Correction to the Traditional Ideal Rocket Thrust Equation

Samuel R. Hedemann*

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The accepted ideal rocket equation for thrust, widely used in rocketry literature, is proved to be incorrect. The correct replacement is then presented and shown to obey the full form of Newton's second law $F=\dot{p}$, so the sum of external forces on the rocket is the time derivative of its momentum. The accepted velocity equation is correct, which may explain why this error has been overlooked.

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INTRODUCTION

The ideal rocket equations are a pair of equations for the *thrust* force $F_{\text{Thr}}\equiv F_{\text{Thr}}(t)$ of the rocket's exhaust acting on the rocket, and the *velocity* $v(t)$ of the rocket.

In this paper, we show that the quantity identified as F_{Thr} is *not* the true thrust, even though the equation for $v(t)$ is correct. This explains why the misidentification of F_{Thr} has not impeded the development of rocketry; as long as formulas are developed from $v(t)$, most predictions of the rocket's motion would still be correct.

However, it is important to fix this long-standing error. Often called the Tsiolkovsky equations, after Konstantin Tsiolkovsky who published them in 1903 [1], they were derived earlier by William Moore in 1813 [2]. Thus, this incorrect form of F_{Thr} has persisted for over 200 years, such as in [3–6], and is indicative of a widespread misapplication of Newton's second law [7].

For simplicity, we consider a classical rocket traveling at speeds much less than the speed of light, and free from any other influences such as gravity, as shown in Fig. 1.

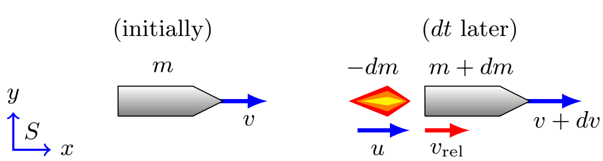

FIG. 1: (color online) Initial and final states of an ideal rocket with changing mass. Its initial mass m has initial velocity v relative to inertial frame S. A time dt later, its mass has changed by $dm<0$ to new rocket mass $m+dm$, with velocity $v+dv$ in S. Its exhaust after dt has mass $-dm$ and velocity u in S, while the rocket's velocity relative to its exhaust is v_{rel}, specified as constant; all other quantities are time-dependent.

Defining the system as rocket-plus-exhaust lets us use conservation of momentum, where m is the time-varying mass of the rocket, $m+dm$ is the mass of the rocket a time dt later, and $dm<0$, so the mass of the exhaust at the nozzle exit after dt is then $-dm$, which has velocity $u\equiv v'_{E,S}$ in an inertial frame S. The rocket's velocity with respect to S at these times is v and $v+dv\equiv v'_{R,S}$, while the relative velocity of the rocket with respect to its exhaust $v_{\text{rel}}\equiv v'_{R,E}$ is specified as constant.

Conservation of momentum (from time integrating the full Newton's second law on the rocket-plus-exhaust system with zero net external force), from t to $t+dt$, is

$$mv=(-dm)u+(m+dm)(v+dv). \tag{1}$$

Using $v'_{R,S}=v'_{R,E}+v'_{E,S}$ yields $u=v+dv-v_{\text{rel}}$, which, put into (1) over dt, gives the first rocket equation,

$$-v_{\text{rel}}\frac{dm}{dt}=ma, \tag{2}$$

which is correct, where $a\equiv\frac{dv}{dt}$ is the acceleration of the rocket in inertial frame S. Then, due to the appearance of ma, (2) is traditionally called the *thrust*,

$$F_{\text{Thr}}\equiv -v_{\text{rel}}\frac{dm}{dt}\qquad\text{(wrong!)}. \tag{3}$$

Although it is true that the mass of the rocket-plus-exhaust system is constant, the m in (2) is not that mass, but rather the changing mass of the rocket itself. Therefore, the ma in (2) which prompted the identification of this quantity as F_{Thr} in (3) is *not* Newton's second law for the rocket-plus-exhaust system (which has *zero* net external force on it), nor is it Newton's second law for the rocket (since $F=ma$ is *only* valid for constant m, which is not the case here), which means that the *true thrust* is generally not equal to $-v_{\text{rel}}\frac{dm}{dt}$.

However, before we explicitly prove that (3) is an incorrect identification of F_{Thr}, the second rocket equation, obtained through time integration of (2), is

$$v(t)=v_i+v_{\text{rel}}\ln\left(\frac{m_i}{m(t)}\right), \tag{4}$$

which is also correct, where $v_i\equiv v(t_i)$ and $m_i\equiv m(t_i)$ are the rocket's velocity and mass at initial time t_i.

PROOF THAT THE TRADITIONAL THRUST EQUATION IS INCORRECT

To see why (3) is wrong, recall that the *impulse* on an object by a net external force $F(t)$ is $J\equiv\int_{t_i}^{t_f}F(t)dt=\Delta p=p(t_f)-p(t_i)$, and thus also gives the change in momentum Δp of the object, a formal solution of the full form of Newton's second law $F=\frac{dp}{dt}$ [7].

Since (4) is correct and the rocket's mass is $m \equiv m(t)$, the change in the rocket's momentum from t_i to any later time t is, using $v(t)$ from (4),

$$\begin{aligned}\Delta p &= m(t)v(t) - m_i v_i \\ &= (m(t) - m_i)v_i + m(t)v_{\rm rel}\ln\left(\frac{m_i}{m(t)}\right),\end{aligned} \tag{5}$$

where, to be clear, this is Δp for the *subsystem of the rocket*, still viewed from inertial frame S. In the rocket subsystem, the sum of external forces is just the thrust.

