Hall term 2nd order correction terms in the field solver's Ohm's law

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Abstract

For review and reference I want to list here the derivation of the correction terms for the Hall term in Ohm's law in the field solver.

1 Variables and principle

The Londrillo & Del Zanna field solver propagates the electric field $\mathbf{E}$ components averaged over a simulation cell's edges and the magnetic field $\mathbf{B}$ components averaged over a cell's faces. See Figure 1 for a schematic view.

The electric field averaged along the cell's edges is computed using Ohm's law

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{1}{\rho_q} \mathbf{j} \times \mathbf{B}. \tag{1}$$

The second-order accuracy of the ideal part $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ is dealt with by the field solver. For the Hall term we need to add correction terms to ensure the second-order accuracy as neither $\mathbf{j}$ nor $\mathbf{B}$ are located on the cell edges.

The plan is:

1. Compute $\mathbf{B}^{\mathrm{E}}$ averaged on the edges starting from the face-averaged values $\mathbf{B}^{\mathrm{F}}$ with interpolations up to second derivatives in order to recover the correct order when taking the components' first derivatives.
2. Compute $\mathbf{j} = \nabla \times \mathbf{B}^{\mathrm{E}}/\mu_0$ on the edge.
3. Compute the Hall term $\mathbf{j} \times \mathbf{B}^{\mathrm{E}}$ for Ohm's law using both $\mathbf{j}$ and $\mathbf{B}^{\mathrm{E}}$ which are located on the cell edges.

For this we need (new features in the code) to:

- Compute all second derivatives of $\mathbf{B}^{\mathrm{F}}$ in order to interpolate to get the $\mathbf{B}^{\mathrm{E}}$ properly;
  - Thus we need an extended stencil and possibly different schemes at boundaries;
- Calculate $\mathbf{B}^{\mathrm{E}}$;
- Communicate $\mathbf{B}^{\mathrm{E}}$;
- Calculate $\mathbf{j}$ using $\mathbf{B}^{\mathrm{E}}$;
- Calculate $\mathbf{j} \times \mathbf{B}^{\mathrm{E}}$ in Ohm's law with the new $\mathbf{j}$ and $\mathbf{B}^{\mathrm{E}}$.

The rest of this document handles some of these aspects in detail.

2 Interpolation of $\mathbf{B}^{\mathrm{F}}$ to $\mathbf{B}^{\mathrm{E}}$ up to second order derivatives

In order for $\mathbf{j}$ to be of the right order, being computed as derivatives of $\mathbf{B}^{\mathrm{E}}$, $\mathbf{B}^{\mathrm{E}}$ must be interpolated from $\mathbf{B}^{\mathrm{F}}$ using first- and second-order derivatives (Taylor expansion). The nomenclature used here is:

$^{\mathrm{E}}$ edge averaged;

$^{\mathrm{F}}$ face average;

$^{\mathrm{En}}$ edge averaged along edge n;

$^{\mathrm{Fn}}$ face averaged on face n;

$_n$ component n;

$_{,n}$ derivative along direction n;

x, y, z coordinates in the cell;

ΔN cell size along N.

The coordinate system used has its origin at the centre of the cell, hence faces n are at coordinates $\pm\Delta n$.

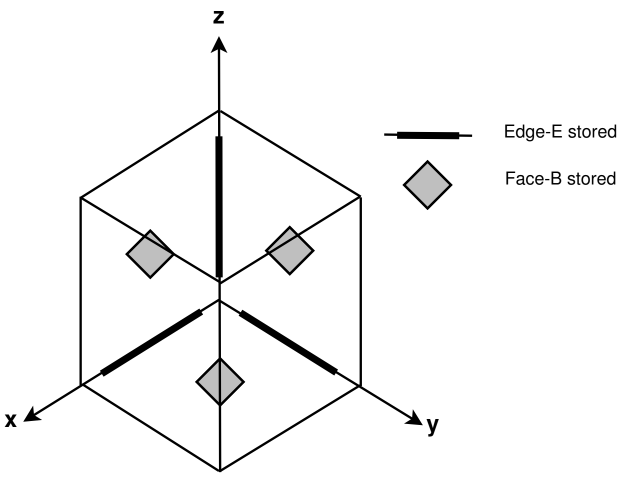

Figure 1: Location of the **E** and **B** components in a spatial cell for the field solver.

