
The property (5) can be also discussed because of the possible particle aggregation. However, this phenomena mainly lead to the particle dimension modification. As highlighted in this paper, the use of higher order statistics allows to validate the basic hypotheses implied in the HPP definition.
The trinomial distribution generalizes the binomial distribution which concerns experiments with two probabilities [7]. For instance, let consider n independent experiments with three possible issues I, J and K with respective probabilities p_I, p_J and $p_K = 1 - p_I - p_J$. If X (respectively Y and Z) is of issues in I (respectively J and K), then a straightforward counting leads to:

$$Pr[X = i, Y = j] = \frac{n!}{i!j!(n-i-j)!} p_I^i p_J^j p_K^{n-i-j} \quad (30)$$

where $0 \le i + j \le n$.
The binomial and trinomial distributions are particular cases of the multinomial distribution, where the number of issues is arbitrary. The generating function can be deduced from the previous definition using the generalization of the binomial formula

$$\begin{aligned} E\left[x^X y^Y\right] &= \sum_{0\le i+j\le n} \frac{n!}{i!j!(n-i-j)!}(p_I x)^i (p_J y)^j p_K^{n-i-j} \\ &= (p_I x + p_J y + p_K)^n \quad (31) \end{aligned}$$

This formula is used above in the particular case where:

$$X = |B|, Y = |C|, p_I = m_{p+q}, p_J = m_p. \quad (32)$$

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