



# HESITANT Triangular Neutrosophic Numbers and Their Applications to MADM

Abhijit Saha <sup>1\*</sup>, Irfan Deli<sup>2</sup>, and Said Broumi <sup>3</sup>

<sup>1</sup> Faculty of Mathematics, Techno College of Engineering Agartala, , Tripura, India, Pin-799004; Email:

abhijit84.math@gmail.com

<sup>2</sup> Muallim Rifat Faculty of Education, Kilis 7 Aralık University, 79000 Kilis, Turkey. E-mail: irfandeli@kilis.edu.tr

<sup>3</sup> Faculty of Science, University of Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco; Email: broumisaid78@gmail.com

\*Correspondence: abhijit84.math@gmail.com

**ABSTRACT:** Hesitant neutrosophic sets can accomodate more uncertainty compare to hesitant fuzzy sets and hesitant intuitionistic sets. On the other hand, triangular neutrosophic numbers are often used by the decision makers to evaluate their opinion in multi-attribute group decision making problems. Based on the combination of triangular neutrosophic numbers and hesitant neutrosophic sets, in this paper, we propose hesitant triangular neutrosophic numbers. Also, we discuss various types of operations between them including some properties. Then, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce score of hesitant triangular neutrosophic numbers to ranking the hesitant triangular neutrosophic numbers. Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making (MADM) approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, we give a practical example, including a comparision study with the other existing method, for enterprise resource planning system selection to verify the application and effectiveness of the proposed method.

**Keywords:** Neutrosophic sets, hesitant triangular neutrosophic numbers, aggregation operators, score value, decision making.

## 1. INTRODUCTION

In our real life, most of the mathematical problems do not contain exact or complete information about the given mathematical modeling. Therefore, fuzzy set theory by introduced Zadeh [01] is a proper tool to process inexact information because it allows the partial belongings of an element in a set with a membership function. Atanassov [02] generalized fuzzy sets to intuitionistic fuzzy sets by adding a non-membership function to overcome problems that contain incomplete information. In case of fuzzy sets and intuitionistic fuzzy sets, the membership (or non-membership) value of an element in a set is a unique value in the closed interval  $[0, 1]$ . But since 2009, researchers begin to investigate, what if the membership (non-membership) value of an element in a set is a discrete finite subset of  $[0, 1]$ . In order to tackle this situation, Torra [03] proposed the concept of a hesitant fuzzy set, which as an extension of a fuzzy set arises from our hesitation among a few different values lying

between the number 0 and 1. Thus the hesitant fuzzy set can more accurately reflect the people's hesitancy in stating their preferences over objectives compared to the fuzzy set and its classical extensions. Beg and Rashid [04] introduced the concept of intuitionistic hesitant fuzzy sets by merging the concept of intuitionistic fuzzy sets and hesitant fuzzy sets. Various researchers have analyzed the decision making problems under fuzzy, hesitant fuzzy, intuitionistic fuzzy and intuitionistic hesitant fuzzy environment in Li [05], Ye [06], Xia and Xu [07], Xu and Xia [08], Wei et al. [09], Xu and Xia [10], Xu and Xia [11], Xu and Zhang [12], Chen et al. [13], Qian et al. [14], Yu [15], Yu [16], Ye [17], Shi et al. [18], Pathinathan and Johnson [19], Joshi and Kumar [20], Liu [21], Nehi [22], Zhang [23], Chen and Huang [24], Yang et al. [25], Lan et al. [26] and Zhang et al. [27].

Although intuitionistic fuzzy sets naturally include hesitancy degree to handle uncertain information, it cannot manage indeterminate information properly because it is dependent on membership and non-membership degrees. To handle this situation, Smarandache [28] introduced the neutrosophic set which is basically a powerful general formal framework that generalizes the concept of the classical set, fuzzy set, intuitionistic fuzzy set. A neutrosophic set is characterized explicitly by truth-membership function, indeterminacy-membership function and falsity membership function and it has applications on image segmentation in Gou and Cheng [29], Gou and Sensur [30], on clustering analysis in Karaaslan [31], on medical diagnosis problem in Ansari et al. [32] etc. The neutrosophic set theory have also studied in Wang et al. [33], Wang et al. [34], Gou et al. [35], Ye [36], Sun et al. [37], Ye [38] and Abdel Basset et al. [39]. The neutrosophic set cannot represent uncertain, imprecise, incomplete and inconsistent information with a few different values assigned by truth-membership degree, indeterminacy-membership degree and falsity-membership degree due to doubts of decision maker. In such a situation, all the decision making algorithms based on neutrosophic sets are difficult to use for such a decision making problem with three kinds of hesitancy information that exists in the real world. To overcome this situation, Ye [40] introduced the concept of hesitant neutrosophic sets which is characterized by three membership degrees, namely-truth membership degrees, indeterminacy membership degrees and falsity membership degrees which is a few different values lying between the number 0 and 1.

Aggregation operators play a vital role in many fields such as decision making, supply chain, personnel evaluation and financial investment to solve multi-criteria group decision making problems. A series of aggregation operators in Xia et al. [41], Wang et al. [42], Zhao et al. [43], and Peng [44] were developed based on fuzzy and hesitant fuzzy information and those were applied in solving decision-making problems. Xu [45], Wan and Dong [46], Wan et al. [47] and Xu and Yager [48] presented an averaging and geometric aggregation operators for aggregating the different intuitionistic fuzzy sets based information. Wang and Liu [49] proposed some Einstein weighted geometric operators for intuitionistic fuzzy sets. Liu et al. [50] proposed some generalized neutrosophic number Hamacher aggregation operators. Liu and Wang [51] defined few neutrosophic normalized, weighted Bonferroni mean operators. Chen and Ye [52] used single-valued neutrosophic dombi weighted aggregation operators for solving a multiple attribute decision-making problem. Some more aggregation operators on neutrosophic environment can be found in Zhao et al. [53], Liu and Shi [54] and Liu and Tang [55].

Since Smarandache put forward the concept of neutrosophic sets, the neutrosophic number is given by Şubaş [56] subsequently, and it has been made much deeper by many authors in Abdel-Basset [57]. As a special neutrosophic number, Şubaş gave two special forms of single valued neutrosophic numbers such as single valued trapezoidal neutrosophic numbers and single valued triangular neutrosophic numbers on the real number set  $R$ . Now the theory of neutrosophic number has become the fundamental of neutrosophic decision making. For example; Deli and Şubaş [58] introduced the concepts of cut sets of neutrosophic numbers and also they applied to single valued trapezoidal neutrosophic numbers and triangular neutrosophic numbers. Finally they presented a ranking method by defining the values and ambiguities of neutrosophic numbers. Also, by using the value and ambiguity index, Biswas et al. [59] presented a multi-attribute decision making method. Broumi et al. [60] gave an application shortest path problem under triangular fuzzy neutrosophic numbers. Deli and Şubaş [61] developed an approach to handle multicriteria decision making

problems under the single valued triangular neutrosophic numbers. Also, they presented some new geometric operators including weighted geometric operator, ordered weighted geometric operator and ordered hybrid weighted geometric operator. Ye [62], Biswas et al. [63] and Deli [64] proposed some weighted arithmetic operators and weighted geometric operators to present some multi attribute decision making methods. Karaaslan [65] introduced Gaussian single valued neutrosophic numbers and applied to a multi attribute decision making. Öztürk [66] and Deli and Öztürk [67, 68] initiated concept of distance measure based on cut sets, magnitude function, 1. and 2. centroid point and 1. and 2. score function. Deli [69] defined concept of centroid point based on single valued trapezoidal neutrosophic numbers and examine several useful properties. Also, he developed hamming ranking value and Euclidean ranking value of single valued trapezoidal neutrosophic numbers. Chakraborty et al. [70] presented a decision making method by introducing different forms of triangular neutrosophic numbers including de-neutrosophication techniques. Fan et al. [71] defined linguistic neutrosophic number Einstein sum, linguistic neutrosophic number Einstein product, and linguistic neutrosophic number Einstein exponentiation operations based on the Einstein operation and used them to develop some MADM problems. Garg and Nancy [72] introduced some linguistic single valued neutrosophic power aggregation operators and presented their applications to group decision making process. Zhao et al. [73] developed induced choquet integral aggregation operators with single valued neutrosophic uncertain linguistic numbers. Recently, Deli and Karaaslan [74] defined generalized trapezoidal hesitant fuzzy numbers and Deli [75] presented a TOPSIS method for multi-criteria decision making problems by using the numbers. Some more trapezoidal/triangular hesitant fuzzy numbers can be found in Zhang et al. [76] and Ye [77].

Motivated by the idea of triangular neutrosophic number, hesitant neutrosophic set and aggregation operators, the aim of this present article is:

- (1) To present the idea of hesitant triangular neutrosophic numbers.
- (2) To define few operations between hesitant triangular neutrosophic numbers and study their basic properties.
- (3) To develop a few weighted aggregation operators such as hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-1, hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2, hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 and hesitant triangular neutrosophic weighted geometric aggregation operator of type-2.
- (4) To propose a decision making method based on the hesitant triangular neutrosophic weighted aggregation operators to handle multicriteria decision making problems with hesitant triangular neutrosophic information.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we propose hesitant triangular neutrosophic number and illustrate it with an example. Also, we discuss various types of operations between them including some properties. In section 4, we propose various types of hesitant triangular neutrosophic weighted aggregation operators to aggregate the hesitant triangular neutrosophic information. Furthermore, we introduce the score of a hesitant triangular neutrosophic number to ranking the hesitant triangular neutrosophic numbers. In section 5, based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the best (most desirable) one. Also, we present a practical example for enterprise resource planning system selection to demonstrate the application and effectiveness of the proposed method. Section 6 is devoted for comparative study. In final section, we present the conclusion of the study.

## 2. PRELIMINARIES:

A neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra and is a powerful general formal framework that generalizes the traditional mathematical tools such as fuzzy sets and intuitionistic fuzzy sets.

**Definition 1: [34]** A single-valued neutrosophic set  $A$  on universe set  $E$  is given by

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in E \}$$

where  $T_A: E \rightarrow [0,1]$ ,  $I_A: E \rightarrow [0,1]$ , and  $F_A: E \rightarrow [0,1]$  satisfy the condition  $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ , for every  $x \in E$ . The functions  $T_A$ ,  $I_A$ , and  $F_A$  define the degree of truth-membership function, indeterminacy-membership function and falsity-membership function, respectively.

**Definition 2: [52]**  $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in E \}$  and  $B = \{ \langle x, T_B(x), I_B(x), F_B(x) \rangle : x \in E \}$  be two single-valued neutrosophic sets and  $\lambda \neq 0$ . Then,

1.  $A + B = \{ \langle x, 1 - \frac{1}{1 + \left\{ \left( \frac{T_A(x)}{1 - T_A(x)} \right)^p + \left( \frac{T_B(x)}{1 - T_B(x)} \right)^p \right\}^{\frac{1}{p}}}, \frac{1}{1 + \left\{ \left( \frac{1 - I_A(x)}{I_A(x)} \right)^p + \left( \frac{1 - I_B(x)}{I_B(x)} \right)^p \right\}^{\frac{1}{p}}}, \frac{1}{1 + \left\{ \left( \frac{1 - F_A(x)}{F_A(x)} \right)^p + \left( \frac{1 - F_B(x)}{F_B(x)} \right)^p \right\}^{\frac{1}{p}}} \rangle : x \in E \}$
2.  $A \times B = \{ \langle x, \frac{1}{1 + \left\{ \left( \frac{1 - T_A(x)}{T_A(x)} \right)^p + \left( \frac{1 - T_B(x)}{T_B(x)} \right)^p \right\}^{\frac{1}{p}}}, 1 - \frac{1}{1 + \left\{ \left( \frac{I_A(x)}{1 - I_A(x)} \right)^p + \left( \frac{I_B(x)}{1 - I_B(x)} \right)^p \right\}^{\frac{1}{p}}}, 1 - \frac{1}{1 + \left\{ \left( \frac{F_A(x)}{1 - F_A(x)} \right)^p + \left( \frac{F_B(x)}{1 - F_B(x)} \right)^p \right\}^{\frac{1}{p}}} \rangle : x \in E \}$
3.  $\lambda.A = \{ \langle x, 1 - \frac{1}{1 + \left\{ \lambda \left( \frac{T_A(x)}{1 - T_A(x)} \right)^p \right\}^{\frac{1}{p}}}, \frac{1}{1 + \left\{ \lambda \left( \frac{1 - I_A(x)}{I_A(x)} \right)^p \right\}^{\frac{1}{p}}}, \frac{1}{1 + \left\{ \lambda \left( \frac{1 - F_A(x)}{F_A(x)} \right)^p \right\}^{\frac{1}{p}}} \rangle : x \in E \}$
4.  $A^\lambda = \{ \langle x, \frac{1}{1 + \left\{ \lambda \left( \frac{1 - T_A(x)}{T_A(x)} \right)^p \right\}^{\frac{1}{p}}}, 1 - \frac{1}{1 + \left\{ \lambda \left( \frac{I_A(x)}{1 - I_A(x)} \right)^p \right\}^{\frac{1}{p}}}, 1 - \frac{1}{1 + \left\{ \lambda \left( \frac{F_A(x)}{1 - F_A(x)} \right)^p \right\}^{\frac{1}{p}}} \rangle : x \in E \}$

By combining single-valued neutrosophic sets and hesitant fuzzy sets, Ye (2015a) introduced the single-valued neutrosophic hesitant fuzzy set as a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single-valued neutrosophic set. He also developed single-valued neutrosophic hesitant fuzzy weighted averaging operator and single-valued neutrosophic hesitant fuzzy weighted geometric operator and applied them to solve a multiple-attribute decision-making problem.