Now comes the key point; if $F_{\rm Thr}$ from (3) is correct, then we should be able to integrate it over time for the impulse on the rocket to get (5), since both (3) and (5) were derived from (2). Thus, integrating (3), we see that

$$\begin{aligned}J_{F_{\rm Thr}} &= \int_{t_i}^{t} F_{\rm Thr}(t')dt' = \int_{t_i}^{t} (-v_{\rm rel}\tfrac{dm(t')}{dt'})dt' \\ &= -v_{\rm rel}\int_{m(t_i)}^{m(t)} dm(t') = -v_{\rm rel}(m(t) - m_i) \\ &\neq \Delta p,\end{aligned} \tag{6}$$

which proves that $F_{\rm Thr}$ from (3) is an incorrect identification of the thrust on the rocket, since $J_{F_{\rm Thr}}$ from (6) does not give the correct impulse derived in (5).

THE CORRECT ROCKET THRUST

The correct rocket thrust is simply obtained by applying the full form of Newton's second law $F = \dot{p}$ to the rocket subsystem viewed in inertial frame S. The only force on the rocket is the thrust, which we label as $F^{\star}_{\rm Thr}$ to distinguish it from (3). The rocket's momentum is simply $p(t) = m(t)v(t)$ where $v(t)$ is given by (4), so then Newton's second law for the rocket is

$$\begin{aligned}F^{\star}_{\rm Thr} &= \frac{dp(t)}{dt} = \frac{dm(t)}{dt}v(t) + m(t)\frac{dv(t)}{dt} \\ &= \left(v_i + v_{\rm rel}\left[\ln\left(\frac{m_i}{m(t)}\right) - 1\right]\right)\frac{dm(t)}{dt},\end{aligned} \tag{7}$$

where we used $v(t)$ from (4) and $m\frac{dv}{dt} = -v_{\rm rel}\frac{dm}{dt}$ from (2) (which is true; it is only the *interpretation* of $-v_{\rm rel}\frac{dm}{dt}$ as thrust that is incorrect).

To prove that $F^{\star}_{\rm Thr}$ is correct, test its impulse as

$$\begin{aligned}J_{F^{\star}_{\rm Thr}} &= \int_{t_i}^{t} F^{\star}_{\rm Thr}(t')dt' \\ &= \int_{t_i}^{t}\left(v_i + v_{\rm rel}\left[\ln\left(\frac{m_i}{m(t')}\right) - 1\right]\right)\frac{dm(t')}{dt'}dt' \\ &= (m(t) - m_i)v_i + m(t)v_{\rm rel}\ln\left(\frac{m_i}{m(t)}\right) \\ &= \Delta p,\end{aligned} \tag{8}$$

where we used $\int_a^b \ln(m)dm = (m\ln(m) - m)|_a^b$. Thus, (8) agrees with (5), proving that $F^{\star}_{\rm Thr}$ from (7) is the correct thrust since it gives the correct impulse (which is guaranteed since Δp is the time integral of $F^{\star}_{\rm Thr} = \frac{dp(t)}{dt}$). This also proves that the *full* Newton's second law $F = \dot{p}$ applied to the rocket subsystem is perfectly valid, contrary to many published warnings of "misunderstandings about Newton's second law" [6]. To see how poorly $F_{\rm Thr}$ performs, Fig. 2 compares (3) to (7) for a simple example. Note that in (7), which can be written as $v(t)\frac{dm}{dt} - v_{\rm rel}\frac{dm}{dt} = v_{E,S}(t)\frac{dm}{dt}$, we generally cannot ignore the "correction term" $v(t)\frac{dm}{dt}$ added to the traditional thrust $-v_{\rm rel}\frac{dm}{dt}$; the traditional thrust of (3) is missing $v(t)\frac{dm}{dt}$ through its misuse of Newton's second law.

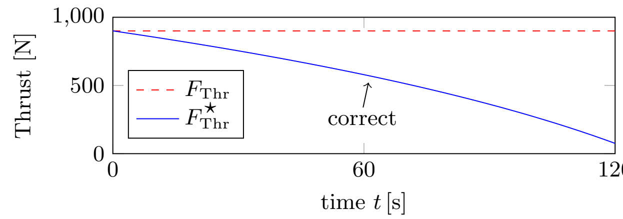

FIG. 2: (color online) Comparison of the incorrect thrust $F_{\rm Thr}$ of (3) to the correct thrust $F^{\star}_{\rm Thr}$ of (7) for an example over 120 [s] with linearly decreasing rocket mass $m(t) = m_i - rt$, where $m_i = 10^4$ [kg], $r = 50\,[\frac{\rm kg}{\rm s}]$, $v_i = 0\,[\frac{\rm m}{\rm s}]$, and $v_{\rm rel} = 18\,[\frac{\rm m}{\rm s}]$, so that 60% of the rocket's initial mass is ejected. This plot shows the possibility of large errors in thrust-dependent quantities using $F_{\rm Thr}$ since $F_{\rm Thr}$ has 1100% error relative to $F^{\star}_{\rm Thr}$ after just two minutes. The thrust *should* decrease; in the discrete case, a rocket throwing out equal boxes at equal velocities $u_{\rm rel} \equiv -v_{\rm rel} < 0$ relative to itself, the momentum change per time (in S) of the rocket and its remaining boxes decreases for successive throws. After each throw, the rocket moves faster in S opposing $u_{\rm rel}$, so the next thrown box's velocity in S is less than that of the previous one, so each box carries away less momentum in S, despite all having the same $u_{\rm rel}$.