2.1 $\mathbf{B}^{\mathrm{E}}$ along the edges

$$
\begin{aligned}
B_x^{\mathrm{E}x} =& B_x^{\mathrm{F}x} \\
&+ B_{x,x}^{\mathrm{F}x}\left(x+\frac{\Delta X}{2}\right) - \frac{1}{2}B_{x,y}^{\mathrm{F}x}\Delta Y - \frac{1}{2}B_{x,z}^{\mathrm{F}x}\Delta Z \\
&+ \frac{1}{2}B_{x,xx}^{\mathrm{F}x}\left(x+\frac{\Delta X}{2}\right)^2 + \frac{1}{8}B_{x,yy}^{\mathrm{F}x}\Delta Y^2 + \frac{1}{8}B_{x,zz}^{\mathrm{F}x}\Delta Z^2 \\
&- \frac{1}{2}B_{x,xy}^{\mathrm{F}x}\Delta Y\left(x+\frac{\Delta X}{2}\right) + \frac{1}{4}B_{x,yz}^{\mathrm{F}x}\Delta Y\Delta Z - \frac{1}{2}B_{x,zx}^{\mathrm{F}x}\Delta Z\left(x+\frac{\Delta X}{2}\right)
\end{aligned} \tag{2}
$$

$$
B_y^{\mathrm{E}x} = B_y^{\mathrm{F}y} + B_{y,x}^{\mathrm{F}y}\cdot x - \frac{1}{2}B_{y,z}^{\mathrm{F}y}\Delta Z + \frac{1}{2}B_{y,xx}^{\mathrm{F}y}\cdot x^2 + \frac{1}{8}B_{y,zz}^{\mathrm{F}y}\Delta Z^2 - \frac{1}{2}B_{y,zx}^{\mathrm{F}y}\Delta Z\cdot x \tag{3}
$$

$$
B_z^{\mathrm{E}x} = B_z^{\mathrm{F}z} + B_{z,x}^{\mathrm{F}z}\cdot x - \frac{1}{2}B_{z,y}^{\mathrm{F}z}\Delta Y + \frac{1}{2}B_{z,xx}^{\mathrm{F}z}\cdot x^2 + \frac{1}{8}B_{z,yy}^{\mathrm{F}z}\Delta Y^2 - \frac{1}{2}B_{z,xy}^{\mathrm{F}z}\Delta Y\cdot x \tag{4}
$$

$$
B_x^{\mathrm{E}y} = B_x^{\mathrm{F}x} + B_{x,y}^{\mathrm{F}x}\cdot y - \frac{1}{2}B_{x,z}^{\mathrm{F}x}\Delta Z + \frac{1}{2}B_{x,yy}^{\mathrm{F}x}\cdot y^2 + \frac{1}{8}B_{x,zz}^{\mathrm{F}x}\Delta Z^2 - \frac{1}{2}B_{x,yz}^{\mathrm{F}x}\Delta Z\cdot y \tag{5}
$$

$$
\begin{aligned}
B_y^{\mathrm{E}y} =& B_y^{\mathrm{F}y} \\
&- B_{y,x}^{\mathrm{F}y}\Delta X + \frac{1}{2}B_{y,y}^{\mathrm{F}y}\left(y+\frac{\Delta Y}{2}\right) - \frac{1}{2}B_{y,z}^{\mathrm{F}y}\Delta Z \\
&+ \frac{1}{8}B_{y,xx}^{\mathrm{F}y}\Delta X^2 + \frac{1}{2}B_{y,yy}^{\mathrm{F}y}\left(y+\frac{\Delta Y}{2}\right)^2 + \frac{1}{8}B_{y,zz}^{\mathrm{F}y}\Delta Z^2 \\
&- \frac{1}{2}B_{y,xy}^{\mathrm{F}y}\Delta X\left(y+\frac{\Delta Y}{2}\right) - \frac{1}{2}B_{y,yz}^{\mathrm{F}y}\Delta Z\left(y+\frac{\Delta Y}{2}\right) + \frac{1}{4}B_{y,zx}^{\mathrm{F}y}\Delta Z\Delta X
\end{aligned} \tag{6}
$$