**Definition 3: [40]** A hesitant neutrosophic set on universe set  $E$  is given by

$$N = \{ \langle x, \tilde{T}_N(x), \tilde{I}_N(x), \tilde{F}_N(x) \rangle : x \in E \}$$

in which  $\tilde{T}_N(x)$ ,  $\tilde{I}_N(x)$  and  $\tilde{F}_N(x)$  are three sets of some values in  $[0,1]$ , denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element  $x \in E$  to the set  $N$ , respectively, with the conditions  $0 \leq \delta, \gamma, \eta \leq 1$  and  $0 \leq \delta^+ + \gamma^+ + \eta^+ \leq 3$ , where

$$\delta \in \tilde{T}_N(x), \gamma \in \tilde{I}_N(x), \eta \in \tilde{F}_N(x), \delta^+ \in \tilde{T}_N^+(x) = \bigcup_{\delta \in \tilde{T}_N(x)} \max\{\delta\}, \gamma^+ \in \tilde{I}_N^+(x) = \bigcup_{\gamma \in \tilde{I}_N(x)} \max\{\gamma\}, \text{ and } \eta^+ \in \tilde{F}_N^+(x) = \bigcup_{\eta \in \tilde{F}_N(x)} \max\{\eta\}, \text{ for } x \in E.$$

For  $N_1 = \{ \langle x, \tilde{T}_{N_1}(x), \tilde{I}_{N_1}(x), \tilde{F}_{N_1}(x) \rangle : x \in E \}$  and  $N_2 = \{ \langle x, \tilde{T}_{N_2}(x), \tilde{I}_{N_2}(x), \tilde{F}_{N_2}(x) \rangle : x \in E \}$  be two hesitant neutrosophic sets and  $\lambda \neq 0$ . Then,

$$\begin{aligned}
1. N_1 \oplus N_1 &= \{ \langle x, \tilde{T}_{N_1}(x) \oplus \tilde{T}_{N_2}(x), \tilde{I}_{N_1}(x) \oplus \tilde{I}_{N_2}(x), \tilde{F}_{N_1}(x) \oplus \tilde{F}_{N_2}(x) \rangle : x \in X \} \\
&= \bigcup_{\substack{\delta_1 \in \tilde{T}_{N_1}(x), \delta_2 \in \tilde{T}_{N_2}(x), \gamma_1 \in \tilde{I}_{N_1}(x), \gamma_2 \in \tilde{I}_{N_2}(x), \eta_1 \in \tilde{F}_{N_1}(x), \eta_2 \in \tilde{F}_{N_2}(x)}} \{ \langle x, \{\delta_1 + \delta_2 - \delta_1 \cdot \delta_2\}, \{\gamma_1 \cdot \gamma_2\}, \{\eta_1 \eta_2\} \rangle : x \in X \} \\
2. N_1 \otimes N_1 &= \{ \langle x, \tilde{T}_{N_1}(x) \otimes \tilde{T}_{N_2}(x), \tilde{I}_{N_1}(x) \otimes \tilde{I}_{N_2}(x), \tilde{F}_{N_1}(x) \otimes \tilde{F}_{N_2}(x) \rangle : x \in X \} \\
&= \bigcup_{\substack{\delta_1 \in \tilde{T}_{N_1}(x), \delta_2 \in \tilde{T}_{N_2}(x), \gamma_1 \in \tilde{I}_{N_1}(x), \gamma_2 \in \tilde{I}_{N_2}(x), \eta_1 \in \tilde{F}_{N_1}(x), \eta_2 \in \tilde{F}_{N_2}(x)}} \{ \langle x, \{\delta_1 \cdot \delta_2\}, \{\gamma_1 + \gamma_2 - \gamma_1 \cdot \gamma_2\}, \{\eta_1 + \eta_2 - \eta_1 \eta_2\} \rangle : x \in X \} \\
3. \lambda.N_1 &= \bigcup_{\substack{\delta_1 \in \tilde{T}_{N_1}(x), \gamma_1 \in \tilde{I}_{N_1}(x), \eta_1 \in \tilde{F}_{N_1}(x)}} \{ \langle x, \{1 - (1 - \delta_1)^\lambda\}, \{\gamma_1^\lambda\}, \{\eta_1^\lambda\} \rangle : x \in X \} (\lambda > 0) \\
4. N_1^\lambda &= \bigcup_{\substack{\delta_1 \in \tilde{T}_{N_1}(x), \gamma_1 \in \tilde{I}_{N_1}(x), \eta_1 \in \tilde{F}_{N_1}(x)}} \{ \langle x, \{\delta_1^\lambda\}, \{1 - (1 - \gamma_1)^\lambda\}, \{1 - (1 - \eta_1)^\lambda\} \rangle : x \in X \} (\lambda > 0)
\end{aligned}$$

**Definition 4:** [56] Let  $a_1 \leq b_1 \leq c_1$  such that  $a_1, b_1, c_1 \in R$ . A triangular neutrosophic number  $\tilde{A} = \langle (a_1, b_1, c_1); w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}} \rangle$  is a special neutrosophic set on the real number set  $R$ , whose truth-membership function  $\mu_{\tilde{A}} : R \rightarrow [0, w_{\tilde{A}}]$ , indeterminacy-membership function  $\nu_{\tilde{A}} : R \rightarrow [u_{\tilde{A}}, 1]$  and falsity-membership function  $\lambda_{\tilde{A}} : R \rightarrow [y_{\tilde{A}}, 1]$  are given as follows;

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a_1)w_{\tilde{A}}}{b_1-a_1}, & a_1 \leq x \leq b_1 \\ \frac{(c_1-x)w_{\tilde{A}}}{c_1-b_1}, & b_1 \leq x \leq c_1 \\ 0, & \text{otherwise} \end{cases}, \quad \nu_{\tilde{A}}(x) = \begin{cases} \frac{b_1-x+u_{\tilde{A}}(x-a_1)}{b_1-a_1}, & a_1 \leq x \leq b_1 \\ \frac{x-b_1+u_{\tilde{A}}(c_1-x)}{c_1-b_1}, & b_1 \leq x \leq c_1 \\ 1, & \text{otherwise} \end{cases}$$

$$\lambda_{\tilde{A}}(x) = \begin{cases} \frac{b_1-x+y_{\tilde{A}}(x-a_1)}{b_1-a_1}, & a_1 \leq x \leq b_1 \\ \frac{x-b_1+y_{\tilde{A}}(c_1-x)}{c_1-b_1}, & b_1 \leq x \leq c_1 \\ 1, & \text{otherwise} \end{cases}$$

Since triangular neutrosophic numbers ([56], [58]) is a special case of trapezoidal neutrosophic numbers (Ye 2017), operations of trapezoidal neutrosophic numbers (Ye 2015b, 2017) based on algebraic sum and algebraic product for triangular neutrosophic numbers can be given as;

If  $\tilde{A} = \langle (a_1, b_1, c_1); w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}} \rangle$  and  $\tilde{B} = \langle (a_2, b_2, c_2); w_{\tilde{B}}, u_{\tilde{B}}, y_{\tilde{B}} \rangle$  be two triangular neutrosophic numbers and  $\gamma \neq 0$ , then we have

$$\begin{aligned}
1. \tilde{A} + \tilde{B} &= \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); w_{\tilde{A}} + w_{\tilde{B}} - w_{\tilde{A}} \cdot w_{\tilde{B}}, u_{\tilde{A}} \cdot u_{\tilde{B}}, y_{\tilde{A}} \cdot y_{\tilde{B}} \rangle \\
2. \tilde{A} \cdot \tilde{B} &= \langle (a_1 a_2, b_1 b_2, c_1 c_2); w_{\tilde{A}} \cdot w_{\tilde{B}}, u_{\tilde{A}} + u_{\tilde{B}} - u_{\tilde{A}} \cdot u_{\tilde{B}}, y_{\tilde{A}} + y_{\tilde{B}} - y_{\tilde{A}} \cdot y_{\tilde{B}} \rangle \\
3. \lambda \tilde{A} &= \langle (\gamma a_1, \gamma b_1, \gamma c_1); 1 - (1 - w_{\tilde{A}})^\lambda, u_{\tilde{A}}^\lambda, y_{\tilde{A}}^\lambda \rangle \\
4. \tilde{A}^\lambda &= \langle (a_1^\lambda, b_1^\lambda, c_1^\lambda); w_{\tilde{A}}^\lambda, 1 - (1 - u_{\tilde{A}})^\lambda, 1 - (1 - y_{\tilde{A}})^\lambda \rangle
\end{aligned}$$

**Definition 5:** [56] Let  $\tilde{A} = \langle (a, b, c); w_{\tilde{A}}, u_{\tilde{A}}, y_{\tilde{A}} \rangle$  be a triangular neutrosophic number. Then, score function of  $\tilde{A}$ , is denoted by  $S_Y(\tilde{A})$ , is defined as:

$$S_Y(\tilde{A}) = \frac{1}{8} [a + b + c] \times (2 + \mu_{\tilde{A}} - \nu_{\tilde{A}} - \gamma_{\tilde{A}})$$

**Definition 6:** [61] Let  $\tilde{A}_j = \langle (a_j, b_j, c_j); w_{\tilde{A}_j}, u_{\tilde{A}_j}, y_{\tilde{A}_j} \rangle$  ( $j = 1, 2, \dots, n$ ) be a collection of triangular neutrosophic numbers. Then,

1. Triangular neutrosophic weighted arithmetic operatoris defined as;

$$N_{ao}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \sum_{j=1}^n w_j \tilde{A}_j$$

2. Triangular neutrosophic weighted geometric operatoris defined as;

$$N_{go}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \prod_{j=1}^n \tilde{A}_j^{w_j}$$

where,  $w = (w_1, w_2, \dots, w_n)^T$  is a weight vector associated with the  $N_{ao}$  or  $N_{go}$  operator, for every  $j$  ( $j = 1, 2, \dots, n$ ) and  $w_j \in [0, 1]$  with  $\sum_{j=1}^n w_j = 1$ .

### 3. HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, the concept of a hesitant triangular neutrosophic number is presented on the basis of the combination of triangular neutrosophic numbers and hesitant fuzzy sets as a further generalization of the concept of triangular neutrosophic numbers. A hesitant triangular neutrosophic number is a special hesitant neutrosophic set on the real number  $R$ , whose truth-membership function, indeterminacy-membership function and falsity-membership function are expressed by several possible functions.

**Definition 7.** Let  $a_1 \leq b_1 \leq c_1$  such that  $a_1, b_1, c_1 \in R$ ,  $w_a^i \in [0, 1] (i \in I_m = \{1, 2, \dots, m\})$ ,  $u_a^i \in [0, 1] (i \in I_n = \{1, 2, \dots, n\})$  and  $y_a^l \in [0, 1] (l \in I_k = \{1, 2, \dots, k\})$ . A hesitant triangular neutrosophic number  $\mathcal{H}_a = \langle (a_1, b_1, c_1); \{w_{\mathcal{H}_a}^i: i \in I_m\}, \{u_{\mathcal{H}_a}^j: j \in I_n\}, \{y_{\mathcal{H}_a}^l: l \in I_k\} \rangle$  is a special hesitant neutrosophic set on the real number  $R$ , whose truth-membership functions  $\mu_a^{HTRI}: R \rightarrow [0, w_a^i] (i \in I_m)$ , indeterminacy-membership function  $\gamma_a^{HTRI}: R \rightarrow [0, u_a^j] (j \in I_n)$  and falsity-membership function  $\delta_a^{HTRI}: R \rightarrow [0, y_a^l] (l \in I_k)$  are given as follows;

$$\begin{aligned} \mu_{\mathcal{H}_a}^{HTRI} &= \begin{cases} \bigcup_{\{a:a \in \{w_{\mathcal{H}_a}^i: i \in I_m\}\}} \left\{ \frac{(x - a_1)a}{b_1 - a_1} \right\}, & a_1 \leq x < b_1 \\ \{a : a \in \{w_{\mathcal{H}_a}^i: i \in I_m\}\}, & x = b_1 \\ \bigcup_{\{a:a \in \{w_{\mathcal{H}_a}^i: i \in I_m\}\}} \left\{ \frac{(c_1 - x)a}{c_1 - b_1} \right\}, & b_1 < x \leq c_1 \\ \{0\}, & \text{otherwise} \end{cases} \\ \gamma_{\mathcal{H}_a}^{HTRI} &= \begin{cases} \bigcup_{\{b:b \in \{u_{\mathcal{H}_a}^j: j \in I_n\}\}} \left\{ \frac{b_1 - x + b(x - a_1)}{b_1 - a_1} \right\}, & a_1 \leq x < b_1 \\ \{b : b \in \{u_{\mathcal{H}_a}^j: j \in I_n\}\}, & x = b_1 \\ \bigcup_{\{b:b \in \{u_{\mathcal{H}_a}^j: j \in I_n\}\}} \left\{ \frac{x - b_1 + b(c_1 - x)}{c_1 - b_1} \right\}, & b_1 < x \leq c_1 \\ \{1\}, & \text{otherwise} \end{cases} \\ \delta_{\mathcal{H}_a}^{HTRI} &= \begin{cases} \bigcup_{\{l:l \in \{y_{\mathcal{H}_a}^l: l \in I_k\}\}} \left\{ \frac{b_1 - x + l(x - a_1)}{b_1 - a_1} \right\}, & a_1 \leq x < b_1 \\ \{l : l \in \{y_{\mathcal{H}_a}^l: l \in I_k\}\}, & x = b_1 \\ \bigcup_{\{l:l \in \{y_{\mathcal{H}_a}^l: l \in I_k\}\}} \left\{ \frac{x - b_1 + l(c_1 - x)}{c_1 - b_1} \right\}, & b_1 < x \leq c_1 \\ \{1\}, & \text{otherwise} \end{cases} \end{aligned}$$

**Example 8.**  $\mathcal{H}_a = \langle (1, 2, 5); \{0.8, 0.9\}, \{0.4, 0.5, 0.6\}, \{0.4\} \rangle$  is a hesitant triangular neutrosophic number whose truth membership function, indeterminacy membership function and falsity membership function are given respectively by:

$$\begin{aligned}
 m_{\theta_0}^{HTRI}(x) &= \begin{cases} \{0.8(x-1), 0.9(x-1)\}, & 1 \leq x < 2 \\ \{0.8, 0.9\}, & x = 2 \\ \{0.8 \frac{(5-x)}{3}, 0.9 \frac{(5-x)}{3}\}, & 2 < x \leq 5 \\ \{0\}, & \text{otherwise} \end{cases}, & d_{\theta_0}^{HTRI}(x) &= \begin{cases} \{1.6-0.6x\}, & 1 \leq x < 2 \\ \{0.4\}, & x = 2 \\ \{0.2x\}, & 2 < x \leq 3 \\ \{1\}, & \text{otherwise} \end{cases} \\
 g_{\theta_0}^{HTRI}(x) &= \begin{cases} \{1.6-0.6x, 1.5-0.5x, 1.4-0.4x\}, & 1 \leq x < 2 \\ \{0.4, 0.5, 0.6\}, & x = 2 \\ \{\frac{0.6x}{3}, 0. \frac{5x+0.5}{3}, \frac{0.4x+1}{3}\}, & 2 < x \leq 5 \\ \{1\}, & \text{otherwise} \end{cases}
 \end{aligned}$$

#### 4. OPERATIONS ON HESITANT TRIANGULAR NEUTROSOPHIC NUMBERS:

In this section, we introduce various operations between hesitant triangular neutrosophic numbers and demonstrate their basic properties.