THE COPS AND ROBBERS EFFECT: RELATIVE MOTION MAKES THRUST VARY

One way to verify that the new thrust $F^{\star}_{\rm Thr}$ is correct is to use the traditional velocity of (4) to show that the thrust may not only decrease, but can also reach zero and become negative, decelerating the rocket, a phenomenon we call the *cops and robbers effect*, illustrated in Fig. 3.

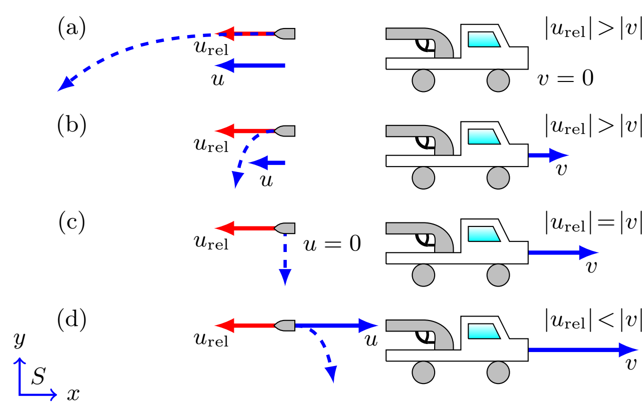

FIG. 3: (color online) *Cops and Robbers Effect:* (a) Robbers at rest $v = 0$ in S. Bullet's S-frame velocity u equals its muzzle velocity $u_{\rm rel}$. (b) Robbers' speed $|v|$ is less than muzzle speed $|u_{\rm rel}|$, but directions oppose, so bullet's $|u|$ in S is less than $|u_{\rm rel}|$. (c) Robbers' v in S is equal and opposite the muzzle velocity, so the bullet just drops straight down in S. (d) Robbers' $|v|$ exceeds muzzle speed $|u_{\rm rel}|$ and directions oppose, so *the bullet chases the robbers!* (Cops not shown.)

If a robber is shooting a gun out the back of a getaway car, when the car goes as fast as the muzzle velocity, the bullets just drop straight down in the ground frame. Faster than that, the bullets actually *follow* the car (more slowly), carrying with them momentum in the forward direction, so the car gains a bit of backwards momentum, hence the net thrust in this region is negative even though the bullets are pointing in the opposite direction!

In each case in Fig. 3, note the bullet's momentum $p_u \equiv m_B u$ in S: in Fig. 3a and Fig. 3b, $p_u < 0$; in Fig. 3c, $p_u = 0$; and in Fig. 3d, $p_u > 0$. Thus, the bullets' momenta decrease in the negative direction, reach zero, then increase in the positive direction, so the forward thrust on the car due to bullets alone decreases to zero and then points in the opposite direction to the robbers' motion, decelerating them. The direction flip of u is purely a relative-motion effect, while the thrust flip is due to both relative motion and conservation of momentum.

To relate this to the rocket problem, let us start by asking: *what is the S-frame velocity of the exhaust $u \equiv v_{E,S}$, and does this velocity ever reach zero?* Confining ourselves to only the traditional equations (1–4), $v_{E,S}$ is

$$v_{E,S} = v_{E,R} + v_{R,S} = -v_{\rm rel} + v, \tag{9}$$

where $v \equiv v_{R,S}$ is the velocity of the rocket relative to S, and $v_{E,R} = -v_{R,E} = -v_{\rm rel} = u_{\rm rel}$ is the velocity of the exhaust relative to the rocket. The conditions for which the exhaust is at rest in S are found by setting (9) to zero at some *critical time* t_c as $v_{E,S}(t_c) \equiv 0$, yielding

$$v(t_c) = v_{\rm rel}, \tag{10}$$

so that putting (10) into (9) at t_c gives, by definition,

$$v_{E,S}(t_c) = 0, \tag{11}$$

meaning t_c is when the rocket has the equal and opposite velocity in S as the exhaust has relative to the rocket (since $v_{\rm rel} = -u_{\rm rel} = -v_{E,R}$), similarly to Fig. 3c. The exact t_c depends on $m(t)$. In general, the traditional velocity in (4) specifies a *critical mass* $m(t_c)$ at which (11) happens. Solving (4) for $m(t)$ and evaluating at t_c gives

$$m(t_c) = m_i e^{\frac{v_i - v(t_c)}{v_{\rm rel}}} = m_i e^{\frac{v_i}{v_{\rm rel}} - 1}. \tag{12}$$

In the case of linear mass depletion, as in Fig. 2,

$$m(t) = \begin{cases} m_i - rt; & t \in [0, \frac{m_i - m_{\rm min}}{r}] \\ m_{\rm min}; & t > \frac{m_i - m_{\rm min}}{r} \end{cases}, \tag{13}$$

where $m_{\rm min}$ is the remaining rocket mass after all fuel is spent, such that $0 < m_{\rm min} < m_i$, and $r > 0$, $t_i = 0$, and $v_i = 0$ for simplicity. Then we can solve (12) for t_c as

$$t_c = (1 - e^{-1})\frac{m_i}{r} \approx 0.632\frac{m_i}{r}, \tag{14}$$

and the critical mass is

$$m(t_c) = m_i e^{-1} \approx 0.368 m_i. \tag{15}$$

Then, putting (4) and (13) into (9) gives the velocity of the exhaust in S as

$$v_{E,S}(t) = v_{\rm rel}\left[\ln\left(\frac{1}{1 - \frac{r}{m_i}t}\right) - 1\right]. \tag{16}$$

So far, (9–16) are purely the results of the traditional problem solution, and nothing has relied on any definition of thrust. With this in mind, Table I checks the sign of $v_{E,S}(t)$ at various times before, including, and after t_c.