$$
B_z^{\mathrm{E}y} = B_z^{\mathrm{F}z} - B_{z,x}^{\mathrm{F}z}\cdot\Delta X + \frac{1}{2}B_{z,y}^{\mathrm{F}z}\cdot y + \frac{1}{8}B_{z,xx}^{\mathrm{F}z}\Delta X^2 + \frac{1}{2}B_{z,yy}^{\mathrm{F}z}\cdot y^2 - \frac{1}{2}B_{z,xy}^{\mathrm{F}z}\Delta X\cdot y \tag{7}
$$

$$
B_x^{\mathrm{E}z} = B_x^{\mathrm{F}x} - \frac{1}{2}B_{x,y}^{\mathrm{F}x}\Delta Y + B_{x,z}^{\mathrm{F}x}\cdot z + \frac{1}{8}B_{x,yy}^{\mathrm{F}x}\Delta Y^2 + \frac{1}{2}B_{x,zz}^{\mathrm{F}x}\cdot z^2 - \frac{1}{2}B_{x,yz}^{\mathrm{F}x}\Delta Y\cdot z \tag{8}
$$

$$
B_y^{\mathrm{E}z} = B_y^{\mathrm{F}y} - \frac{1}{2}B_{y,x}^{\mathrm{F}y}\Delta X + B_{y,z}^{\mathrm{F}y}\cdot z + \frac{1}{8}B_{y,xx}^{\mathrm{F}y}\Delta X^2 + \frac{1}{2}B_{y,zz}^{\mathrm{F}y}\cdot z^2 - \frac{1}{2}B_{y,zx}^{\mathrm{F}y}\Delta X\cdot z \tag{9}
$$

$$
\begin{aligned}
B_z^{\mathrm{E}z} =& B_z^{\mathrm{F}z} \\
&- B_{z,x}^{\mathrm{F}z}\Delta X - \frac{1}{2}B_{z,y}^{\mathrm{F}z}\Delta Y + \frac{1}{2}B_{z,z}^{\mathrm{F}z}\left(z+\frac{\Delta Z}{2}\right) \\
&+ \frac{1}{8}B_{z,xx}^{\mathrm{F}z}\Delta X^2 + \frac{1}{8}B_{z,yy}^{\mathrm{F}z}\Delta Y^2 + \frac{1}{2}B_{z,zz}^{\mathrm{F}z}\left(z+\frac{\Delta Z}{2}\right)^2 \\
&+ \frac{1}{4}B_{z,xy}^{\mathrm{F}z}\Delta X\Delta Y - \frac{1}{2}B_{z,yz}^{\mathrm{F}z}\Delta Y\left(z+\frac{\Delta Z}{2}\right) - \frac{1}{2}B_{z,zx}^{\mathrm{F}z}\Delta X\left(z+\frac{\Delta Z}{2}\right)
\end{aligned} \tag{10}
$$

2.2 Edge-averaged $\bar{\mathbf{B}}^{\mathrm{E}}$

The general formula to calculate the edge-averaged value of component n along edge m is

$$\bar{B}_n^{\mathrm{E}m} = \frac{1}{\Delta M} \int_{-\frac{\Delta M}{2}}^{\frac{\Delta M}{2}} B_n^{\mathrm{E}m}(m)\ \mathrm{d}m. \tag{11}$$

If I did not do too many mistakes or an even number of sign errors, the edge-averaged $\mathbf{B}^{\mathrm{E}}$ components are the following.

$$\begin{aligned}
\bar{B}_x^{\mathrm{E}x} =& B_x^{\mathrm{F}x} \\
&+ \frac{1}{2}B_{x,x}^{\mathrm{F}x}\Delta X - \frac{1}{2}B_{x,y}^{\mathrm{F}x}\Delta Y - \frac{1}{2}B_{x,z}^{\mathrm{F}x}\Delta Z \\
&+ \frac{1}{6}B_{x,xx}^{\mathrm{F}x}\Delta X^2 + \frac{1}{8}B_{x,yy}^{\mathrm{F}x}\Delta Y^2 + \frac{1}{8}B_{x,zz}^{\mathrm{F}x}\Delta Z^2 \\
&- \frac{1}{4}B_{x,xy}^{\mathrm{F}x}\Delta X\Delta Y + \frac{1}{4}B_{x,yz}^{\mathrm{F}x}\Delta Y\Delta Z - \frac{1}{4}B_{x,zx}^{\mathrm{F}x}\Delta Z\Delta X && (12)\\
\bar{B}_y^{\mathrm{E}x} =& B_y^{\mathrm{F}y} - \frac{1}{2}B_{y,z}^{\mathrm{F}y}\Delta Z + \frac{1}{8}B_{y,zz}^{\mathrm{F}y}\Delta Z^2 + \frac{1}{24}B_{y,xx}^{\mathrm{F}y}\Delta X^2 && (13)\\
\bar{B}_z^{\mathrm{E}x} =& B_z^{\mathrm{F}z} - \frac{1}{2}B_{z,y}^{\mathrm{F}z}\Delta Y + \frac{1}{8}B_{z,yy}^{\mathrm{F}z}\Delta Y^2 + \frac{1}{24}B_{z,xx}^{\mathrm{F}z}\Delta X^2 && (14)
\end{aligned}$$