**Definition 9.** Let  $\theta_0 = \langle (a_1, b_1, c_1); \{w_{\theta_0}^i: i \in I_{m_1}\}, \{u_{\theta_0}^j: j \in I_{n_1}\}, \{y_{\theta_0}^l: l \in I_{k_1}\} \rangle$  and

$\theta_0' = \langle (a_2, b_2, c_2); \{w_{\theta_0'}^i: i \in I_{m_2}\}, \{u_{\theta_0'}^j: j \in I_{n_2}\}, \{y_{\theta_0'}^l: l \in I_{k_2}\} \rangle$  be two hesitant triangular neutrosophic numbers and  $s > 0$ , then

1.  $s \odot \theta_0 = \langle (s a_1, s b_1, s c_1); \{1 - (1 - a_1)^s: a_1 \in \{w_{\theta_0}^i: i \in I_{m_1}\}\}, \{1 - (1 - b_1)^s: b_1 \in \{u_{\theta_0}^j: j \in I_{n_1}\}\}, \{1 - (1 - c_1)^s: c_1 \in \{y_{\theta_0}^l: l \in I_{k_1}\}\} \rangle$
2.  $s \odot \theta_0' = \langle (s a_2, s b_2, s c_2); \{1 - (1 - a_2)^s: a_2 \in \{w_{\theta_0'}^i: i \in I_{m_2}\}\}, \{1 - (1 - b_2)^s: b_2 \in \{u_{\theta_0'}^j: j \in I_{n_2}\}\}, \{1 - (1 - c_2)^s: c_2 \in \{y_{\theta_0'}^l: l \in I_{k_2}\}\} \rangle$
3.  $s \odot \theta_0 \odot \theta_0' = \langle (s a, s b, s c); \{1 - (1 - a)^s: a \in \{w_{\theta_0}^i: i \in I_{m_1}\}\}, \{1 - (1 - b)^s: b \in \{u_{\theta_0}^j: j \in I_{n_1}\}\}, \{1 - (1 - c)^s: c \in \{y_{\theta_0}^l: l \in I_{k_1}\}\} \rangle$
4.  $s \odot \theta_0 = \langle (a^s, b^s, c^s); \{a^s: a \in \{w_{\theta_0}^i: i \in I_{m_1}\}\}, \{1 - (1 - b)^s: b \in \{u_{\theta_0}^j: j \in I_{n_1}\}\}, \{1 - (1 - c)^s: c \in \{y_{\theta_0}^l: l \in I_{k_1}\}\} \rangle$

**Theorem 10.** Let  $\theta_0 = \langle (a_1, b_1, c_1); \{w_{\theta_0}^i: i \in I_{m_1}\}, \{u_{\theta_0}^j: j \in I_{n_1}\}, \{y_{\theta_0}^l: l \in I_{k_1}\} \rangle$ ,

$\theta_0' = \langle (a_2, b_2, c_2); \{w_{\theta_0'}^i: i \in I_{m_2}\}, \{u_{\theta_0'}^j: j \in I_{n_2}\}, \{y_{\theta_0'}^l: l \in I_{k_2}\} \rangle$  and

$\theta_0'' = \langle (a_3, b_3, c_3); \{w_{\theta_0''}^i: i \in I_{m_3}\}, \{u_{\theta_0''}^j: j \in I_{n_3}\}, \{y_{\theta_0''}^l: l \in I_{k_3}\} \rangle$  be three hesitant triangular neutrosophic numbers

and  $s, s_1, s_2 > 0$ , then

1.  $s \odot \theta_0 = \theta_0 \odot s$
2.  $s \odot \theta_0' = \theta_0' \odot s$
3.  $s \odot (\theta_0 \odot \theta_0') = (s \odot \theta_0) \odot \theta_0'$
4.  $s \odot (\theta_0' \odot \theta_0'') = (\theta_0' \odot \theta_0'') \odot s$
5.  $s \odot (\theta_0 \odot \theta_0') = (s \odot \theta_0) \odot \theta_0'$
6.  $s \odot (\theta_0' \odot \theta_0'') = (s \odot \theta_0') \odot \theta_0''$
7.  $s \odot (\theta_0 \odot \theta_0'') = (s \odot \theta_0) \odot \theta_0''$
8.  $(s_1 + s_2) \odot \theta_0 = (s_1 \odot \theta_0) \odot (s_2 \odot \theta_0)$

**Proof:** 1-2 straight forward.

$$\begin{aligned}
3. \mathcal{A}(\mathcal{B}) &= \langle (a_1, b_1, c_1); w_{\mathcal{B}} u_{\mathcal{B}} y_{\mathcal{B}} \rangle \mathcal{A} \mathcal{B} \mathcal{C} = \langle (a_2 + a_3, b_2 + b_3, c_2 + c_3); \{a_2 + a_3 - a_2 a_3 : a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \\
&\quad a_3 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{b_2 b_3 : b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{l_2 l_3 : l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}, \\
&\quad l_3 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle \\
&= \langle (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3), c_1 + (c_2 + c_3)); \{a_1 + (a_2 + a_3 - a_2 a_3) - a_1(a_2 + a_3 - a_2 a_3) : \\
&\quad a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{b_1(b_2 b_3) : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : \\
&\quad j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{l_1(l_2 l_3) : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle \\
&= \langle (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{a_1 + a_2 + a_3 - a_2 a_3 - a_1 a_2 + a_1 a_3 - a_1 a_2 a_3\} : \\
&\quad a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{b_1 b_2 b_3 : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, \\
&\quad b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{l_1 l_2 l_3 : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}, \\
&\quad l_3 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle & (1)
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{A} \mathcal{B}) \mathcal{A} \mathcal{B} \mathcal{C} &= \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{a_1 + a_2 - a_1 a_2 : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \\
&\quad \{b_1 b_2 : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_1 l_2 : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle \\
&\mathcal{A} \mathcal{B} \mathcal{C} = \langle (a_3, b_3, c_3); \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle \\
&= \langle ((a_1 + a_2) + a_3, (b_1 + b_2) + b_3, (c_1 + c_2) + c_3); \{(a_1 + a_2 - a_1 a_2) + a_3 - (a_1 + b_1 - a_1 b_1) a_3 : \\
&\quad a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{(b_1 b_2) b_3 : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : \\
&\quad j \hat{I} I_{n_2}\} b_3 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{(l_1 l_2) l_3 : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle \\
&= \langle (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{a_1 + a_2 + a_3 - a_2 a_3 - a_1 a_2 + a_1 a_3 - a_1 a_2 a_3\} : \\
&\quad a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_3}\}, \{b_1 b_2 b_3 : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, \\
&\quad b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_3}\}, \{l_1 l_2 l_3 : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}, \\
&\quad l_3 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_3}\} \rangle & (2)
\end{aligned}$$

Hence from eq. 1-2, we have,  $\mathcal{A}(\mathcal{B}) \mathcal{A} \mathcal{B} \mathcal{C} = (\mathcal{A} \mathcal{B}) \mathcal{A} \mathcal{B} \mathcal{C}$ .

4. Proof is similar to 3

$$\begin{aligned}
5. s \in \mathcal{A}(\mathcal{B}) &= s \in \mathcal{C} \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{a_1 + a_2 - a_1 a_2 : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \\
&\quad \{b_1 b_2 : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_1 l_2 : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle \\
&= \langle (s(a_1 + a_2), s(b_1 + b_2), s(c_1 + c_2)); \{1 - (1 - (a_1 + a_2 - a_1 a_2))^s : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, \\
&\quad a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \{(b_1 b_2)^s : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{(l_1 l_2)^s : \\
&\quad l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle \\
&= \langle (s a_1 + s a_2, s b_1 + s b_2, s c_1 + s c_2); \{1 - (1 - a_1)^s (1 - a_2)^s : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \\
&\quad \{b_1^s b_2^s : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_1^s l_2^s : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle & (3)
\end{aligned}$$

and

$$\begin{aligned}
(s \in \mathcal{A}) \mathcal{A} \mathcal{B} \mathcal{C} &= \langle (s a_1, s b_1, s c_1); \{1 - (1 - a_1)^s : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, \{b_1^s : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, \\
&\quad \{l_1^s : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\} \rangle \mathcal{A} \mathcal{B} \mathcal{C} = \langle (s a_2, s b_2, s c_2); \{1 - (1 - a_2)^s : a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \\
&\quad \{b_2^s : b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_2^s : l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle \\
&= \langle (s a_1 + s a_2, s b_1 + s b_2, s c_1 + s c_2); \{(1 - (1 - a_1)^s) + (1 - (1 - a_2)^s) - \\
&\quad (1 - (1 - a_1)^s)(1 - (1 - a_2)^s) : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \{b_1^s b_2^s : b_1 \hat{I} \\
&\quad \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_1^s l_2^s : l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle \\
&= \langle (s a_1 + s a_2, s b_1 + s b_2, s c_1 + s c_2); \{1 - (1 - a_1)^s (1 - a_2)^s : a_1 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_1}\}, \\
&\quad a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \{b_1^s b_2^s : b_1 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{l_1^s l_2^s : \\
&\quad l_1 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} \rangle & (4)
\end{aligned}$$



Hence from eq. 3-4, we have  $s \in \mathcal{A}(\mathcal{A} \hat{\mathcal{A}} \mathcal{B}) = (s \in \mathcal{A}) \hat{\mathcal{A}} (s \in \mathcal{B})$ .

6. Proof is similar to 5

$$\begin{aligned} 7. (s_1 + s_2) \in \mathcal{A} &= ((s_1 + s_2)a_1, (s_1 + s_2)b_1, (s_1 + s_2)c_1); \{1 - (1 - a_1)^{s_1+s_2} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \{b_1^{s_1+s_2} : \\ & b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \{l_1^{s_1+s_2} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \\ &= < (s_1 a_1 + s_2 a_1, s_1 b_1 + s_2 b_1, s_1 c_1 + s_2 c_1); \{1 - (1 - a_1)^{s_1}(1 - a_1)^{s_2} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \\ & \{b_1^{s_1} b_1^{s_2} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \{l_1^{s_1} l_1^{s_2} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \end{aligned} \quad (5)$$

And

$$\begin{aligned} (s_1 \in \mathcal{A}) \hat{\mathcal{A}} (s_2 \in \mathcal{A}) &= < (s_1 a_1, s_1 b_1, s_1 c_1); \{1 - (1 - a_1)^{s_1} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \{b_1^{s_1} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \\ & \{l_1^{s_1} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \hat{\mathcal{A}} < (s_2 a_1, s_2 b_1, s_2 c_1); \{1 - (1 - a_1)^{s_2} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \\ & \{b_1^{s_2} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \{l_1^{s_2} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \\ &= < (s_1 a_1 + s_2 a_1, s_1 b_1 + s_2 b_1, s_1 c_1 + s_2 c_1); \{1 - (1 - a_1)^{s_1} + (1 - a_1)^{s_2} - \\ & (1 - a_1)^{s_1} (1 - a_1)^{s_2} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \{b_1^{s_1} b_1^{s_2} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \\ & \{l_1^{s_1} l_1^{s_2} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \\ &= < (s_1 a_1 + s_2 a_1, s_1 b_1 + s_2 b_1, s_1 c_1 + s_2 c_1); \{1 - (1 - a_1)^{s_1} (1 - a_1)^{s_2} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j\}, \\ & \{b_1^{s_1} b_1^{s_2} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i\}, \{l_1^{s_1} l_1^{s_2} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j\} \} > \end{aligned} \quad (6)$$

Hence from eq. 5-6, we have  $(s_1 + s_2) \in \mathcal{A} = (s_1 \in \mathcal{A}) \hat{\mathcal{A}} (s_2 \in \mathcal{A})$ .

8. Proof is similar to 7

**Definition 11.** Let  $\mathcal{A} = \langle (a_1, b_1, c_1); \{w_{\mathcal{A}}^i : i \hat{I}_{m_1}\}, \{u_{\mathcal{A}}^j : j \hat{I}_{n_1}\}, \{y_{\mathcal{A}}^l : l \hat{I}_{k_1}\} \rangle$  and

$\mathcal{B} = \langle (a_2, b_2, c_2); \{w_{\mathcal{B}}^i : i \hat{I}_{m_2}\}, \{u_{\mathcal{B}}^j : j \hat{I}_{n_2}\}, \{y_{\mathcal{B}}^l : l \hat{I}_{k_2}\} \rangle$  be two hesitant triangular neutrosophic numbers and  $s > 0$ , then

$$\begin{aligned} 1. \mathcal{A} \hat{\mathcal{A}} \mathcal{B} &= \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{1 - \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j, a_2 \hat{I}_{m_2}^i : i \hat{I}_{m_2}^j\}, \\ & \{ \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i, b_2 \hat{I}_{m_2}^j : j \hat{I}_{m_2}^i\}, \\ & \{ \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j, l_2 \hat{I}_{k_2}^l : l \hat{I}_{k_2}^j \} \} > \\ 2. \mathcal{A} \hat{\mathcal{A}} \mathcal{B} &= \langle (a_1 a_2, b_1 b_2, c_1 c_2); \{ \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : a_1 \hat{I}_{m_1}^i : i \hat{I}_{m_1}^j, a_2 \hat{I}_{m_2}^i : i \hat{I}_{m_2}^j\}, \\ & \{ 1 - \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : b_1 \hat{I}_{m_1}^j : j \hat{I}_{m_1}^i, b_2 \hat{I}_{m_2}^j : j \hat{I}_{m_2}^i\}, \\ & \{ 1 - \frac{1}{1 + \frac{a_1^2 \hat{\sigma}^2}{1 - a_1^2 \hat{\sigma}^2} + \frac{a_2^2 \hat{\sigma}^2}{1 - a_2^2 \hat{\sigma}^2}} : l_1 \hat{I}_{k_1}^l : l \hat{I}_{k_1}^j, l_2 \hat{I}_{k_2}^l : l \hat{I}_{k_2}^j \} \} > \end{aligned}$$

$$\begin{aligned}
3. s \in \mathcal{A} &= (s_{a_1}, s_{b_1}, s_{c_1}); \{1 - \frac{1}{1 + s \frac{a^2 - a^2}{1 - a^2}} : a_1 \hat{I} \{w_{\mathcal{A}}^i : i \hat{I} I_{m_1}\}\}, \{1 - \frac{1}{1 + s \frac{b^2 - b^2}{1 - b^2}} : b_1 \hat{I} \{u_{\mathcal{A}}^j : j \hat{I} I_{n_1}\}\}, \\
&\quad \{1 - \frac{1}{1 + s \frac{l^2 - l^2}{1 - l^2}} : l_1 \hat{I} \{y_{\mathcal{A}}^l : l \hat{I} I_{k_1}\}\} > \\
4. s \in \mathcal{B} &= (s_{a_2}, s_{b_2}, s_{c_2}); \{1 - \frac{1}{1 + s \frac{a^2 - a^2}{1 - a^2}} : a_2 \hat{I} \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}\}, \{1 - \frac{1}{1 + s \frac{b^2 - b^2}{1 - b^2}} : b_2 \hat{I} \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}\}, \\
&\quad \{1 - \frac{1}{1 + s \frac{l^2 - l^2}{1 - l^2}} : l_2 \hat{I} \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\}\} >
\end{aligned}$$

**Theorem 12.** Let  $\mathcal{A} = (a_1, b_1, c_1); \{w_{\mathcal{A}}^i : i \hat{I} I_{m_1}\}, \{u_{\mathcal{A}}^j : j \hat{I} I_{n_1}\}, \{y_{\mathcal{A}}^l : l \hat{I} I_{k_1}\} >$ ,

$\mathcal{B} = (a_2, b_2, c_2); \{w_{\mathcal{B}}^i : i \hat{I} I_{m_2}\}, \{u_{\mathcal{B}}^j : j \hat{I} I_{n_2}\}, \{y_{\mathcal{B}}^l : l \hat{I} I_{k_2}\} >$  and