TABLE I: Values of $v_{E,S}(t)$ from (16) at key times relative to t_c, where $m_{\rm min} \equiv 0.1 m_i$ and $t \leq \frac{m_i - m_{\rm min}}{r}$ so that only the top row of (13) applies, so $m(t) \geq m_{\rm min}$ at all of these times, and $v_{\rm rel} > 0$, as in Fig. 2. These results prove by example that the exhaust can change direction in S.

t	$v_{E,S}(t)$
$(1 - e^{-0})\frac{m_i}{r} = 0.000\frac{m_i}{r} < t_c$	$-v_{\rm rel} < 0$
$(1 - e^{-\frac{1}{2}})\frac{m_i}{r} \approx 0.393\frac{m_i}{r} < t_c$	$-\frac{1}{2}v_{\rm rel} < 0$
$(1 - e^{-1})\frac{m_i}{r} \approx 0.632\frac{m_i}{r} = t_c$	0
$(1 - e^{-\frac{3}{2}})\frac{m_i}{r} \approx 0.777\frac{m_i}{r} > t_c$	$+\frac{1}{2}v_{\rm rel} > 0$
$(1 - e^{-2})\frac{m_i}{r} \approx 0.865\frac{m_i}{r} > t_c$	$+v_{\rm rel} > 0$

Thus, Table I shows that, *using only the traditional velocity equation and Galilean relativity, the velocity of the exhaust leaving the rocket with respect to the ground frame S must change sign as time increases.*

Therefore, all exhaust leaving the rocket after t_c has a *positive momentum* in S, so the exhaust caries momentum out of the rocket subsystem in the $+x$ direction after t_c. Thus, by conservation of momentum, the rocket must lose momentum in the $+x$ direction after t_c, meaning the rocket's rate of change of momentum is negative after t_c, so *there must be a force on the rocket in the $-x$ direction after t_c*. Furthermore, since any exhaust leaving the rocket at t_c carries *zero* momentum out of the rocket, its momentum does not need to change to compensate, so *there can only be zero force on the rocket at t_c*.

Therefore, the *correct* thrust $F(t)$ must satisfy

$$\text{sgn}[F(t)] = \text{sgn}(t_c - t), \tag{17}$$

which gives us a necessary test *based only on (4) (derived from conservation of momentum) and Galilean relativity.* (A complete test would use the full Newton's second law $F = \dot{p}$, but we avoid that here to show that even if you do not believe it, $F_{\rm Thr}$ must still be wrong.)

Test of Traditional Thrust: To see if the *traditional thrust* $F_{\rm Thr}$ of (3) is compatible with the actual physics of the situation derived above, putting (13) into (3) gives

$$F_{\rm Thr}(t) = -v_{\rm rel}\frac{d(m_i - rt)}{dt} = v_{\rm rel} r = \text{constant} > 0, \tag{18}$$

showing that the traditional form claims that the thrust is constant and positive at all times, including at t_c, when no momentum leaves the rocket momentarily, yielding

$$F_{\rm Thr}(t_c) = v_{\rm rel} r \neq 0, \tag{19}$$

contradicting (17) and thus the traditional velocity of (4) and Galilean relativity, and therefore $F_{\rm Thr}$ must be false.

Test of New Thrust: Putting (13) into our *new thrust* $F^\star_{\rm Thr}$ of (7) and evaluating at t_c of (14) gives

$$\begin{aligned} F^\star_{\rm Thr}(t_c) &= v_{\rm rel} r\left[1-\ln\left(\frac{1}{1-\frac{r}{m_i}t_c}\right)\right] \\ &= v_{\rm rel} r\left[1-\ln\left(\frac{1}{1-(1-e^{-1})}\right)\right] = v_{\rm rel} r[1-\ln(e)] \\ &= 0, \end{aligned} \tag{20}$$

which is exactly the behavior required by conservation of momentum, and furthermore, $F^\star_{\rm Thr}(t)$ has the correct sign behavior to satisfy (17), as seen in Fig. 4.

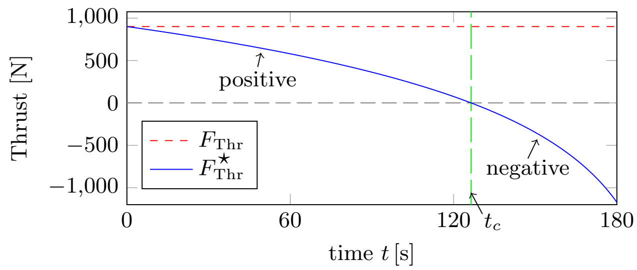

FIG. 4: (color online) The same situation as in Fig. 2, with time extended to include both the critical time t_c from (14) and $t = (1-e^{-2})\frac{m_i}{r}$, the final time from Table I. This plot shows that the correct thrust $F^\star_{\rm Thr}$ flips direction at the same time t_c as the S-frame velocity of the exhaust $v_{E,S} \equiv u$, and thus $F^\star_{\rm Thr}$ behaves in a way consistent with (17), (4), and Galilean relativity, while the traditional thrust $F_{\rm Thr}$ does not.

Therefore, the new thrust $F^\star_{\rm Thr}$ has the correct physical behavior, while the traditional thrust $F_{\rm Thr}$ does not.