$$\begin{aligned}
\bar{B}_x^{\mathrm{E}y} =& B_x^{\mathrm{F}x} - \frac{1}{2}B_{x,z}^{\mathrm{F}x}\Delta Z + \frac{1}{8}B_{x,zz}^{\mathrm{F}x}\Delta Z^2 + \frac{1}{24}B_{x,yy}^{\mathrm{F}x}\Delta Y^2 && (15)\\
\bar{B}_y^{\mathrm{E}y} =& B_y^{\mathrm{F}y} \\
&- \frac{1}{2}B_{y,x}^{\mathrm{F}y}\Delta X + \frac{1}{2}B_{y,y}^{\mathrm{F}y}\Delta Y - \frac{1}{2}B_{y,z}^{\mathrm{F}y}\Delta Z \\
&+ \frac{1}{8}B_{y,xx}^{\mathrm{F}y}\Delta X^2 + \frac{1}{6}B_{y,yy}^{\mathrm{F}y}\Delta Y^2 + \frac{1}{8}B_{y,zz}^{\mathrm{F}y}\Delta Z^2 \\
&- \frac{1}{4}B_{y,xy}^{\mathrm{F}y}\Delta X\Delta Y - \frac{1}{4}B_{y,yz}^{\mathrm{F}y}\Delta Y\Delta Z + \frac{1}{4}B_{y,zx}^{\mathrm{F}y}\Delta Z\Delta X && (16)\\
\bar{B}_z^{\mathrm{E}y} =& B_z^{\mathrm{F}z} - \frac{1}{2}B_{z,x}^{\mathrm{F}z}\Delta X + \frac{1}{8}B_{z,xx}^{\mathrm{F}z}\Delta X^2 + \frac{1}{24}B_{z,yy}^{\mathrm{F}z}\Delta Y^2 && (17)
\end{aligned}$$

$$\begin{aligned}
\bar{B}_x^{\mathrm{E}z} =& B_x^{\mathrm{F}x} - \frac{1}{2}B_{x,y}^{\mathrm{F}x}\Delta Y + \frac{1}{8}B_{x,yy}^{\mathrm{F}x}\Delta Y^2 + \frac{1}{24}B_{x,zz}^{\mathrm{F}x}\Delta Z^2 && (18)\\
\bar{B}_y^{\mathrm{E}z} =& B_y^{\mathrm{F}y} - \frac{1}{2}B_{y,x}^{\mathrm{F}y}\Delta X + \frac{1}{8}B_{y,xx}^{\mathrm{F}y}\Delta X^2 + \frac{1}{24}B_{y,zz}^{\mathrm{F}y}\Delta Z^2 && (19)\\
\bar{B}_z^{\mathrm{E}z} =& B_z^{\mathrm{F}z} \\
&- \frac{1}{2}B_{z,x}^{\mathrm{F}z}\Delta X - \frac{1}{2}B_{z,y}^{\mathrm{F}z}\Delta Y + \frac{1}{2}B_{z,z}^{\mathrm{F}z}\Delta Z \\
&+ \frac{1}{8}B_{z,xx}^{\mathrm{F}z}\Delta X^2 + \frac{1}{8}B_{z,yy}^{\mathrm{F}z}\Delta Y^2 + \frac{1}{6}B_{z,zz}^{\mathrm{F}z}\Delta Z^2 \\
&+ \frac{1}{4}B_{z,xy}^{\mathrm{F}z}\Delta X\Delta Y - \frac{1}{4}B_{z,yz}^{\mathrm{F}z}\Delta Y\Delta Z - \frac{1}{4}B_{z,zx}^{\mathrm{F}z}\Delta Z\Delta X && (20)
\end{aligned}$$