$\mathcal{C} = (a_3, b_3, c_3); \{w_{\mathcal{C}}^i : i \hat{I} I_{m_3}\}, \{u_{\mathcal{C}}^j : j \hat{I} I_{n_3}\}, \{y_{\mathcal{C}}^l : l \hat{I} I_{k_3}\} >$  be three hesitant triangular neutrosophic numbers

and  $s, s_1, s_2 > 0$ , then

1.  $\mathcal{A} \dot{\mathcal{A}} \mathcal{B} = \mathcal{B} \dot{\mathcal{A}} \mathcal{A}$
2.  $\mathcal{A} \ddot{\mathcal{A}} \mathcal{B} = \mathcal{B} \ddot{\mathcal{A}} \mathcal{A}$
3.  $\mathcal{A} \dot{\mathcal{A}} (\mathcal{B} \dot{\mathcal{A}} \mathcal{C}) = (\mathcal{A} \dot{\mathcal{A}} \mathcal{B}) \dot{\mathcal{A}} \mathcal{C}$
4.  $\mathcal{A} \ddot{\mathcal{A}} (\mathcal{B} \ddot{\mathcal{A}} \mathcal{C}) = (\mathcal{A} \ddot{\mathcal{A}} \mathcal{B}) \ddot{\mathcal{A}} \mathcal{C}$
5.  $s \in (\mathcal{A} \dot{\mathcal{A}} \mathcal{B}) = (s \in \mathcal{A}) \dot{\mathcal{A}} (s \in \mathcal{B})$
6.  $s \in (\mathcal{A} \ddot{\mathcal{A}} \mathcal{B}) = (s \in \mathcal{A}) \ddot{\mathcal{A}} (s \in \mathcal{B})$
7.  $(s_1 + s_2) \in \mathcal{A} = (s_1 \in \mathcal{A}) \dot{\mathcal{A}} (s_2 \in \mathcal{A})$
8.  $(s_1 + s_2) \in \mathcal{A} = (s_1 \in \mathcal{A}) \ddot{\mathcal{A}} (s_2 \in \mathcal{A})$

**Proof:** 1.-2. Straight forward.

3.  $\tilde{A} \oplus \tilde{B} \oplus \tilde{C}$ 

$$= \langle (a_1, b_1, c_1); \{w_{\tilde{a}_6}^i: i \in I_{n_1}\}, \{u_{\tilde{b}_6}^j: j \in I_{n_1}\}, \{y_{\tilde{c}_6}^l: l \in I_{k_1}\} \rangle \tilde{A} \oplus \langle (a_2 + a_3, b_2 + b_3, c_2 + c_3);$$

$$\{1 - \frac{1}{1 + \frac{a_2^2}{1 - a_2^2} + \frac{a_3^2}{1 - a_3^2}}; a_2 \{w_{\tilde{a}_6}^i: i \in I_{m_2}\}, a_3 \{w_{\tilde{a}_6}^i: i \in I_{m_3}\}\},$$

$$\{1 - \frac{1}{1 + \frac{b_2^2}{1 - b_2^2} + \frac{b_3^2}{1 - b_3^2}}; b_2 \{u_{\tilde{b}_6}^j: j \in I_{n_2}\}, b_3 \{u_{\tilde{b}_6}^j: j \in I_{n_3}\}\},$$

$$\{1 - \frac{1}{1 + \frac{l_2^2}{1 - l_2^2} + \frac{l_3^2}{1 - l_3^2}}; l_2 \{y_{\tilde{c}_6}^l: l \in I_{k_2}\}, l_3 \{y_{\tilde{c}_6}^l: l \in I_{k_3}\}\} \rangle$$

$$= \langle (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3), c_1 + (c_2 + c_3));$$

$$\{1 - \frac{1}{1 + \frac{a_1^2}{1 - a_1^2} + \frac{a_2^2}{1 - a_2^2} + \frac{a_3^2}{1 - a_3^2}}; a_1 \{w_{\tilde{a}_6}^i: i \in I_{m_1}\}, a_2 \{w_{\tilde{a}_6}^i: i \in I_{m_2}\}, a_3 \{w_{\tilde{a}_6}^i: i \in I_{m_3}\}\},$$

$$\{1 - \frac{1}{1 + \frac{b_1^2}{1 - b_1^2} + \frac{b_2^2}{1 - b_2^2} + \frac{b_3^2}{1 - b_3^2}}; b_1 \{u_{\tilde{b}_6}^j: j \in I_{n_1}\}, b_2 \{u_{\tilde{b}_6}^j: j \in I_{n_2}\}, b_3 \{u_{\tilde{b}_6}^j: j \in I_{n_3}\}\},$$

$$\{1 - \frac{1}{1 + \frac{l_1^2}{1 - l_1^2} + \frac{l_2^2}{1 - l_2^2} + \frac{l_3^2}{1 - l_3^2}}; l_1 \{y_{\tilde{c}_6}^l: l \in I_{k_1}\}, l_2 \{y_{\tilde{c}_6}^l: l \in I_{k_2}\}, l_3 \{y_{\tilde{c}_6}^l: l \in I_{k_3}\}\} \rangle$$

$$\begin{aligned}
&= \langle (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{1 - \frac{1}{1 + \frac{a_1^2 \sigma_1^2}{1 - a_1^2 \sigma_1^2} + \frac{a_2^2 \sigma_2^2}{1 - a_2^2 \sigma_2^2} + \frac{a_3^2 \sigma_3^2}{1 - a_3^2 \sigma_3^2}} \} : \\
&a_1 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_3}\}\}, \\
&\{ \frac{1}{1 + \frac{b_1^2 \sigma_1^2}{1 - b_1^2 \sigma_1^2} + \frac{b_2^2 \sigma_2^2}{1 - b_2^2 \sigma_2^2} + \frac{b_3^2 \sigma_3^2}{1 - b_3^2 \sigma_3^2}} : b_1 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_3}\}\}, \\
&\{ \frac{1}{1 + \frac{l_1^2 \sigma_1^2}{1 - l_1^2 \sigma_1^2} + \frac{l_2^2 \sigma_2^2}{1 - l_2^2 \sigma_2^2} + \frac{l_3^2 \sigma_3^2}{1 - l_3^2 \sigma_3^2}} : l_1 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_3}\} \} \rangle
\end{aligned} \quad (7)$$

(\(\mathcal{A} \otimes \mathcal{B}\)) \(\mathcal{A} \otimes \mathcal{B}\)

$$\begin{aligned}
&= \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{1 - \frac{1}{1 + \frac{a_1^2 \sigma_1^2}{1 - a_1^2 \sigma_1^2} + \frac{a_2^2 \sigma_2^2}{1 - a_2^2 \sigma_2^2}} \} : a_1 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_2}\}\}, \\
&\{ \frac{1}{1 + \frac{b_1^2 \sigma_1^2}{1 - b_1^2 \sigma_1^2} + \frac{b_2^2 \sigma_2^2}{1 - b_2^2 \sigma_2^2}} : b_1 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_2}\}\}, \\
&\{ \frac{1}{1 + \frac{l_1^2 \sigma_1^2}{1 - l_1^2 \sigma_1^2} + \frac{l_2^2 \sigma_2^2}{1 - l_2^2 \sigma_2^2}} : l_1 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_2}\} \} \rangle \mathcal{A} \otimes \langle (a_3, b_3, c_3); w_{\theta_0} u_{\theta_0} y_{\theta_0} \rangle \\
&= \langle ((a_1 + a_2) + a_3, (b_1 + b_2) + b_3, (c_1 + c_2) + c_3); \\
&\{1 - \frac{1}{1 + \frac{a_1^2 \sigma_1^2}{1 - a_1^2 \sigma_1^2} + \frac{a_2^2 \sigma_2^2}{1 - a_2^2 \sigma_2^2} + \frac{a_3^2 \sigma_3^2}{1 - a_3^2 \sigma_3^2}} \} : a_1 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\theta_0}^i : i \hat{I} I_{m_3}\}\}, \\
&\{ \frac{1}{1 + \frac{b_1^2 \sigma_1^2}{1 - b_1^2 \sigma_1^2} + \frac{b_2^2 \sigma_2^2}{1 - b_2^2 \sigma_2^2} + \frac{b_3^2 \sigma_3^2}{1 - b_3^2 \sigma_3^2}} : b_1 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\theta_0}^j : j \hat{I} I_{n_3}\}\}, \\
&\{ \frac{1}{1 + \frac{l_1^2 \sigma_1^2}{1 - l_1^2 \sigma_1^2} + \frac{l_2^2 \sigma_2^2}{1 - l_2^2 \sigma_2^2} + \frac{l_3^2 \sigma_3^2}{1 - l_3^2 \sigma_3^2}} : l_1 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\theta_0}^l : l \hat{I} I_{k_3}\} \} \rangle
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1}{1 + \frac{1 - l_1^2 \sigma^2}{l_1^2 \sigma^2} + \frac{1 - l_2^2 \sigma^2}{l_2^2 \sigma^2} + \frac{1 - l_3^2 \sigma^2}{l_3^2 \sigma^2}} : l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_3}\} \right\} > \\
& = \langle (a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3); \{1 - \frac{1}{1 + \frac{1 - a_1^2 \sigma^2}{a_1^2 \sigma^2} + \frac{1 - a_2^2 \sigma^2}{a_2^2 \sigma^2} + \frac{1 - a_3^2 \sigma^2}{a_3^2 \sigma^2}} : a_1 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}, \\
& \quad a_2 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_2}\}, a_3 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_3}\}\}, \{ \frac{1}{1 + \frac{1 - b_1^2 \sigma^2}{b_1^2 \sigma^2} + \frac{1 - b_2^2 \sigma^2}{b_2^2 \sigma^2} + \frac{1 - b_3^2 \sigma^2}{b_3^2 \sigma^2}} : \\
& \quad b_1 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_2}\}, b_3 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_3}\}\}, \{ \frac{1}{1 + \frac{1 - l_1^2 \sigma^2}{l_1^2 \sigma^2} + \frac{1 - l_2^2 \sigma^2}{l_2^2 \sigma^2} + \frac{1 - l_3^2 \sigma^2}{l_3^2 \sigma^2}} : \\
& \quad l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\}, l_3 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_3}\} \} \rangle \quad (8)
\end{aligned}$$

Hence from eq. 7-8, we have,  $\mathcal{A} \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\} = (\mathcal{A} \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}) \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}$ .

4. Proof is similar to 3.

5.  $s \in \mathcal{A} \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}$

$$\begin{aligned}
& = s \in \mathcal{A} \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2); \{1 - \frac{1}{1 + \frac{1 - a_1^2 \sigma^2}{a_1^2 \sigma^2} + \frac{1 - a_2^2 \sigma^2}{a_2^2 \sigma^2}} : a_1 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_2}\}\}, \\
& \quad \{ \frac{1}{1 + \frac{1 - b_1^2 \sigma^2}{b_1^2 \sigma^2} + \frac{1 - b_2^2 \sigma^2}{b_2^2 \sigma^2}} : b_1 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_2}\}\}, \{ \frac{1}{1 + \frac{1 - l_1^2 \sigma^2}{l_1^2 \sigma^2} + \frac{1 - l_2^2 \sigma^2}{l_2^2 \sigma^2}} : l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, \\
& \quad l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\} \} \rangle \\
& = \langle (s a_1 + s a_2, s b_1 + s b_2, s c_1 + s c_2); \{1 - \frac{1}{1 + \frac{1 - (s a_1)^2 \sigma^2}{(s a_1)^2 \sigma^2} + \frac{1 - (s a_2)^2 \sigma^2}{(s a_2)^2 \sigma^2}} : a_1 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}, \\
& \quad a_2 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_2}\}\},
\end{aligned}$$

$$\begin{aligned}
& \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{b_1^2 \sigma_1^2} + \frac{1}{b_2^2 \sigma_2^2}} \right)} : b_1 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_2}\} \right\}, \\
& \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{l_1^2 \sigma_1^2} + \frac{1}{l_2^2 \sigma_2^2}} \right)} : l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\} \right\} > \\
& = \langle (S a_1 + S a_2, S b_1 + S b_2, S c_1 + S c_2); \left( 1 - \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{a_1^2 \sigma_1^2} + \frac{1}{a_2^2 \sigma_2^2}} \right)} : a_1 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}, \right. \\
& \quad \left. a_2 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_2}\} \right), \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{b_1^2 \sigma_1^2} + \frac{1}{b_2^2 \sigma_2^2}} \right)} : b_1 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_2}\} \right\}, \\
& \quad \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{l_1^2 \sigma_1^2} + \frac{1}{l_2^2 \sigma_2^2}} \right)} : l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\} \right\} \rangle \\
& = \langle (S a_1 + S a_2, S b_1 + S b_2, S c_1 + S c_2); \left( 1 - \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{a_1^2 \sigma_1^2} + \frac{1}{a_2^2 \sigma_2^2}} \right)} : \right. \\
& \quad \left. a_1 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\beta_0}^i : i \hat{I} I_{m_2}\} \right), \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{b_1^2 \sigma_1^2} + \frac{1}{b_2^2 \sigma_2^2}} \right)} : b_1 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j : j \hat{I} I_{n_2}\} \right\}, \\
& \quad \left\{ \frac{1}{1 + s \left( \frac{1}{1 + \frac{1}{l_1^2 \sigma_1^2} + \frac{1}{l_2^2 \sigma_2^2}} \right)} : l_1 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l : l \hat{I} I_{k_2}\} \right\} \rangle
\end{aligned}$$

$$\begin{aligned}
& b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_2}\}, \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : a_1 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_1}\}\}, \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_2^2}{1 - a_2^2} \frac{\hat{\sigma}_2^2}{\hat{\sigma}_2^2} \frac{1}{\hat{p}}}} : b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}\}, \\
& l_1 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_2}\} \} > \\
& = \langle (s a_1, s b_1, s c_1); \{1 - \frac{1}{1 + \frac{1}{1 + s \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : a_1 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_1}\}\}, \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}\}, \\
& \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : l_1 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_1}\}\} \rangle \hat{\Delta} \langle (s a_2, s b_2, s c_2); \{1 - \frac{1}{1 + \frac{1}{1 + s \frac{a_2^2}{1 - a_2^2} \frac{\hat{\sigma}_2^2}{\hat{\sigma}_2^2} \frac{1}{\hat{p}}}} : a_2 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_2}\}\}, \\
& \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_2^2}{1 - a_2^2} \frac{\hat{\sigma}_2^2}{\hat{\sigma}_2^2} \frac{1}{\hat{p}}}} : b_2 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_2}\}\}, \{ \frac{1}{1 + \frac{1}{1 + s \frac{a_2^2}{1 - a_2^2} \frac{\hat{\sigma}_2^2}{\hat{\sigma}_2^2} \frac{1}{\hat{p}}}} : l_2 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_2}\}\} \rangle \\
& = (s e \frac{a_1}{a_1} \hat{\Delta} (s e \frac{b_1}{b_1})
\end{aligned}$$

6. Proof is similar to 5

7.  $(s_1 + s_2)e \frac{a_1}{a_1}$

$$\begin{aligned}
& = \langle ((s_1 + s_2)a_1, (s_1 + s_2)b_1, (s_1 + s_2)c_1); \{1 - \frac{1}{1 + \frac{1}{1 + (s_1 + s_2) \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : a_1 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_1}\}\}, \\
& \{ \frac{1}{1 + \frac{1}{1 + (s_1 + s_2) \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}\}, \{ \frac{1}{1 + \frac{1}{1 + (s_1 + s_2) \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : l_1 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_1}\}\} \\
& = \langle ((s_1 + s_2)a_1, (s_1 + s_2)b_1, (s_1 + s_2)c_1); \\
& \{1 - \frac{1}{1 + \frac{1}{1 + s_1 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : a_1 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_1}\}\}, \\
& \{ \frac{1}{1 + \frac{1}{1 + s_1 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}\}, \{ \frac{1}{1 + \frac{1}{1 + s_1 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : l_1 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_1}\}\} \\
& + \{1 - \frac{1}{1 + \frac{1}{1 + s_2 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : a_1 \hat{I} \{w_{\beta_0}^i: i \hat{I} I_{m_1}\}\}, \\
& \{ \frac{1}{1 + \frac{1}{1 + s_2 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : b_1 \hat{I} \{u_{\beta_0}^j: j \hat{I} I_{n_1}\}\}, \{ \frac{1}{1 + \frac{1}{1 + s_2 \frac{a_1^2}{1 - a_1^2} \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2} \frac{1}{\hat{p}}}} : l_1 \hat{I} \{y_{\beta_0}^l: l \hat{I} I_{k_1}\}\} \rangle
\end{aligned}$$

8. Proof is similar to 7.

This section deals with various types of hesitant triangular neutrosophic weighted aggregation operators along with their basic properties.

$$HTNWAAO_{T_i}(a_0, a_0, a_0, \dots, a_0) = (w_1 \in \mathcal{A}_0) \wedge (w_2 \in \mathcal{A}_0) \wedge (w_3 \in \mathcal{A}_0) \wedge \dots \wedge (w_n \in \mathcal{A}_0)$$

where  $w_j$  is the weight of  $\theta_j$  ( $j=1,2,3,\dots,n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .



**Theorem 14:** Let  $\mathcal{A}_j = \langle (a_j, b_j, c_j); \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}, \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}, \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then  $HTNWAAO_{T_1}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n)$  is a hesitant triangular neutrosophic number and

$$HTNWAAO_{T_1}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n) = \langle (\bigotimes_{j=1}^n w_j a_j, \bigotimes_{j=1}^n w_j b_j, \bigotimes_{j=1}^n w_j c_j); \{1 - \bigotimes_{j=1}^n (1 - a_{jk})^{w_j} : a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{\bigotimes_{j=1}^n b_{jm}^{w_j} : b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \{\bigotimes_{j=1}^n l_{jr}^{w_j} : l_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle$$

where  $w_j$  is the weight of  $\mathcal{A}_j$  ( $j=1, 2, 3, \dots, n$ ) such that  $w_j \geq 0$  and  $\bigotimes_{j=1}^n w_j = 1$ .

**Proof:** Let us prove the result using the method of mathematical induction. For  $n=2$ ,  $HTNWAAO_{T_1}(\mathcal{A}_1, \mathcal{A}_2)$

$$\begin{aligned} &= (w_1 e_{\mathcal{A}_1}) \mathring{\wedge} (w_2 e_{\mathcal{A}_2}) \\ &= \langle (w_1 a_1, w_1 b_1, w_1 c_1); \{1 - (1 - a_1)^{w_1} : a_1 \in \{w_{\mathcal{A}_1}^i : i \in I_{m_1}\}\}, \{b_1^{w_1} : b_1 \in \{u_{\mathcal{A}_1}^r : r \in I_{n_1}\}\}, \\ &\quad \{l_1^{w_1} : l_1 \in \{y_{\mathcal{A}_1}^l : l \in I_{k_1}\}\} \rangle \mathring{\wedge} \langle (w_2 a_2, w_2 b_2, w_2 c_2); \{1 - (1 - a_2)^{w_2} : a_2 \in \{w_{\mathcal{A}_2}^i : i \in I_{m_2}\}\}, \\ &\quad \{b_2^{w_2} : b_2 \in \{u_{\mathcal{A}_2}^r : r \in I_{n_2}\}\}, \{l_2^{w_2} : l_2 \in \{y_{\mathcal{A}_2}^l : l \in I_{k_2}\}\} \rangle \\ &= \langle (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); (1 - (1 - a_1)^{w_1}) + (1 - (1 - a_2)^{w_2}) - \\ &\quad \{(1 - (1 - a_1)^{w_1})' (1 - (1 - a_2)^{w_2}) : a_1 \in \{w_{\mathcal{A}_1}^i : i \in I_{m_1}\}, a_2 \in \{w_{\mathcal{A}_2}^i : i \in I_{m_2}\}\}, \{b_1^{w_1} b_2^{w_2} : \\ &\quad b_1 \in \{u_{\mathcal{A}_1}^r : r \in I_{n_1}\}, b_2 \in \{u_{\mathcal{A}_2}^r : r \in I_{n_2}\}\}, \{l_1^{w_1} l_2^{w_2} : l_1 \in \{y_{\mathcal{A}_1}^l : l \in I_{k_1}\}, l_2 \in \{y_{\mathcal{A}_2}^l : l \in I_{k_2}\}\} \rangle \\ &= \langle (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); \{1 - (1 - a_1)^{w_1} (1 - a_2)^{w_2} : a_1 \in \{w_{\mathcal{A}_1}^i : i \in I_{m_1}\}, \\ &\quad a_2 \in \{w_{\mathcal{A}_2}^i : i \in I_{m_2}\}\}, \{b_1^{w_1} b_2^{w_2} : b_1 \in \{u_{\mathcal{A}_1}^r : r \in I_{n_1}\}, b_2 \in \{u_{\mathcal{A}_2}^r : r \in I_{n_2}\}\}, \{l_1^{w_1} l_2^{w_2} : \\ &\quad l_1 \in \{y_{\mathcal{A}_1}^l : l \in I_{k_1}\}, l_2 \in \{y_{\mathcal{A}_2}^l : l \in I_{k_2}\}\} \rangle \\ &= \langle (\bigotimes_{j=1}^2 w_j a_j, \bigotimes_{j=1}^2 w_j b_j, \bigotimes_{j=1}^2 w_j c_j); \{1 - \bigotimes_{j=1}^2 (1 - a_j)^{w_j} : a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{\bigotimes_{j=1}^2 b_j^{w_j} : b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \\ &\quad \{\bigotimes_{j=1}^2 l_j^{w_j} : l_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle \end{aligned}$$

Thus the result is true for  $n=2$ . Let us assume that the result is true for  $n=s$ . Then  $HTNWAAO_{T_1}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_s)$

$$\begin{aligned} &= \langle (\bigotimes_{j=1}^s w_j a_j, \bigotimes_{j=1}^s w_j b_j, \bigotimes_{j=1}^s w_j c_j); \{1 - \bigotimes_{j=1}^s (1 - a_j)^{w_j} : a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{\bigotimes_{j=1}^s b_j^{w_j} : b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \\ &\quad \{\bigotimes_{j=1}^s l_j^{w_j} : l_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle \end{aligned}$$

Now for  $n=s+1$ , we have,  $HTNWAAO_{T_1}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_{s+1})$

$$\begin{aligned} &= \langle (\bigotimes_{j=1}^s w_j a_j, \bigotimes_{j=1}^s w_j b_j, \bigotimes_{j=1}^s w_j c_j); \{1 - \bigotimes_{j=1}^s (1 - a_j)^{w_j} : a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{\bigotimes_{j=1}^s b_j^{w_j} : b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \\ &\quad \{\bigotimes_{j=1}^s l_j^{w_j} : l_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle \mathring{\wedge} (w_{s+1} e_{\mathcal{A}_{s+1}}) \\ &= \langle (\bigotimes_{j=1}^s w_j a_j, \bigotimes_{j=1}^s w_j b_j, \bigotimes_{j=1}^s w_j c_j); \{1 - \bigotimes_{j=1}^s (1 - a_j)^{w_j} : a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{\bigotimes_{j=1}^s b_j^{w_j} : b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \\ &\quad \{\bigotimes_{j=1}^s l_j^{w_j} : l_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle \mathring{\wedge} \langle (w_{s+1} a_{s+1}, w_{s+1} b_{s+1}, w_{s+1} c_{s+1}); \{1 - (1 - a_{s+1})^{w_{s+1}} : \end{aligned}$$

$$\begin{aligned}
& a_{s+1} \hat{I} \{w_{\theta_{s+1}}^i : i \hat{I} I_{m_{s+1}}\} \{b_{s+1}^{w_{s+1}} : b_{s+1} \hat{I} \{u_{\theta_{s+1}}^r : r \hat{I} I_{n_{s+1}}\}\}, \{l_{s+1}^{w_{s+1}} : l_{s+1} \hat{I} \{y_{\theta_{s+1}}^l : l \hat{I} I_{k_{s+1}}\}\} > \\
& < (\hat{a}_{j=1}^s w_j a_j + w_{s+1} a_{s+1}, \hat{a}_{j=1}^s w_j b_j + w_{s+1} b_{s+1}, \hat{a}_{j=1}^s w_j c_j + w_{s+1} c_{s+1}); \\
& \{(1 - \hat{O}_{j=1}^s (1 - a_j)^{w_j}) + (1 - (1 - a_{(s+1)})^{w_{s+1}}) - (1 - \hat{O}_{j=1}^s (1 - a_j)^{w_j})' (1 - (1 - a_{(s+1)})^{w_{s+1}}) : a_j \hat{I} \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \\
& a_{s+1} \hat{I} \{w_{\theta_{s+1}}^i : i \hat{I} I_{m_{s+1}}\}, \{(\hat{O}_{j=1}^s b_j^{w_j}) b_{s+1}^{w_{s+1}} : b_j \hat{I} \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}, b_{s+1} \hat{I} \{u_{\theta_{s+1}}^r : r \hat{I} I_{n_{s+1}}\}\}, \\
& \{(\hat{O}_{j=1}^s l_j^{w_j}) l_{s+1}^{w_{s+1}} : l_j \hat{I} \{y_{\theta_j}^l : l \hat{I} I_{k_j}\}, l_{s+1} \hat{I} \{y_{\theta_{s+1}}^l : l \hat{I} I_{k_{s+1}}\}\} > \\
& = < (\hat{a}_{j=1}^s w_j a_j + w_{s+1} a_{s+1}, \hat{a}_{j=1}^s w_j b_j + w_{s+1} b_{s+1}, \hat{a}_{j=1}^s w_j c_j + w_{s+1} c_{s+1}); \{1 - (\hat{O}_{j=1}^s (1 - a_j)^{w_j}) a_{s+1}^{w_{s+1}} : a_j \hat{I} \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \\
& a_{s+1} \hat{I} \{w_{\theta_{s+1}}^i : i \hat{I} I_{m_{s+1}}\}\}, \{(\hat{O}_{j=1}^s b_j^{w_j}) b_{s+1}^{w_{s+1}} : b_j \hat{I} \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}, b_{s+1} \hat{I} \{u_{\theta_{s+1}}^r : r \hat{I} I_{n_{s+1}}\}\}, \\
& \{(\hat{O}_{j=1}^s l_j^{w_j}) l_{s+1}^{w_{s+1}} : l_j \hat{I} \{y_{\theta_j}^l : l \hat{I} I_{k_j}\}, l_{s+1} \hat{I} \{y_{\theta_{s+1}}^l : l \hat{I} I_{k_{s+1}}\}\} > \\
& = < (\hat{a}_{j=1}^{s+1} w_j a_j, \hat{a}_{j=1}^{s+1} w_j b_j, \hat{a}_{j=1}^{s+1} w_j c_j), \{1 - \hat{O}_{j=1}^{s+1} (1 - a_j)^{w_j} : a_j \hat{I} \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}\}, \{\hat{O}_{j=1}^{s+1} b_j^{w_j} : b_j \hat{I} \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}\}, \\
& \{\hat{O}_{j=1}^{s+1} l_j^{w_j} : l_j \hat{I} \{y_{\theta_j}^l : l \hat{I} I_{k_j}\}\} >
\end{aligned}$$

Thus the result is true for  $n=s+1$  also. Hence by the principle of mathematical induction, the result is true for any natural number  $n$ .

**Theorem 15:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}, \{y_{\theta_j}^l : l \hat{I} I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number  $\theta^0$ , we have,

$$(i) HTNWAAO_{T_1}(\theta^0 \hat{\Delta} \theta_1, \theta^0 \hat{\Delta} \theta_2, \theta^0 \hat{\Delta} \theta_3, \dots, \theta^0 \hat{\Delta} \theta_n) = \theta^0 \hat{\Delta} HTNWAAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$$

$$(ii) HTNWAAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n) = \theta^0 \text{ if } \theta_j = \theta^0 \text{ for each } j$$

**Proof:** Straight Forward.

**Definition 16:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}, \{y_{\theta_j}^l : l \hat{I} I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted geometric aggregation operator of type-1 ( $HTNWGAO_{T_1}$  for short) is defined as:

$$HTNWGAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n) = (w_1 * \theta_1) \hat{\Delta} (w_2 * \theta_2) \hat{\Delta} (w_3 * \theta_3) \hat{\Delta} \dots \hat{\Delta} (w_n * \theta_n)$$

where  $w_j$  is the weight of  $\theta_j$  ( $j=1, 2, 3, \dots, n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Theorem 17:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}, \{y_{\theta_j}^l : l \hat{I} I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then  $HTNWGAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$  is a hesitant triangular neutrosophic number and  $HTNWGAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$

$$\begin{aligned}
& = < (\hat{O}_{j=1}^n a_j^{w_j}, \hat{O}_{j=1}^n b_j^{w_j}, \hat{O}_{j=1}^n c_j^{w_j}); \hat{O}_{j=1}^n a_j^{w_j} : a_j \hat{I} \{w_{\theta_j}^i : i \hat{I} I_{m_j}\}, \{1 - \hat{O}_{j=1}^n (1 - b_j)^{w_j} : b_j \hat{I} \{u_{\theta_j}^r : r \hat{I} I_{n_j}\}\}, \\
& \{1 - \hat{O}_{j=1}^n (1 - l_j)^{w_j} : l_j \hat{I} \{y_{\theta_j}^l : l \hat{I} I_{k_j}\}\} >
\end{aligned}$$

where  $w_j$  is the weight of  $\theta_j$  ( $j=1,2,3,\dots,n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Proof:** Similar to the proof of Theorem 14.