VARIABLE-MASS WORK-KINETIC ENERGY THEOREM: ALTERNATIVE PROOF

Before we can examine the energy of variable mass systems, we must acknowledge that the popular equations for energy are derived from the constant-mass version of Newton's second law $F = ma$. Therefore we must first derive variable-mass versions using $F = \dot{p}$, which are in App. A, the results of which we summarize here.

In one dimension, the net external work W on an object of mass m due to net external force $F(t)$ on it is

$$W \equiv W(t) \equiv \textstyle\int_{x(t_i)}^{x(t)} F(t')dx(t'). \tag{21}$$

Using the full Newton's second law $F = \dot{p}$ in (21) yields

$$\begin{aligned} W &= \textstyle\int_{t_i}^{t} v^2(t')\frac{dm(t')}{dt'}dt' + \int_{t_i}^{t} p(t')a(t')dt' \\ &= \textstyle\int_{t_i}^{t} v^2\frac{dm}{dt'}dt' + \int_{t_i}^{t} padt', \end{aligned} \tag{22}$$

where $p(t) \equiv m(t)v(t)$, $a(t) \equiv \frac{dv(t)}{dt}$, and we will suppress time-dependence in integrands from now on, with the understanding that it is there in general, and we keep time as the variable of integration for consistency. When m is constant, (22) simplifies to the usual work-kinetic energy theorem $W = \frac{1}{2}mv^2(t)|_{t_i}^{t} \equiv \Delta K$. Thus, (22) is the *variable-mass work-kinetic energy theorem.*

Defining kinetic energy $K(t) \equiv \frac{1}{2}m(t)v^2(t)$ and $\Delta K \equiv \Delta K(t) \equiv K(t) - K(t_i)$, alternative forms of (22) are

$$W = 2\Delta K - \textstyle\int_{t_i}^{t} padt', \tag{23}$$

involving momentum p, and

$$W = \Delta K + \tfrac{1}{2}\textstyle\int_{t_i}^{t} v^2\frac{dm}{dt'}dt', \tag{24}$$

involving mass current $\frac{dm}{dt}$. We can also solve these for a "work-free" version of the kinetic energy change as

$$\Delta K = \tfrac{1}{2}\textstyle\int_{t_i}^{t} v^2\frac{dm}{dt'}dt' + \int_{t_i}^{t} padt'. \tag{25}$$

For our purposes, we want a general test that will enable us to verify whether a given net external force F is consistent with (22). A simple way to achieve this is to use (24) with (21) in the left side as

$$W = \textstyle\int_{x(t_i)}^{x(t)} Fdx = \Delta K + \frac{1}{2}\int_{t_i}^{t} v^2\frac{dm}{dt'}dt'. \tag{26}$$

Since the right side of (26) can be expanded entirely with functions that do not explicitly involve F, the left side gives us an operation to test a given candidate F.

For an ideal rocket, solving (4) as $m(t) = m_i e^{\frac{v_i - v(t)}{v_{\rm rel}}}$ and putting that into (26) yields, from App. B,

$$W = \textstyle\int_{x(t_i)}^{x(t)} Fdx = 2\Delta K + v_{\rm rel}\Delta p + v_{\rm rel}^2\Delta m, \tag{27}$$

so (27) is our test to verify which thrust is correct for an ideal rocket, regardless of the particular form of $m(t)$.

Test of Traditional Thrust: Putting $F_{\rm Thr}$ into (27) gives, as shown in App. C,

$$\begin{aligned} \textstyle\int_{x(t_i)}^{x(t)} F_{\rm Thr}dx &= -v_{\rm rel}\Delta p - v_{rel}^2\Delta m \\ &\neq W, \end{aligned} \tag{28}$$

so the traditional thrust $F_{\rm Thr}$ of (3) *fails* to produce the correct work as required by (22) and (27).

Test of New Thrust: Putting $F^\star_{\rm Thr}$ into (27) gives

$$\begin{aligned} \textstyle\int_{x(t_i)}^{x(t)} F^\star_{\rm Thr}dx &= 2\Delta K + v_{\rm rel}\Delta p + v_{\rm rel}^2\Delta m \\ &= W, \end{aligned} \tag{29}$$

as in App. D, so the new thrust $F^\star_{\rm Thr}$ of (7) *produces the exact correct work* required by (22) and (27).

Therefore, we have just proved, using the variable-mass work-kinetic energy theorem of (22–25), that the traditional thrust $F_{\rm Thr}$ of (3) is *wrong* while our new thrust $F^\star_{\rm Thr}$ of (7) is *correct* since it does the proper work.

CONCLUSIONS

We have shown that the traditional identification of rocket thrust in (3) is incorrect, and have provided and proven the correct thrust to be $F^{\star}_{\mathrm{Thr}}$ in (7), also demonstrating that the full form of Newton's second law $F = \dot{p}$ is the correct version to use. The original derivation starting from the rocket-exhaust system is still completely valid, as is the equation in (2); it is merely the identification of that quantity as thrust that is incorrect.

Since most models are simply solved for the *trajectory* of the rocket, obtainable from (4) which is correct, that is likely why this error has not been noticed. The only time it would cause a problem would be if F_{Thr} from (3) were used to calculate something thrust-dependent, such as thrust-to-weight ratio or specific impulse (not the same quantity as the impulse used earlier), popular rocket-performance metrics. However, since the correct method of obtaining a *trajectory* is to solve Newton's full second law which includes a sum of the forces acting on the rocket, then the *trajectory is generally thrust-dependent*. Therefore, using the correct thrust presented here is imperative to obtain correct solutions in nearly all aspects of rocketry for real-world applications. Large errors in F_{Thr}, as in Fig. 4, may lead to costly false conclusions about rocket designs.