**Theorem 18:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \in I_{m_j}\}, \{u_{\theta_j}^r : r \in I_{n_j}\}, \{y_{\theta_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number  $\theta$ , we have,

$$(i) HTNWGAO_{T_1}(\theta \otimes \theta_1, \theta \otimes \theta_2, \theta \otimes \theta_3, \dots, \theta \otimes \theta_n) = \theta \otimes HTNWGAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$$

$$(ii) HTNWGAO_{T_1}(\theta_1, \theta_2, \theta_3, \dots, \theta_n) = \theta \text{ if } \theta_j = \theta \text{ for each } j$$

**Proof:** Straight forward.

**Definition 19:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \in I_{m_j}\}, \{u_{\theta_j}^r : r \in I_{n_j}\}, \{y_{\theta_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted arithmetic aggregation operator of type-2 is denoted by  $HTNWAAO_{T_2}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$  and is defined by:

$$HTNWAAO_{T_2}(\theta_1, \theta_2, \theta_3, \dots, \theta_n) = (w_1 \otimes \theta_1) \oplus (w_2 \otimes \theta_2) \oplus (w_3 \otimes \theta_3) \oplus \dots \oplus (w_n \otimes \theta_n)$$

where  $w_j$  is the weight of  $\theta_j$  ( $j=1,2,3,\dots,n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Theorem 20:** Let  $\theta_j = \langle (a_j, b_j, c_j); \{w_{\theta_j}^i : i \in I_{m_j}\}, \{u_{\theta_j}^r : r \in I_{n_j}\}, \{y_{\theta_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then  $HTNWAAO_{T_2}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$  is a hesitant triangular neutrosophic number and  $HTNWAAO_{T_2}(\theta_1, \theta_2, \theta_3, \dots, \theta_n)$

$$= \langle \left( \sum_{j=1}^n w_j a_j, \sum_{j=1}^n w_j b_j, \sum_{j=1}^n w_j c_j \right), \left\{ 1 - \frac{1}{1 + \sum_{j=1}^n w_j \left( \frac{a_j^2}{a_j^2 - a_j^2} \right)^{\frac{2}{\theta}}}} : a_j \in \{w_{\theta_j}^i : i \in I_{m_j}\} \right\}, \left\{ 1 - \frac{1}{1 + \sum_{j=1}^n w_j \left( \frac{b_j^2}{b_j^2 - b_j^2} \right)^{\frac{2}{\theta}}}} : b_j \in \{u_{\theta_j}^r : r \in I_{n_j}\} \right\}, \left\{ 1 - \frac{1}{1 + \sum_{j=1}^n w_j \left( \frac{c_j^2}{c_j^2 - c_j^2} \right)^{\frac{2}{\theta}}}} : c_j \in \{y_{\theta_j}^l : l \in I_{k_j}\} \right\} \rangle$$

where  $w_j$  is the weight of  $\theta_j$  ( $j=1,2,3,\dots,n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Proof:** For  $n=2$ , we have,

$$HTNWAAO_{T_2}(\theta_1, \theta_2)$$

$$= (w_1 \otimes \theta_1) \oplus (w_2 \otimes \theta_2)$$

$$= \langle (w_1 a_1, w_1 b_1, w_1 c_1); \left\{ 1 - \frac{1}{1 + w_1 \left( \frac{a_1^2}{a_1^2 - a_1^2} \right)^{\frac{2}{\theta}}} \right\} : a_1 \in \{w_{\theta_1}^i : i \in I_{m_1}\} \right\}, \left\{ 1 - \frac{1}{1 + w_1 \left( \frac{b_1^2}{b_1^2 - b_1^2} \right)^{\frac{2}{\theta}}} \right\} : b_1 \in \{u_{\theta_1}^r : r \in I_{n_1}\} \right\}, \left\{ 1 - \frac{1}{1 + w_1 \left( \frac{c_1^2}{c_1^2 - c_1^2} \right)^{\frac{2}{\theta}}} \right\} : c_1 \in \{y_{\theta_1}^l : l \in I_{k_1}\} \right\} \rangle \oplus \langle (w_2 a_2, w_2 b_2, w_2 c_2); \left\{ 1 - \frac{1}{1 + w_2 \left( \frac{a_2^2}{a_2^2 - a_2^2} \right)^{\frac{2}{\theta}}} \right\} : a_2 \in \{w_{\theta_2}^i : i \in I_{m_2}\} \right\}, \left\{ 1 - \frac{1}{1 + w_2 \left( \frac{b_2^2}{b_2^2 - b_2^2} \right)^{\frac{2}{\theta}}} \right\} : b_2 \in \{u_{\theta_2}^r : r \in I_{n_2}\} \right\}, \left\{ 1 - \frac{1}{1 + w_2 \left( \frac{c_2^2}{c_2^2 - c_2^2} \right)^{\frac{2}{\theta}}} \right\} : c_2 \in \{y_{\theta_2}^l : l \in I_{k_2}\} \right\} \rangle$$

$$\begin{aligned}
& \{a_2 \hat{I} \{w_{\theta_2}^i : i \hat{I} I_{m_2}\}\}, \left\{ \frac{1}{1 + \frac{w_1 \frac{a_1^2 - b_1^2 \frac{\sigma^2}{\theta^2}}{b_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - b_2^2 \frac{\sigma^2}{\theta^2}}{b_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : b_2 \hat{I} u_{\theta_2} \right\}, \left\{ \frac{1}{1 + \frac{w_1 \frac{a_1^2 - l_1^2 \frac{\sigma^2}{\theta^2}}{l_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - l_2^2 \frac{\sigma^2}{\theta^2}}{l_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : l_2 \hat{I} \{y_{\theta_2}^l : l \hat{I} I_{k_2}\} \right\} > \\
& = (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); \\
& \left\{ 1 - \frac{1}{1 + \frac{w_1 \frac{a_1^2 - b_1^2 \frac{\sigma^2}{\theta^2}}{b_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - b_2^2 \frac{\sigma^2}{\theta^2}}{b_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : a_1 \hat{I} \{w_{\theta_1}^i : i \hat{I} I_{m_1}\}, a_2 \hat{I} \{w_{\theta_2}^i : i \hat{I} I_{m_2}\} \right\}, \\
& \left\{ 1 - \frac{1}{1 + \frac{w_1 \frac{a_1^2 - b_1^2 \frac{\sigma^2}{\theta^2}}{b_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - b_2^2 \frac{\sigma^2}{\theta^2}}{b_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : b_1 \hat{I} \{u_{\theta_1}^r : r \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\theta_2}^r : j \hat{I} I_{n_2}\} \right\}, \\
& \left\{ \frac{1}{1 + \frac{w_1 \frac{a_1^2 - l_1^2 \frac{\sigma^2}{\theta^2}}{l_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - l_2^2 \frac{\sigma^2}{\theta^2}}{l_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : l_1 \hat{I} \{y_{\theta_1}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\theta_2}^l : l \hat{I} I_{k_2}\} \right\} > \\
& = (w_1 a_1 + w_2 a_2, w_1 b_1 + w_2 b_2, w_1 c_1 + w_2 c_2); \left\{ 1 - \frac{1}{1 + \frac{w_1 \frac{a_1^2 - b_1^2 \frac{\sigma^2}{\theta^2}}{b_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - b_2^2 \frac{\sigma^2}{\theta^2}}{b_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : a_1 \hat{I} \{w_{\theta_1}^i : i \hat{I} I_{m_1}\}, \right. \\
& \left. a_2 \hat{I} \{w_{\theta_2}^i : i \hat{I} I_{m_2}\} \right\}, \left\{ \frac{1}{1 + \frac{w_1 \frac{a_1^2 - b_1^2 \frac{\sigma^2}{\theta^2}}{b_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - b_2^2 \frac{\sigma^2}{\theta^2}}{b_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : b_1 \hat{I} \{u_{\theta_1}^r : r \hat{I} I_{n_1}\}, b_2 \hat{I} \{u_{\theta_2}^r : j \hat{I} I_{n_2}\} \right\}, \\
& \left\{ \frac{1}{1 + \frac{w_1 \frac{a_1^2 - l_1^2 \frac{\sigma^2}{\theta^2}}{l_1^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_1^2}{\theta^2}} + \frac{w_2 \frac{a_2^2 - l_2^2 \frac{\sigma^2}{\theta^2}}{l_2^2 \frac{\sigma^2}{\theta^2}}}{1 - \frac{a_2^2}{\theta^2}}} : l_1 \hat{I} \{y_{\theta_1}^l : l \hat{I} I_{k_1}\}, l_2 \hat{I} \{y_{\theta_2}^l : l \hat{I} I_{k_2}\} \right\} >
\end{aligned}$$



[illegible]

Thus the result is true for  $n=s+1$  also. Hence by the principle of mathematical induction, the result is true for any natural number  $n$ .

**Theorem 21:** Let  $\mathcal{A}_j = \langle (a_j, b_j, c_j); \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}, \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}, \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number  $\mathcal{A}$ , we have,

$$(i) HTNWAAO_{T_2}(\mathcal{A} \# \mathcal{A}_1, \mathcal{A} \# \mathcal{A}_2, \mathcal{A} \# \mathcal{A}_3, \dots, \mathcal{A} \# \mathcal{A}_n) = \mathcal{A} \# HTNWAAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n)$$

$$(ii) HTNWAAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n) = \mathcal{A} \text{ if } \mathcal{A}_j = \mathcal{A} \text{ for each } j$$

**Proof:** Straight forward.

**Definition 22:** Let  $\mathcal{A}_j = \langle (a_j, b_j, c_j); \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}, \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}, \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then the hesitant triangular neutrosophic weighted geometric aggregation operator of type-2 is denoted by  $HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n)$  and is defined by:

$$HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n) = (w_1 * \mathcal{A}_1) \# (w_2 * \mathcal{A}_2) \# (w_3 * \mathcal{A}_3) \# \dots \# (w_n * \mathcal{A}_n)$$

where  $w_j$  is the weight of  $\mathcal{A}_j$  ( $j=1, 2, 3, \dots, n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Theorem 23:** Let  $\mathcal{A}_j = \langle (a_j, b_j, c_j); \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}, \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}, \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then  $HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n)$  is a hesitant triangular neutrosophic number and

$$HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n) = \langle (\bigotimes_{j=1}^n a_j^{w_j}, \bigotimes_{j=1}^n b_j^{w_j}, \bigotimes_{j=1}^n c_j^{w_j}); \{ \frac{1}{1 + \frac{1}{\sum_{j=1}^n w_j \frac{a_j^2 - a_j^{\frac{2}{\sigma}}}{\sigma}}}: a_j \in \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}\}, \{1 - \frac{1}{1 + \frac{1}{\sum_{j=1}^n w_j \frac{b_j^2 - b_j^{\frac{2}{\sigma}}}{\sigma}}}: b_j \in \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}\}, \{1 - \frac{1}{1 + \frac{1}{\sum_{j=1}^n w_j \frac{c_j^2 - c_j^{\frac{2}{\sigma}}}{\sigma}}}: c_j \in \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\}\} \rangle$$

where  $w_j$  is the weight of  $\mathcal{A}_j$  ( $j=1, 2, 3, \dots, n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .

**Proof:** Similar to the proof of Theorem 20.

**Theorem 24:** Let  $\mathcal{A}_j = \langle (a_j, b_j, c_j); \{w_{\mathcal{A}_j}^i : i \in I_{m_j}\}, \{u_{\mathcal{A}_j}^r : r \in I_{n_j}\}, \{y_{\mathcal{A}_j}^l : l \in I_{k_j}\} \rangle$  ( $j=1, 2, 3, \dots, n$ ) be a collection of hesitant triangular neutrosophic numbers. Then for any hesitant triangular neutrosophic number  $\mathcal{A}$ , we have,

$$(i) HTNWGAO_{T_2}(\mathcal{A} \# \mathcal{A}_1, \mathcal{A} \# \mathcal{A}_2, \mathcal{A} \# \mathcal{A}_3, \dots, \mathcal{A} \# \mathcal{A}_n) = \mathcal{A} \# HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n)$$

$$(ii) HTNWGAO_{T_2}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n) = \mathcal{A} \text{ if } \mathcal{A}_j = \mathcal{A} \text{ for each } j$$

**Proof:** Straight forward.

**Definition 25:** Let  $\mathcal{A} = \langle (a_1, b_1, c_1); \{w_{\mathcal{A}}^i : i \in I_m\}, \{u_{\mathcal{A}}^r : r \in I_n\}, \{y_{\mathcal{A}}^l : l \in I_k\} \rangle$  be a hesitant triangular neutrosophic number. Then the score of  $\mathcal{A}$  is defined by:

$$S(\mathcal{A}) = \frac{1}{3 \max\{a_1, b_1, c_1\}} (a_1 + b_1 + c_1) \left( \frac{1}{\sigma} + \frac{1}{m} \sum_{j=1}^m a_j - \frac{1}{n} \sum_{j=1}^n b_j - \frac{1}{k} \sum_{j=1}^k y_j \right)$$

Where  $a_j \hat{=} \{w_{\alpha_j}^i : i \hat{=} I_{m_j}\}, b_j \hat{=} \{u_{\alpha_j}^r : r \hat{=} I_{n_j}\}, l_j \hat{=} \{y_{\alpha_j}^l : l \hat{=} I_{k_j}\}$

If  $\alpha_j = \langle (a_j, b_j, c_j); \{w_{\alpha_j}^i : i \hat{=} I_{m_j}\}, \{u_{\alpha_j}^r : r \hat{=} I_{n_j}\}, \{y_{\alpha_j}^l : l \hat{=} I_{k_j}\} \rangle$  ( $j = 1, 2$ ) be two hesitant triangular neutrosophic numbers, then, the comparison method is given as;

- I. If  $S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2)$  then  $\tilde{\alpha}_1 \succ \tilde{\alpha}_2$
- II. If  $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$  then  $\tilde{\alpha}_1 = \tilde{\alpha}_2$

## 5. APPLICATION OF HESITANT TRAPEZOIDAL NEUTROSOPHIC NUMBERS:

In this section, we apply the weighted aggregation operators and the score function of hesitant triangular neutrosophic numbers to the multi-attribute decision-making problem with hesitant triangular neutrosophic information.

Let  $X = \{A_1, A_2, A_3, \dots, A_m\}$  be a set of alternatives,  $A = \{c_1, c_2, c_3, \dots, c_n\}$  be a set of attributes and  $w = \{w_1, w_2, w_3, \dots, w_n\}$  be a set of weights ( $w_j$  is the weight of attribute  $C_j$  ( $j=1,2,3,\dots,n$ ) such that  $w_j \geq 0$  and  $\sum_{j=1}^n w_j = 1$ .) In this case, the characteristic of the alternative  $A_i$  ( $i = 1, 2, \dots, m$ ) on attribute

$C_j$  ( $j = 1, 2, \dots, n$ ) is represented by the following form of a hesitant triangular neutrosophic number:

$$A_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}); \{w_{\alpha_{ij}}^p : p \hat{=} I_{m_{ij}}\}, \{u_{\alpha_{ij}}^r : r \hat{=} I_{n_{ij}}\}, \{y_{\alpha_{ij}}^l : l \hat{=} I_{k_{ij}}\} \rangle.$$

Now, we construct a multi-attribute decision making method by the following algorithm:

### • ALGORITHM:

**Step-1:** Express the evaluation results of the expert based on the alternative  $A_i$  ( $i = 1, 2, \dots, m$ ) on attribute  $c_j$  ( $1, 2, \dots, n$ ) in terms of hesitant triangular neutrosophic numbers  $x_{ij}$  as a  $m \times n$  Table.

**Step-2:** Compute the aggregation values  $g_i^{T^A}$  ( $i = 1, 2, \dots, m$ ) ( $k = 1, 2$ ) or  $g_i^{T^G}$  ( $i = 1, 2, \dots, m$ ) ( $k = 1, 2$ ) of  $A_i$  ( $i = 1, 2, \dots, m$ ) as;

$$g_i^{T^A} = HTNWAAO_{T_k}(A_{i1}, A_{i2}, \dots, A_{in}) \quad (i = 1, 2, \dots, m) \quad (k = 1, 2)$$

or

$$g_i^{T^G} = HTNWGAO_{T_k}(A_{i1}, A_{i2}, \dots, A_{in}) \quad (i = 1, 2, \dots, m) \quad (k = 1, 2)$$

**Step-3:** Calculate the score values of  $g_i^{T^A}$  ( $i = 1, 2, \dots, m$ ) ( $k = 1, 2$ ) or  $g_i^{T^G}$  ( $i = 1, 2, \dots, m$ ) ( $k = 1, 2$ ) of  $A_i$  ( $i = 1, 2, \dots, m$ ) based on Definition 25.

**Step-4:** Rank the alternatives by using definition 25.

### Example 22:

Let us consider a decision making problem adapted from Wei et al. (2017). Suppose an organisation plans to implement enterprise resource planning (ERP) system. The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project team chooses five potential ERP systems  $A_i$  ( $i=1, 2, 3, 4, 5$ ) as candidates. The company employs some external professional organizations (or experts) to aid this decision making. The project team selects four attributes to evaluate the alternatives: function and technology  $c_1$ , strategic fitness  $c_2$ , vendor's ability  $c_3$  and vendor's reputation  $c_4$ . The five possible ERP systems  $A_i$  ( $i=1, 2, 3, 4, 5$ ) are to be evaluate during the hesitant triangular neutrosophic numbers by the decision makers under the above four attributes whose weighting vector is  $w = (0.3, 0.3, 0.2, 0.2)$ .

**Step-1:** We express the initial evaluation results of the expert for five possible alternatives based on four attributes by the form of hesitant triangular neutrosophic numbers, as shown in Table 1.



**Table 1:** The evaluation result by the expert is shown in the below table

	$c_1$	$c_2$	$c_3$	$c_4$
$A_1$	$\langle(0.3,0.4,0.2); \{0.5,0.6\}, \{0.1,0.3\}, \{0.4,0.7\}\rangle$	$\langle(0.1,0.5,0.6); \{0.3\}, \{0.4,0.5\}, \{0.3,0.6\}\rangle$	$\langle(0.5,0.6,0.7); \{0.2,0.4\}, \{0.5\}, \{0.3,0.4\}\rangle$	$\langle(0.4,0.6,0.7); \{0.8\}, \{0.6,0.7\}, \{0.1,0.2\}\rangle$
$A_2$	$\langle(0.2,0.3,0.4); \{0.2\}, \{0.5,0.7\}, \{0.4\}\rangle$	$\langle(0.4,0.6,0.6); \{0.7,0.9\}, \{0.2\}, \{0.5,0.7\}\rangle$	$\langle(0.5,0.7,0.9); \{0.1,0.2\}, \{0.6,0.8\}, \{0.5,0.6\}\rangle$	$\langle(0.2,0.3,0.5); \{0.6\}, \{0.4,0.5\}, \{0.2,0.4\}\rangle$
$A_3$	$\langle(0.6,0.6,0.7); 0.1,0.4\}, \{0.6,0.8\}, \{0.4,0.5,0.7\}\rangle$	$\langle(0.3,0.4,0.4); \{0.6,0.7\}, \{0.5\}, \{0.5,0.8\}\rangle$	$\langle(0.1,0.2,0.3); \{0.2,0.3\}, \{0.4,0.6\}, \{0.5,0.6\}\rangle$	$\langle(0.5,0.5,0.6); \{0.3,0.5\}, \{0.3,0.5\}, \{0.2\}\rangle$
$A_4$	$\langle(0.1,0.3,0.3); \{0.7\}, \{0.6,0.9\}, \{0.4\}\rangle$	$\langle(0.5,0.5,0.5); \{0.1,0.2\}, \{0.4,0.7\}, \{0.2,0.5\}\rangle$	$\langle(0.4,0.5,0.6); \{0.2,0.6\}, \{0.2,0.4\}, \{0.3,0.4\}\rangle$	$\langle(0.1,0.1,0.2); \{0.3,0.5\}, \{0.7,0.8\}, \{0.1,0.2\}\rangle$
$A_5$	$\langle(0.5,0.6,0.6); \{0.3,0.6\}, \{0.4,0.5\}, \{0.2,0.4\}\rangle$	$\langle(0.2,0.3,0.3); \{0.3,0.5\}, \{0.4,0.5\}, \{0.4,0.6\}\rangle$	$\langle(0.7,0.8,0.8); \{0.5,0.7\}, \{0.3,0.6\}, \{0.1,0.3\}\rangle$	$\langle(0.2,0.3,0.5); \{0.4,0.5\}, \{0.2,0.3\}, \{0.6,0.7,0.8\}\rangle$

**Step-2:** We compute the aggregation values  $g_i^{T_1^A}$  ( $i = 1, 2, \dots, 5$ ) of  $A_i$  ( $i = 1, 2, \dots, 5$ ) as;

$$\begin{aligned}
 g_1^{T_1^A} &= HTNWAAO_{T_1}(A_{11}, A_{12}, A_{13}, A_{14}) \\
 &= \langle(0.30, 0.51, 0.52); \{0.494124, 0.522409, 0.526880, 0.553333\}, \{0.299254, 0.308624, 0.319973, 0.329992, 0.416080, \\
 &\quad 0.429108, 0.444888, 0.458818\}, \{0.262529, 0.301566, 0.278077, 0.319426, 0.323211, 0.371272, \\
 &\quad 0.342353, 0.393260, 0.310519, 0.356693, 0.328909, 0.377818, 0.382294, 0.439141, 0.412567, 0.465148\}\rangle
 \end{aligned}$$

$$\begin{aligned}
 g_2^{T_1^A} &= HTNWAAO_{T_1}(A_{21}, A_{22}, A_{23}, A_{24}) \\
 &= \langle(0.32, 0.47, 0.58); \{0.468719, 0.481089, 0.617891, 0.626787\}, \{0.376740, 0.393934, 0.399052, 0.417264, 0.416754, \\
 &\quad \{0.389321, 0.447212, 0.403779, 0.435774, 0.441436, 0.461583\}, 0.463821, 0.430672, 0.494712, 0.446666, 0.513085\}\rangle
 \end{aligned}$$

$$\begin{aligned}
 g_3^{T_1^A} &= HTNWAAO_{T_1}(A_{31}, A_{32}, A_{33}, A_{34}) \\
 &= \langle(0.39, 0.44, 0.51); \{0.419636, 0.457406, 0.434930, 0.471704, 0.467623, 0.502270, 0.481653, 0.515387, 0.344568, \\
 &\quad 0.387223, 0.361840, 0.403371, 0.398762, 0.437891, 0.414606, 0.452704\}, \{0.456007, 0.505058, 0.494527, \\
 &\quad 0.547722, 0.497111, 0.550583, 0.539102, 0.597092\}, \{0.389321, 0.448274, 0.479310, 0.460489, \\
 &\quad 0.416275, 0.530219, 0.403779, 0.464922, 0.497111, 0.477590, 0.431735, 0.549910\}\rangle
 \end{aligned}$$

$$\begin{aligned}
 g_4^{T_1^A} &= HTNWAAO_{T_1}(A_{41}, A_{42}, A_{43}, A_{44}) \\
 &= \langle(0.39, 0.44, 0.51); \{0.398762, 0.437891, 0.476592, 0.510656, 0.419636, 0.457406, 0.494764, 0.527644\}, \{0.439833, \\
 &\quad 0.451737, 0.505235, 0.518910, 0.520235, 0.534315, 0.597593, 0.613767, 0.496724, 0.510168, 0.570586, 0.586030, 0.587525, \\
 &\quad 0.603427, 0.674889, 0.693156\}, \{0.232461, 0.267027, 0.282842, 0.246228, 0.306007, 0.351510, 0.372328, 0.324130\}\rangle
 \end{aligned}$$

$$\begin{aligned}
 g_5^{T_1^A} &= HTNWAAO_{T_1}(A_{51}, A_{52}, A_{53}, A_{54}) \\
 &= \langle(0.39, 0.49, 0.53); \{0.365425, 0.388147, 0.427055, 0.447570, 0.426353, 0.446894, \\
 &\quad 0.482066, 0.500612, 0.463497, 0.482708, 0.515603, 0.532948, 0.515010, 0.532376, 0.562112, 0.577792\}, \{0.328749, \\
 &\quad 0.356519, 0.377634, 0.409533, 0.351510, 0.381202, 0.403779, 0.437887, 0.375847, 0.407595, 0.431735, 0.468204\}, \\
 &\quad \{0.267027, 0.275388, 0.282842, 0.332644, 0.343059, 0.352345, 0.301566, 0.311008, 0.319426, 0.375670, 0.387433, \\
 &\quad 0.397920, 0.328749, 0.339042, 0.348219, 0.409533, 0.422356, 0.433787, 0.371272, 0.382897, 0.393260, 0.462505, \\
 &\quad 0.476986, 0.489897\}\rangle
 \end{aligned}$$

**Step-3:** We calculate the score values of  $g_i^{T_k^A}$  ( $i = 1, 2, \dots, 5$ ) of  $A_i$  ( $i = 1, 2, \dots, 5$ ) as

$$\begin{aligned}
 S(A_1) &= \frac{(0.30 + 0.51 + 0.52)}{3 \cdot 0.52} \cdot \left[ 2 + \frac{1}{4} (0.494124 + 0.522409 + 0.526880 + 0.553333) \right. \\
 &\quad - \frac{1}{8} (0.299254 + 0.308624 + 0.319973 + 0.329992 + 0.416080 + 0.429108 + 0.444888 + 0.458818) \\
 &\quad - \frac{1}{16} (0.262529 + 0.301566 + 0.278077 + 0.319426 + 0.323211 + 0.371272 + 0.342353 + 0.393260 \\
 &\quad + 0.310519 + 0.356693 + 0.328909 + 0.377818 + 0.382294 + 0.439141 + 0.412567 + 0.465148) \\
 &= \frac{1.33}{1.56} \cdot 2 + 0.524186 - 0.375842 - 0.354048 \\
 &= 1.5297,
 \end{aligned}$$

Similarly, we have;  $S(A_2) = 1.3244$ ,  $S(A_3) = 1.2687$ ,  $S(A_4) = 1.4110$ ,  $S(A_5) = 1.5235$ .

**Step-4:** Since  $S(A_1) > S(A_5) > S(A_4) > S(A_2) > S(A_3)$ , So  $A_1 \succ A_5 \succ A_4 \succ A_2 \succ A_3$ .

Thus we conclude that  $A_1$  is the best (most desirable) ERP system. On the other hand, if we apply the other proposed weighted aggregation operators such as  $HTNWGAO_{T_1}$ ,  $HTNWAAO_{T_2}$ ,  $HTNWGAO_{T_2}$  for computing the best alternative(s), then step 2 of the proposed approach has been executed for each weighted aggregation operators and hence their corresponding hesitant triangular neutrosophic number has been constructed. Finally, based on these, the score values of the aggregated hesitant triangular neutrosophic numbers are computed and ranking has been done which are summarized in table-2. We can conclude from table-2 that although the ranking orders of the alternatives are slightly different; the best (most desirable) alternative is still  $A_1$  in all cases.

**Table-2:** Ranking order of alternatives

Weighted aggregation operators	Ranking	Best alternative
$HTNWAAO_{T_1}$	$A_1 \succ A_5 \succ A_4 \succ A_2 \succ A_3$	$A_1$
$HTNWGAO_{T_1}$	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$	$A_1$
$HTNWAAO_{T_2}$	$A_1 \succ A_2 \succ A_5 \succ A_3 \succ A_4$	$A_1$
$HTNWGAO_{T_2}$	$A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$	$A_1$

## 6. COMPARATIVE STUDY:

In order to compare the performance of the proposed method with some existing methods (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017, Wei et al. 2017), a comparative study is presented and their corresponding final ranking are summarized in table 3. From table-3, it is clear that although the ranking order of the alternatives are slightly different, but the best (most desirable) alternative is the same as found in the existing approaches (Ye 2013a, Ye 2014, Ye 2015a, Ye 2015b, Liu 2016, Abdel-Basset et al. 2017). Thus, our proposed method can be suitably utilized to solve the multi attribute decision making problems than the other existing methods due to the fact that more fuzziness and uncertainties are involved in our proposed approach.

**Table 3:** Comparative study

Existing approach	Ranking	Our proposed method	Ranking	Best alternative
Ye [36]	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$A_2$
		$HTNWGAO_{T_1}$	$A_2 \text{ f } A_3 \text{ f } A_4 \text{ f } A_1$	$A_2$
		$HTNWAAO_{T_2}$	$A_2 \text{ f } A_4 \text{ f } A_1 \text{ f } A_3$	$A_2$
		$HTNWGAO_{T_2}$	$A_2 \text{ f } A_1 \text{ f } A_4 \text{ f } A_3$	$A_2$
Ye [17]	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$A_2$
		$HTNWGAO_{T_1}$	$A_2 \text{ f } A_3 \text{ f } A_4 \text{ f } A_1$	$A_2$
		$HTNWAAO_{T_2}$	$A_2 \text{ f } A_4 \text{ f } A_1 \text{ f } A_3$	$A_2$
		$HTNWGAO_{T_2}$	$A_2 \text{ f } A_1 \text{ f } A_4 \text{ f } A_3$	$A_2$
Ye [40]	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_2 \text{ f } A_4 \text{ f } A_3 \text{ f } A_1$	$A_2$
		$HTNWGAO_{T_1}$	$A_2 \text{ f } A_3 \text{ f } A_4 \text{ f } A_1$	$A_2$
		$HTNWAAO_{T_2}$	$A_2 \text{ f } A_4 \text{ f } A_1 \text{ f } A_3$	$A_2$
		$HTNWGAO_{T_2}$	$A_2 \text{ f } A_1 \text{ f } A_4 \text{ f } A_3$	$A_2$
Ye [38]	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$A_4$
		$HTNWGAO_{T_1}$	$A_4 \text{ f } A_3 \text{ f } A_2 \text{ f } A_1$	$A_4$
		$HTNWAAO_{T_2}$	$A_4 \text{ f } A_2 \text{ f } A_1 \text{ f } A_3$	$A_4$
		$HTNWGAO_{T_2}$	$A_4 \text{ f } A_1 \text{ f } A_2 \text{ f } A_3$	$A_4$
Liu [21]	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$A_4$
		$HTNWGAO_{T_1}$	$A_4 \text{ f } A_3 \text{ f } A_2 \text{ f } A_1$	$A_4$
		$HTNWAAO_{T_2}$	$A_4 \text{ f } A_2 \text{ f } A_1 \text{ f } A_3$	$A_4$
		$HTNWGAO_{T_2}$	$A_4 \text{ f } A_1 \text{ f } A_2 \text{ f } A_3$	$A_4$
Abdel-Basset et al. [57]	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$HTNWAAO_{T_1}$	$A_4 \text{ f } A_2 \text{ f } A_3 \text{ f } A_1$	$A_4$
		$HTNWGAO_{T_1}$	$A_4 \text{ f } A_3 \text{ f } A_2 \text{ f } A_1$	$A_4$
		$HTNWAAO_{T_2}$	$A_4 \text{ f } A_2 \text{ f } A_1 \text{ f } A_3$	$A_4$
		$HTNWGAO_{T_2}$	$A_4 \text{ f } A_1 \text{ f } A_2 \text{ f } A_3$	$A_4$

## 7. CONCLUSION

In this paper, hesitant triangular neutrosophic numbers and their basic properties are presented. Also, various types of operations between the hesitant triangular neutrosophic numbers are discussed. Then, various types of hesitant triangular neutrosophic weighted aggregation operators are proposed to aggregate the hesitant triangular neutrosophic information. Furthermore, score of hesitant triangular neutrosophic numbers is proposed to ranking the hesitant triangular neutrosophic numbers. Based on the hesitant triangular neutrosophic weighted aggregation operators and score of hesitant triangular neutrosophic numbers, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of hesitant triangular neutrosophic numbers and the alternatives are ranked according to the values of the score of hesitant triangular neutrosophic numbers to select the most desirable one. Finally, a practical example for enterprise resource planning (ERP) system selection is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems with hesitant triangular neutrosophic information because hesitant triangular neutrosophic numbers can handle indeterminate and inconsistent information and are the extensions of hesitant

triangular fuzzy numbers, hesitant triangular intuitionistic fuzzy numbers as well as triangular neutrosophic numbers.