Rocket designers who use thrust-to-weight ratio or specific impulse as fitness indicators must now revisit all conclusions based on the incorrect thrust of (3), and all measured data taken on the basis of (3) needs to be reinterpreted. Furthermore, publishers of mechanics textbooks now need to rewrite their variable-mass problems. While (2) is still valid, *interpreting* it as thrust is incorrect, and is a misapplication of Newton's second law that must now be remedied.

For reference, in the slightly more general version of this problem where v_{rel} is time-dependent, the velocity of the rocket becomes, generalizing (4),

$$v(t) = v_i - \int_{t_i}^{t} \frac{v_{\mathrm{rel}}(t')}{m(t')} \frac{dm(t')}{dt'} dt', \tag{30}$$

where (2) still holds except that v_{rel} is time-dependent, and the correct thrust force is, generalizing (7),

$$\begin{aligned} F^{\star}_{\mathrm{Thr}} &= \tfrac{dp}{dt} = \left[v_i - v_{\mathrm{rel}}(t) - \int_{t_i}^{t} \tfrac{v_{\mathrm{rel}}(t')}{m(t')} \tfrac{dm(t')}{dt'} dt' \right] \tfrac{dm(t)}{dt} \\ &= [v(t) - v_{\mathrm{rel}}(t)] \tfrac{dm(t)}{dt} = v_{E,S}(t) \tfrac{dm(t)}{dt}. \end{aligned} \tag{31}$$

This generalizes to vectors simply as

$$\mathbf{F}^{\star}_{\mathrm{Thr}}(t) = \left[\mathbf{v}_i - \mathbf{v}_{\mathrm{rel}}(t) - \int_{t_i}^{t} \frac{\mathbf{v}_{\mathrm{rel}}(t')}{m(t')} \frac{dm(t')}{dt'} dt' \right] \frac{dm(t)}{dt}, \tag{32}$$

so that (32) is the force for the rocket thrust that appears as one of the forces in the sum of external forces on the rocket (where the rocket includes its remaining fuel) within the full Newton's second law as $\sum_k \mathbf{F}_k = \dot{\mathbf{p}}$, where the sum generally includes gravitational forces, friction, and any other forces on the rocket.

Note that although the ideal velocity equation in (4) is correct, its popular adaptation to include gravity is wrong because that is derived from $\sum_k \mathbf{F}_k = m\mathbf{a}$. Only the full Newton's second law $\sum_k \mathbf{F}_k = \dot{\mathbf{p}}$ can give the proper equation of motion when multiple forces are involved in variable-mass systems. For example, one must solve $\mathbf{F}^{\star}_{\mathrm{Thr}} + \mathbf{F}_{\mathrm{G}} = \dot{\mathbf{p}}$ where $\mathbf{F}_{\mathrm{G}}$ is the force of gravity on the rocket and the thrust is from (7) or (32). For the case of constant gravitational acceleration g, the velocity can then be solved for by time-integrating this equation of motion which incurs a mass-memory term as $v(t) = v_i + v_{\mathrm{rel}} \ln(\frac{m_i}{m(t)}) - \frac{g}{m(t)} \int_{t_i}^{t} m(t') dt'$, whereas the traditional version of this is $v(t) = v_i + v_{\mathrm{rel}} \ln(\frac{m_i}{m(t)}) - g \cdot (t - t_i)$, indicating that its gravity term is really the constant-mass version of what it should be. More realistic models, such as those using Newton's full gravitation law, will likely require numerical methods of solving coupled systems of differential equations to obtain solutions, and may even require the gravitational Jefimenko equations since mass is changing and Newton's gravitation law is for static situations.

APPENDIX A: Derivation of the Variable-Mass Work-Kinetic Energy Theorem

Starting with the full Newton's second law, $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, where $\mathbf{F}$ is the sum of external forces acting on the object of momentum $\mathbf{p} \equiv m\mathbf{v}$ of mass m and velocity $\mathbf{v}$, if we take the dot product of $\mathbf{F}$ with infinitesimal displacement $d\mathbf{r}$ and integrate, we get

$$\int_{\mathbf{r}(t_i)}^{\mathbf{r}(t)} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}(t_i)}^{\mathbf{r}(t)} \tfrac{d\mathbf{p}}{dt} \cdot d\mathbf{r}. \tag{33}$$

Then, using $\mathbf{p} \equiv m\mathbf{v}$ and $\mathbf{v} \equiv \frac{d\mathbf{r}}{dt}$ in (33) and noting that $\int_{\mathbf{r}(t_i)}^{\mathbf{r}(t)} \mathbf{F} \cdot d\mathbf{r} = W(t) \equiv W$ is the net external work on the object (the work due to the sum of the external forces on the object), (33) becomes

$$\begin{aligned} W &= \int_{\mathbf{r}(t_i)}^{\mathbf{r}(t)} \left(\tfrac{dm}{dt}\mathbf{v} + m\tfrac{d\mathbf{v}}{dt} \right) \cdot d\mathbf{r} \\ &= \int_{\mathbf{r}(t_i)}^{\mathbf{r}(t)} (\mathbf{v} \cdot \mathbf{v} dm + m\mathbf{v} \cdot d\mathbf{v}) \\ &= \int_{m(t_i)}^{m(t)} v^2 dm + \int_{v^2(t_i)}^{v^2(t)} m \tfrac{1}{2} d(v^2) \\ &= \int_{m(t_i)}^{m(t)} v^2 dm + \int_{v(t_i)}^{v(t)} mv dv, \end{aligned} \tag{34}$$

where we used $d(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot d\mathbf{v} \Rightarrow \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2} d(v^2)$. Then, adjusting (34) for time as variable of integration,

$$\begin{aligned} W &= \int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' + \int_{t_i}^{t} mv \tfrac{dv}{dt'} dt' \\ &= \int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' + \int_{t_i}^{t} pa dt', \end{aligned} \tag{35}$$

which gives the result in (22).

Next, to introduce kinetic energy $K \equiv K(t) \equiv \frac{1}{2} m(t) v^2(t)$, noting that $d(mv^2) = v^2 dm + 2mv dv$, integrating, and rearranging gives

$$\begin{aligned} \int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' &= \int_{t_i}^{t} \tfrac{d(mv^2)}{dt'} dt' - 2 \int_{t_i}^{t} mv \tfrac{dv}{dt'} dt' \\ &= (mv^2)\big|_{t_i}^{t} - 2 \int_{t_i}^{t} pa dt', \end{aligned} \tag{36}$$

which, when put into (35) gives

$$\begin{aligned} W &= (mv^2)|_{t_i}^{t} - 2\int_{t_i}^{t} padt' + \int_{t_i}^{t} padt' \\ &= 2\Delta K - \int_{t_i}^{t} padt', \end{aligned} \tag{37}$$

which is the result in (23), where $\Delta K \equiv K(t) - K(t_i)$.

Then, solving for $\int_{t_i}^{t} padt' = 2\Delta K - W$ in (37) and putting that into (35) gives

$$\begin{aligned} W &= \int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' + 2\Delta K - W \\ 2W &= 2\Delta K + \int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' \\ W &= \Delta K + \tfrac{1}{2}\int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt', \end{aligned} \tag{38}$$

which is the result in (24).

To get a "work-free" form of ΔK, put (37) into (38) as

$$\begin{aligned} 2\Delta K - \int_{t_i}^{t} padt' &= \Delta K + \tfrac{1}{2}\int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' \\ \Delta K &= \tfrac{1}{2}\int_{t_i}^{t} v^2 \tfrac{dm}{dt'} dt' + \int_{t_i}^{t} padt', \end{aligned} \tag{39}$$

which is the form in (25).

Thus, we have derived the essential results of the variable-mass work-kinetic energy theorem.

APPENDIX B: Derivation of the Thrust Test by the Variable-Mass Work-Kinetic Energy Theorem

Solving (4) for $m(t)$ gives

$$m(t) = m_i e^{\frac{v_i - v(t)}{v_{\rm rel}}}, \tag{40}$$

and time-differentiating (40) yields

$$\frac{dm}{dt} = -\frac{m_i}{v_{\rm rel}} e^{\frac{v_i - v(t)}{v_{\rm rel}}} \frac{dv}{dt}. \tag{41}$$

Putting (40) and (41) into (27) or (38) gives

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F dx &= \tfrac{1}{2} m_i e^{\frac{v_i - v}{v_{\rm rel}}} v^2|_{t_i}^{t} - \tfrac{m_i}{2v_{\rm rel}} \int_{v(t_i)}^{v(t)} v^2 e^{\frac{v_i - v}{v_{\rm rel}}} dv \\ &= \tfrac{m_i}{2}\left(e^{\frac{v_i - v}{v_{\rm rel}}} v^2|_{v_i}^{v(t)} - \tfrac{1}{v_{\rm rel}} \int_{v_i}^{v(t)} v^2 e^{\frac{v_i - v}{v_{\rm rel}}} dv \right). \end{aligned} \tag{42}$$

Letting $w \equiv \frac{-v}{v_{\rm rel}}$, so $v = -v_{\rm rel} w$ and $dv = -v_{\rm rel} dw$, then

$$\begin{aligned} \int_{v_i}^{v(t)} v^2 e^{\frac{v_i - v}{v_{\rm rel}}} dv &= -v_{\rm rel}^3 e^{\frac{v_i}{v_{\rm rel}}} \int_{w(t_i)}^{w(t)} w^2 e^w dw \\ &= -v_{\rm rel}^3 e^{\frac{v_i}{v_{\rm rel}}} (w^2 e^w - 2we^w + 2e^w)|_{w_i}^{w} \\ &= -v_{\rm rel}^3 e^{\frac{v_i}{v_{\rm rel}}} \left(\tfrac{v^2}{v_{\rm rel}^2} + 2\tfrac{v}{v_{\rm rel}} + 2\right) e^{\frac{-v}{v_{\rm rel}}} |_{v_i}^{v(t)} \\ &= -(v_{\rm rel} v^2 + 2v_{\rm rel}^2 v + 2v_{\rm rel}^3) e^{\frac{v_i - v}{v_{\rm rel}}} |_{v_i}^{v(t)}. \end{aligned} \tag{43}$$

Finally, putting (43) into (42) gives

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F dx &= \tfrac{m_i}{2}\Big(e^{\frac{v_i - v}{v_{\rm rel}}} v^2 \\ &\quad + \tfrac{1}{v_{\rm rel}} (v_{\rm rel} v^2 + 2v_{\rm rel}^2 v + 2v_{\rm rel}^3) e^{\frac{v_i - v}{v_{\rm rel}}} \Big) |_{v_i}^{v(t)} \\ &= \tfrac{1}{2} m_i e^{\frac{v_i - v}{v_{\rm rel}}} (2v^2 + 2v_{\rm rel} v + 2v_{\rm rel}^2)|_{v_i}^{v(t)} \\ &= (mv^2 + v_{\rm rel} mv + v_{\rm rel}^2 m)|_{t_i}^{t} \\ &= 2\Delta K + v_{\rm rel}\Delta p + v_{\rm rel}^2 \Delta m, \end{aligned} \tag{44}$$

which is the thrust test of (27).