In the future, we will develop another approach called linguistic hesitant triangular neutrosophic number as a further generalization of it and this will be applied in different practical problems.

**FUNDING:** This research received no external funding.

**ACKNOWLEDGEMENTS:** Nil.

**CONFLICTS OF INTEREST:** The authors declare no conflict of interest.

## REFERENCES:

- [01] Zadeh, L.A. Fuzzy Sets. *Information and Control* 1965, 8, 338-353.
- [02] Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems* 1986, 20, 87-96.
- [03] Torra, V. Hesitant fuzzy sets. *International Journal of Intelligent Systems* 2009, 25, 529-539.
- [04] Beg, I.; Rashid, T. Group decision making using intuitionistic hesitant fuzzy sets. *International Journal of Fuzzy Logic and Intelligent Systems* 2014, 14(3), 181-187.
- [05] Li, D.F. A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems. *Computers and Mathematics with Applications* 2010, 60, 1557-1570.
- [06] Ye, J. Multi-criteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. *Expert Systems with Applications* 2009, 36, 6899-6902.
- [07] Xia, M.M.; Xu, Z.S. Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning* 2011, 52, 395-407.
- [08] Xu, Z.S.; Xia, M.M. Distance and similarity measures for hesitant fuzzy sets. *Information Sciences* 2011, 181, 2128-2138.
- [09] Wei, G.; Alsaadi, F.E.; Hayat, T.; Alsaedi, A. Bipolar fuzzy Hamacher Aggregation operators in multiple attribute decision making. *International Journal of Fuzzy Systems* 2018, 20(1), 1-12.
- [10] Xu, Z.S.; Xia, M.M. Hesitant fuzzy entropy and cross-entropy and their use in multi attribute decision making. *International Journal of Intelligent Systems* 2012, 27, 799-822.
- [11] Xu, Z.S., Xia, M.M. On distance and correlation measures of hesitant fuzzy information. *International Journal of Intelligent Systems* 2013, 26, 410-425.
- [12] Xu, Z.S.; Zhang, X.L. Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information. *Knowledge-Based Systems* 2013, 52, 53-64.
- [13] Chen N., Xu, Z.S., Xia, M.M. Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. *Applied Mathematical Modeling* 2013, 37, 2197-2211.
- [14] Qian, G.; Wang, H.; Feng, X. Generalized of hesitant fuzzy sets and their application in decision support system. *Knowledge Based Systems* 2013, 37, 357-365.
- [15] Yu, D. Triangular hesitant fuzzy set and its application to teaching quality evaluation. *Journal of Information and Computational Science* 2013, 10(7), 1925-1934.
- [16] Yu, D. Intuitionistic trapezoidal fuzzy information aggregation methods and their applications to teaching quality evaluation. *Journal of Information and Computational Science* 2013, 10(6), 1861-1869.
- [17] Ye, J. Correlation coefficient of dual hesitant fuzzy sets and its application to multi attribute decision making. *Applied Mathematical Modelling* 2014, 38, 659-666.
- [18] Shi, J.; Meng, C.; Liu, Y. Approach to multiple attribute decision making based on the intelligence computing with hesitant triangular fuzzy information and their application. *Journal of Intelligent and Fuzzy Systems* 2014, 27, 701-707.
- [19] Pathinathan, T.; Johnson, S.S. Trapezoidal hesitant fuzzy multi-attribute decision making based on TOPSIS. *International Archive of Applied Sciences and Technology* 2015, 3(6), 39-49.
- [20] Joshi, D.; Kumar, S. Interval valued intuitionistic hesitant fuzzy Choquet integral based TOPSIS method for multi-criteria group decision making. *European Journal of Operational Research* 2016, 248, 183-191.
- [21] Liu, P. The aggregation operators based on Archimedean t-Conorm and t-Norm for single-valued neutrosophic numbers and their application to decision making. *International Journal of Fuzzy Systems* 2016, 18(5), 849-863.

- [22] Nehi, H.M. A new ranking method for intuitionistic fuzzy numbers. *International Journal of Fuzzy Systems* 2010, 12(1), 80–86.
- [23] Zhang, Z. Hesitant triangular multiplicative aggregation operators and their application to multiple attribute group decision making. *Neural Computing and Applications* 2017, 28, 195-217.
- [24] Chen, J.J.; Huang, X.J. Hesitant triangular intuitionistic fuzzy information and its application to multi-attribute decision making problem. *Journal of Non-linear Science and Applications* 2017, 10, 1012-1029.
- [25] Yang, Y.; Hu, J.; An, Q.; Chen, X. Group decision making with multiplicative triangular hesitant fuzzy preference relations and cooperative games method. *International Journal for Uncertainty Quantification* 2017, 3(7), 271-284.
- [26] Lan, J.; Yang, M.; Hu, M.; Liu, F. Multi attribute group decision making based on hesitant fuzzy sets, Topsis method and fuzzy preference relations. *Technological and Economic Development of Economy* 2017, 24(6), 2295–2317.
- [27] Zhang, X.; Yang, T.; Liang, W.; Xiong, M. Closeness degree based hesitant trapezoidal fuzzy multi-criteria decision making method for evaluating green suppliers with qualitative information. *Hindawi Discrete Dynamics in Nature and Society*, 2018 (Article ID: 3178039, <https://doi.org/10.1155/2018/3178039>).
- [28] Smarandache, F. A Unifying Field in Logics. *Neutrosophy: Neutrosophic probability, set and logic*, 1999. American Research Press, Rehoboth.
- [29] Gou, Y.; Cheng, H.D. New neutrosophic approach to image segmentation. *Pattern Recognition* 2009, 42, 587-595.
- [30] Guo, Y.H.; Sensur, A. A novel image segmentation algorithm based on neutrosophic similarity clustering. *Applied Soft Computing* 2014, 25, 391-398.
- [31] Karaaslan, F. Correlation coefficients of single valued neutrosophic refined soft sets and their applications in clustering analysis. *Neural Computing and Applications* 2017, 28, 2781-2793.
- [32] Ansari, A.Q.; Biswas, R.; Aggarwal, S. Proposal for applicability of neutrosophic set theory in medical AI. *International Journal of Computer Applications* 2011, 27(5), 5-11.
- [33] Wang, H.; Smarandache, F.; Zhang, Y.Q. Interval neutrosophic sets and logic: Theory and applications in computing 2005. Hexis, Phoenix.
- [34] Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multi-space and Multi-structure* 2010, 4, 410–413.
- [35] Gou, Y.; Cheng, H.D.; Zhang, Y.; Zhao, W. A new neutrosophic approach to image de-noising. *New Mathematics and Natural Computation* 2009, 5, 653-662.
- [36] Ye, J. Multi-criteria decision making method using the correlation coefficient under single valued neutrosophic environment. *International Journal of General Systems* 2013, 42, 386-394.
- [37] Sun, H.X.; Yang, H.X.; Wu, J.Z.; Yao, O.Y. Interval neutrosophic numbers choquet integral operator for multi-criteria decision making. *Journal of Intelligent and Fuzzy Systems* 2015, 28, 2443-2455.
- [38] Ye, J. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. *Neural Computing and Applications* 2015, 26, 1157–1166.
- [39] Abdel-Basset, M.; Chang, V.; Gamal, A. Evaluation of the green supply chain management practices: A novel neutrosophic approach. *Computers in Industry* 2019, 108, 210-220.
- [40] Ye, J. Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment. *Journal of Intelligent Systems* 2015, 24(1), 23–36.
- [41] Xia, M.M.; Xu, Z.S.; Chen, N.; Some hesitant fuzzy aggregation operators with their application in group decision making. *Group Decision and Negotiation* 2013, 22, 259-279.
- [42] Wang, C.Y.; Li, Q.; Zhou, Yang, X.Q. Hesitant triangular fuzzy information aggregation operators based on Bonferroni means and their application to multiple attribute decision making. *Scientific World Journal* 2014 (Article ID: 648516, <http://dx.doi.org/10.1155/2014/648516>).
- [43] Zhao, X.F.; Lin, R.; Wei, G. Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making. *Expert Systems with Applications* 2014, 41, 1086-1094.
- [44] Peng, X. Hesitant trapezoidal fuzzy aggregation operators based on Archimedean t-norm and t-conorm and their application in MADM with completely unknown weight information. *International Journal for Uncertainty Quantification* 2017, 6(7), 475-510
- [45] Xu, Z.S. Intuitionistic fuzzy aggregation operators. *IEEE Transactions on Fuzzy Systems* 2007, 15, 1179-1187.
- [46] Wan, S.P.; Dong, J.Y. Power geometric operators of trapezoidal intuitionistic fuzzy numbers and application to multi attribute group decision making. *Applied Soft Computing* 2015, 29, 153-168.

- [47] Wan, S.P.; Wang, F.; Li, L.; Dong, J.Y. Some generalized aggregation operators for triangular intuitionistic fuzzy numbers and application to multi attribute group decision making. *Computer and Industrial Engineering* 2016, 93, 286-301.
- [48] Xu, Z.S.; Yager, R. R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems* 2006, 35, 417-433.
- [49] Wang, W.Z.; Liu, X.W. Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. *Internal Journal of Intelligent Systems* 2011, 26, 1049-1075.
- [50] Liu, P.D.; Chu, Y. C.; Li, Y. W.; Chen, Y. B. Some generalized neutrosophic number Hamacher aggregation operators and their applications to group decision making. *International Journal of Fuzzy Systems* 2014, 16(2), 242-255.
- [51] Liu, P. D.; Wang, Y. M. Multiple attribute decision making method based on single valued neutrosophic normalized weighted Bonferroni mean. *Neural Computing and Applications* 2014, 25, 2001-2010.
- [52] Chen, J.Q.; Ye, J. Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision making, *Symmetry* 2017, 9, doi:10.3390/sym9060082.
- [53] Zhao, A.W.; Du, J.G.; Guan, H.J. Interval valued neutrosophic sets and multi attribute decision making based on generalized weighted aggregation operators. *Journal of Intelligent and Fuzzy Systems* 2015, 29, 2697-2706.
- [54] Liu, P. D.; Shi, L. L. The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multi attribute decision making. *Neural Computing and Applications* 2015, 26, 457-471.
- [55] Liu, P.D.; Tang, G.L. Some power generalized aggregation operators based on the interval neutrosophic sets and their applications to decision making. *Journal of Intelligent and Fuzzy Systems* 2016, 30, 2517-2528.
- [56] Şubaş, Y. Neutrosophic numbers and their application to multi-attribute decision making problems. (In Turkish, 2015) (Masters Thesis, 7 Aralık University, Graduate School of Natural and Applied Science).
- [57] Abdel-Basset M.; Mohamed, M.; Hussien, A.N.; Sangaiah, A.K. A novel group decision-making model based on triangular neutrosophic numbers. *Soft Computing* 2017, 22, 6629–6643.
- [58] Deli, I.; Şubaş, Y. A ranking method of single valued neutrosophic numbers and its application to multi-attribute decision making problems. *International Journal of Machine Learning and Cybernetics* 2017, 8, 1309–1322.
- [59] Biswas, P.; Pramanik, S.; Giri, B.C. Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. *Neutrosophic Sets and Systems* 2006, 12, 127-138.
- [60] Broumi, S.; Talea, M.; Bakali, A.; Smarandache, F.; Vladareanu, L. Shortest Path Problem Under Triangular Fuzzy Neutrosophic Information. 10th International Conference on Software, Knowledge, Information Management and Applications 2016 (SKIMA).
- [61] Deli, I.; Şubaş, Y. Some weighted geometric operators with SVTrN-numbers and their applications to multi-criteria decision making problems. *Journal of Intelligent and Fuzzy Systems* 2017, 32(1), 291-301.
- [62] Ye, J. Some Weighted Aggregation Operators of Trapezoidal Neutrosophic Numbers and Their Multiple Attribute Decision Making Method, *Informatica* 2017, 28(2), 387–402.
- [63] Biswas, P.; Pramanik, S.; Giri, B.C. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. *Neutrosophic Sets and Systems* 2016, 12, 20-38.
- [64] Deli, I. Operators on single valued trapezoidal neutrosophic numbers and SVTN-group decision making. *Neutrosophic Sets and Systems* 2018, 22, 131-151.
- [65] Karaaslan, F. Gaussian single valued neutrosophic numbers and its applications in multi attribute decision making. *Neutrosophic sets and Systems* 2018, 22, 101-117
- [66] Öztürk, E.K. Some new approaches on single valued trapezoidal neutrosophic numbers and their applications to multiple criteria decision making problems. (In Turkish 2018) (Masters Thesis, Kilis 7 Aralık University, Graduate School of Natural and Applied Science).
- [67] Deli, I.; Öztürk E.K. Neutrosophic Graph Theory and Algorithms, Chapter 10: Two Centroid Point for SVTN-Numbers and SVTrN-Numbers: SVN-MADM Method, IGI Global Publisher 2020, 279-307.
- [68] Deli I.; Öztürk E.K. Neutrosophic Theories in Communication, Management and Information Technology, Chapter 19: Some New Concepts On Normalized Svtn-Number And Multiple Criteria Decision Making, Nova (Publisher) 2020.
- [69] Deli, I. A novel defuzzification method of SV-trapezoidal neutrosophic numbers and multi-attribute decision making: a comparative analysis, *Soft Computing* 2020, 23, 12529–12545.

- [70] Chakraborty, A.; Mondal, S.P.; Ahmadian, A.; Senu, N.; Alam, S.; Salahshour, S. Different forms of triangular neutrosophic numbers, de-neutrosophication techniques, and their applications. *Symmetry* 2018, 10, (doi: 10.3390/sym10080327).
- [71] Fan, C.; Feng, C.; Hu, C. Linguistic neutrosophic numbers Einstein operator and its applications in decision making. *Mathematics* 2019, 7, 1-11.
- [72] Garg, H. ; Nancy. Linguistic single valued neutrosophic power aggregation operators and their applications to group decision making process. *CAA Journal of Automatica Sinica* 2019 , 1-13 (doi: 10.1109/ JAS. 2019.1911522).
- [73] Zhao, S.; Wang, D.; Changyong, D.; Lu, W. Induced choquet integral aggregation operators with single valued neutrosophic uncertain linguistic numbers and their applications in multi-attribute group decision making. *Mathematical problems in Engineering* 2019 (doi.org/10.1155/2019/9143624).
- [74] Deli, I.; Karaaslan, F. Generalized trapezoidal hesitant fuzzy numbers and their applications to multiple criteria decision making problems. *Soft Computing* 2019 (submitted).
- [75] Deli, I. A TOPSIS method by using generalized trapezoidal hesitant fuzzy numbers and application to a robot selection problem. *Journal of Intelligent & Fuzzy Systems* 2019 (doi:10.3233/JIFS-179448).
- [76] Zhang, X.; Xu, Z.; Liu, M. Hesitant trapezoidal fuzzy QUALIFLEX method and its application in the evaluation of green supply Chain Initiatives. *Sustainability* 2016, 8 (doi:10.3390/su8090952).
- [77] Ye, J. Multi-criteria decision making method using expected values in trapezoidal hesitant fuzzy setting. *Journal of Convergence Information Technology* 2013, 8(11) (doi:10.4156/jcit.vol8.issue11.16).

Received: Apr 11, 2020. Accepted: July 2 2020