APPENDIX C: Test of Traditional Thrust

Putting the traditional thrust $F_{\rm Thr}$ of (3) into (27) and using (41) gives

$$\int_{x(t_i)}^{x(t)} F_{\rm Thr} dx = \int_{x(t_i)}^{x(t)} (-v_{\rm rel} \tfrac{dm}{dt}) dx = m_i \int_{v_i}^{v(t)} e^{\frac{v_i - v}{v_{\rm rel}}} v dv. \tag{45}$$

Using $w \equiv \frac{-v}{v_{\rm rel}}$, so $v = -v_{\rm rel} w$ and $dv = -v_{\rm rel} dw$, then

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F_{\rm Thr} dx &= v_{\rm rel}^2 m_i e^{\frac{v_i}{v_{\rm rel}}} \int_{w_i}^{w} e^w w dw \\ &= v_{\rm rel}^2 m_i e^{\frac{v_i}{v_{\rm rel}}} (we^w - e^w)|_{w_i}^{w} \\ &= v_{\rm rel}^2 m_i e^{\frac{v_i - v}{v_{\rm rel}}} \left(\tfrac{-v}{v_{\rm rel}} - 1\right)|_{v_i}^{v} \\ &= -(v_{\rm rel} mv + v_{\rm rel}^2 m)|_{t_i}^{t} \\ &= -v_{\rm rel}\Delta p - v_{\rm rel}^2 \Delta m, \end{aligned} \tag{46}$$

which is the result in (28), and is *not* the correct work, which was derived in (44).

APPENDIX D: Test of New Thrust

Putting the new thrust $F_{\rm Thr}^{\star}$ of (7) into (27) gives

$$\int_{x(t_i)}^{x(t)} F_{\rm Thr}^{\star} dx = \int_{x(t_i)}^{x(t)} \left(v_i + v_{\rm rel}\left[\ln\left(\tfrac{m_i}{m}\right) - 1\right]\right) \tfrac{dm}{dt} dx. \tag{47}$$

Then, putting (40) into (47) yields

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F_{\rm Thr}^{\star} dx &= \int_{x(t_i)}^{x(t)} \left(v_i + v_{\rm rel}\left[\ln\left(\frac{m_i}{m_i e^{\frac{v_i - v}{v_{\rm rel}}}} \right) - 1 \right] \right) \tfrac{dm}{dt} dx \\ &= \int_{x(t_i)}^{x(t)} \left(v_i + v_{\rm rel}\left[-\tfrac{v_i - v}{v_{\rm rel}} - 1\right]\right) \tfrac{dm}{dt} dx \\ &= \int_{x(t_i)}^{x(t)} (v - v_{\rm rel}) \tfrac{dm}{dt} dx, \end{aligned} \tag{48}$$

and using (41) in (48), we get

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F_{\rm Thr}^{\star} dx &= \int_{x(t_i)}^{x(t)} (v - v_{\rm rel}) \left(-\tfrac{m_i}{v_{\rm rel}} e^{\frac{v_i - v}{v_{\rm rel}}} \tfrac{dv}{dt} \right) dx \\ &= -\tfrac{m_i}{v_{\rm rel}} \Big(\int_{v_i}^{v(t)} v^2 e^{\frac{v_i - v}{v_{\rm rel}}} dv \\ &\quad - v_{\rm rel} \int_{v_i}^{v(t)} e^{\frac{v_i - v}{v_{\rm rel}}} v dv \Big). \end{aligned} \tag{49}$$

Putting (43) into (49) for the first integral, and for the second integral, noting from (45) and (46) that

$$\int_{v_i}^{v(t)} e^{\frac{v_i - v}{v_{\rm rel}}} v dv = e^{\frac{v_i - v}{v_{\rm rel}}} (-v_{\rm rel} v - v_{\rm rel}^2)|_{v_i}^{v}, \tag{50}$$

then (49) becomes

$$\begin{aligned} \int_{x(t_i)}^{x(t)} F_{\rm Thr}^{\star} dx &= -\tfrac{m_i}{v_{\rm rel}} \Big[-(v_{\rm rel} v^2 + 2v_{\rm rel}^2 v + 2v_{\rm rel}^3) e^{\frac{v_i - v}{v_{\rm rel}}} |_{v_i}^{v} \\ &\quad - v_{\rm rel} e^{\frac{v_i - v}{v_{\rm rel}}} (-v_{\rm rel} v - v_{\rm rel}^2)|_{v_i}^{v} \Big] \\ &= -m(-v^2 - 2v_{\rm rel} v - 2v_{\rm rel}^2 + v_{\rm rel} v + v_{\rm rel}^2)|_{t_i}^{t} \\ &= (mv^2 + v_{\rm rel} mv + v_{\rm rel}^2 m)|_{t_i}^{t} \\ &= 2\Delta K + v_{\rm rel}\Delta p + v_{\rm rel}^2 \Delta m, \end{aligned} \tag{51}$$

which is the result in (29), and is the correct work of (44).

* samuel.hedemann@gmail.com

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